# THE $0^{++}$ GLUEBALL MASS IN SU(3) LATTICE GAUGE THEORY: TOWARDS DEFINITIVE RESULTS

Ph. DE FORCRAND
*Centre de Physique Théorique, Ecole Polytechnique, Palaiseau, France*

G. SCHIERHOLZ
*Deutsches Elektronen-Synchrotron DESY, Hamburg, Fed. Rep. Germany*
*and Institut für Kernphysik, Kernforschungsanlage Jülich, Jülich, Fed. Rep. Germany*

H. SCHNEIDER
*Max-Planck-Institut für Physik und Astrophysik, Werner-Heisenberg-Institut für Physik, Munich, Fed. Rep. Germany*

and

M. TEPER
*L.A.P.P., Annecy, France*
*and Institut für Kernphysik, Kernforschungsanlage Jülich, Jülich, Fed. Rep. Germany*

Received 30 November 1984

We calculate the $0^{++}$ glueball mass for $\beta = 5.5$, 5.7 and 5.9 on lattices ranging in size from $6^3 \times 12$ to $10^3 \times 20$ using the source method. The calculation is accurate enough to identify the asymptotic exponential decay of the correlation functions which makes sure that we are extracting the ground state. It also allows us to determine the infinite volume limit. We find the mass to be remarkably consistent with asymptotic scaling: its $\beta$ dependence definitely differs from that of the string tension. If we take the $\beta = 5.7$, 5.9 data only, we are $\approx 1\sigma$ from asymptotic scaling and $\approx 1\sigma$ from the same $\beta$ dependence as the string tension.

*1. Introduction.* Recent "second generation" calculations of the string tension [1,2] $K$ and deconfining temperature [3] indicate appreciable deviations from asymptotic scaling

$$\sqrt{K}a(\beta) = (\sqrt{K}/\Lambda_L)[(8\pi^2/33)\,\beta]^{51/121} \times \exp[-(4\pi^2/33)\,\beta] \qquad (1)$$

in the interesting coupling region $5.5 \lesssim \beta \lesssim 6.0$, where accurate calculations are presently feasible. This implies that either we are not yet in the continuum region (of these quantities) or higher order corrections to the two-loop perturbative $\beta$-function

$$\beta(g) = -(11/16\pi^2)\,g^3 - [102/(16\pi^2)^2]\,g^5\,, \qquad (2)$$

are not small in the regime where the corrections subject to the cut-off are already negligible (or both). In the latter case we would expect $\beta(g)$ to be universal and to gradually approach the asymptotic form (2). To confirm this property, and finally to extract the $\beta$-function for a correct interpretation of the results, it will be necessary to calculate a variety of (other) physical quantities within the pure glue theory.

In this paper we are concerned with obtaining similarly definitive results for the $0^{++}$ glueball mass. Such a calculation must satisfy the following conditions:

(i) Typically the glueball mass is extracted from the asymptotic behaviour of the correlation function. To be sure that one is extracting the true ground state the correlation function must be "measured" far enough that the *asymptotic* exponential decay is accurately displayed.

0370-2693/85/$ 03.30 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

(ii) The calculation should be performed over a sufficient range of lattices to determine the infinite volume limit.

(iii) We should be aiming for statistical errors (of the masses) $<10\%$ at all $\beta$.

Previous glueball mass calculations do not satisfy (all of) the above conditions. Two classes of methods have been used: the variational method [4] "measures" correlations between fluctuations in the vacuum, with a variational improvement of the wave-functional designed to maximise the projection onto the lowest glueball state; the source method [5] disturbs the vacuum and "measures" the asymptotic decay with distance of this disturbance. The best results at larger $\beta$ are with the former method. Accurate "measurements" of correlation functions up to 2 lattice spacings have been performed on $4^3 \cdot 8$ [6], $5^3 \cdot 8$ [7] and $8^4$ [8] lattices. The "measurements" at 3 lattice spacings (except at $\beta = 5.9$ [6,8]) are generally too inaccurate to be useful. From these "measurements" the conclusion [8] was that the glueball mass satisfied asymptotic scaling for $5.1 \lesssim \beta \lesssim 5.9$ with minor finite size effects. However, there was no hard numerical evidence that the lowest glueball mass was indeed being "measured" at two lattice spacings.

To perform a calculation of the type being envisaged here with the variational method would be very expensive. The calculations in this paper use the source method to which we now turn.

2. *The source method.* To introduce a source alter the path integral

$$\int [\mathrm{d}U]\, \mathrm{e}^{-\beta S(U)} \rightarrow \int [\mathrm{d}U]\,[\mathrm{d}\tilde{U}]\, F_J(\tilde{U}) \exp[-\beta S(U) - S_J(U, \tilde{U})] \,. \quad (3)$$

Typically, we might choose

$$S_J(U, \tilde{U}) = \gamma_J \sum_{\substack{n \\ \mu \neq \nu}} [1 - \tfrac{1}{3} \operatorname{Re} \operatorname{Tr}(U_{n,\mu} \tilde{U}_{n+\hat{\mu},\nu} \tilde{U}_{n+\hat{\mu}+\hat{\nu},\mu} \tilde{U}_{n+\hat{\nu},\nu})] \,. \quad (4)$$

To retain the transfer matrix of the undisturbed theory choose $J$ to be localized at $t = 0$, i.e. $\gamma_J = 0$ unless $t = 0$. Let $\tilde{\phi}_t(U)$ be a wave-functional of links at time $t$, then by inserting intermediate eigenstates in the usual way

$$\langle 0|\tilde{\phi}_t(U)\, \tilde{\phi}_0(U)|J\rangle = \sum_n \langle 0|\tilde{\phi}|n\rangle\langle n|\tilde{\phi}|J\rangle \exp(-E_n t) \underset{t\to\infty}{\longrightarrow} \langle 0|\tilde{\phi}|\mathrm{g}\rangle\langle \mathrm{g}|\tilde{\phi}|J\rangle \exp(-E_{\mathrm{g}} t) \,, \quad (5)$$

where we explicitly have chosen $\tilde{\phi}$ such that $\langle 0|\tilde{\phi}|0\rangle = 0$, and $|\mathrm{g}\rangle$ is the state of lowest energy above the vacuum. Choosing a translationally invariant (zero-momentum) source and $\tilde{\phi}$ we thus obtain the glueball mass $m_{\mathrm{g}} = E_{\mathrm{g}}$.

Some comments on advantages and disadvantages of the method:

(i) For a strong source the fluctuations of $\tilde{\phi}_0(U)$ in (5) are negligible, as compared to the variational method where one "measures" the correlation of the fluctuations. This means that the source method should have a much better signal/error ratio, and that (naively) the computer time needed for a given such ratio will be independent of (spatial) volume in contrast to the variational method where it increases with the volume [8]. This is the crucial feature of the source method that will allow us to "measure" correlation functions at large distances.

(ii) The important disadvantage of the source method is that the coefficients in the expansion (5) need not be positive. This is in contrast to the variational method where the corresponding coefficients are $|\langle n|\tilde{\phi}|0\rangle|^2$, and so at non-asymptotic times the correlation function always gives an *upper bound* to the glueball mass. The source method will in general give us *no information* on $m_{\mathrm{g}}$ at non-asymptotic distances: it is only worth using if we pursue it to the point where we accurately identify the asymptotic exponential decay.

The source used in the bulk of the present calculations consists of setting all space-like links at $t = 0$ to unity [9,10]. (Further below we discuss some other sources.) This is equivalent to no source, but integrating only over those gauge configurations which have $U_{n,\mu=1,2,3} = 1$ at $t = 0$, i.e. the state at $t = 0$, $|\chi\rangle$, is the unit eigenstate of the field operators $U_{n,\mu=1,2,3}$. So if we integrate over a Wilson loop $\tilde{\phi}_t(U) = \phi_t(U) - \langle\phi(U)\rangle$ at time $t$, we have

$$\langle 0|\tilde{\phi}_t(U)|\chi\rangle \underset{t\to\infty}{\longrightarrow} \langle 0|\phi|\mathrm{g}\rangle\langle \mathrm{g}|\chi\rangle \exp(-m_{\mathrm{g}} t) \,. \quad (6)$$

Note that $|\chi\rangle$ is a $0^{++}$, zero-momentum state.

Of course, such a structureless "white" source projects on all $0^{++}$ states and a priori the lowest glueball might dominate only at inaccessibly large distances. However, the variational calculations [6–8] teach us that for $\beta$ not too much greater than 5.5 the smallest loops appear to be the most efficient projectors onto the lowest mass glueball, i.e. it is relatively small and structureless and should be adequately projected upon by even a "white" source. A further defect of such a source, which simultaneously extends through the whole spatial volume V, is that we expect $\langle g_1 \ldots g_N|\chi\rangle$ ($|g_1 \ldots g_N\rangle$ is a state with $N$ glueballs) to peak for $N \sim V$ and hence $\langle g|\chi\rangle$ to decrease (rapidly?) with $V$ for large enough $V$. Hence our strategy is to begin calculations at $\beta = 5.5$ and to increase $\beta$ (and $V$) until the method fails (which it does by $\beta = 6.0$).

*3. Calculations and results.* We work with the standard Wilson action and the "white" source as described above on periodic $L_s^3 \cdot L_t$ lattices. We "measure" expectation values of $1 \times 1$, $1 \times 2$, $2 \times 2$, $1 \times 3$, $2 \times 3$ and, on the larger lattices, $1 \times 4$, $2 \times 4$ Wilson loops as a function of distance (in time) from the source. For a periodic lattice we expect an asymptotic behaviour

$$\langle 0|\phi_t|\chi\rangle \underset{t\ \text{large}}{=} C \cosh[(\hat{t} - \tfrac{1}{2}L_t)\, m_g a] + \langle\phi\rangle\,, \qquad (7)$$

where $t = \hat{t}a$. If one wishes to use the value of $\langle\phi\rangle$ as "measured" in source-free calculations at the same $\beta$, there are finite volume ambiguities that can be significant at our level of accuracy. The most consistent method is to use lattices with large $L_t$ and to fit $m_g$, $\langle\phi\rangle$ and $C$ simultaneously on the same data, which is what we do.

We compute correlation functions on a $6^3 \cdot 12$ lattice at $\beta = 5.5$, on $6^3 \cdot 16$ and $8^3 \cdot 16$ lattices at $\beta = 5.7$ and on a $10^3 \cdot 20$ lattice at $\beta = 5.9$. Attempts to extract the glueball mass from correlation functions on a $12^3 \cdot 24$ lattice at $\beta = 6.0$ have not been successful so far with this source. Typically we use about 1000 sweeps to reach equilibrium and about 20 000 sweeps for "measurements" (in most cases we perform more sweeps as we get further from the source – see ref. [11] for details), and these come in three sequences from independent starting configurations.

A given correlation function is fitted by the functional form (7) first for all $1 \leqslant \hat{t} \leqslant L_t - 1$, then for $2 \leqslant \hat{t} \leqslant L_t - 2$ and so on until a $\chi^2$ is achieved that is acceptable given the number of degrees of freedom. The error on the mass can be determined by seeing how far we can vary the mass without dropping the $\chi^2$ below, say, a 20% confidence level. This procedure for estimating errors turns out to be complicated by the fact that the fluctuations far along the correlation function are highly correlated. For a more detailed discussion we refer to ref. [11]: one estimates the effective number of degrees of freedom (or the error correlation matrix) and gets the error accordingly. Where there may be ambiguities we split the data into sequential subsets to check the errors.

In figs. 1a–1c we present the data for typical loops (which according to the variational calculation [6] have a large projection on $|g\rangle$) with the best fits. We obtain the masses from analysing all loops. The results are given in table 1 and, to make deviations from asymptotic scaling more visible, are plotted in units of $\Lambda_L$ in fig. 2 by using the two-loop perturbative expression (1) for $a(\beta)$. The corresponding (fitted) vacuum expectation values of some of the Wilson loops are listed in table 2.

*4. Volume and β dependence.* The (spatial) volume dependence of $m_g$ has been determined by Lüscher [12] to be

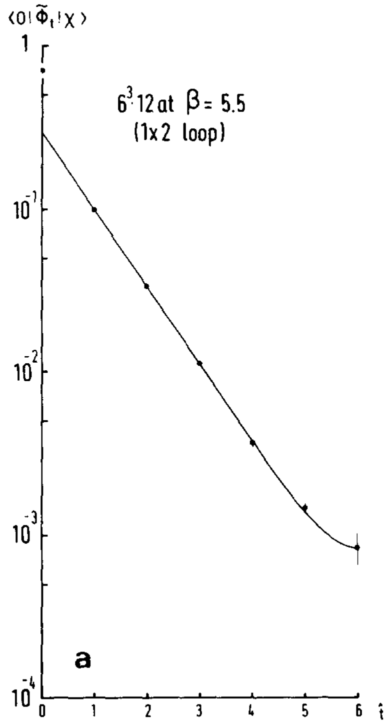
$$m_g(L_s) = m_g(\infty)\{1 - (3/16\pi)[\lambda/m_g(\infty)]^2$$
$$\times \{\exp[-\tfrac{1}{2}\sqrt{3}m_g(\infty)\, aL_s]/m_g(\infty)\, aL_s\}$$
$$\times [1 + O(L^{-1})]\}\,, \qquad (8)$$

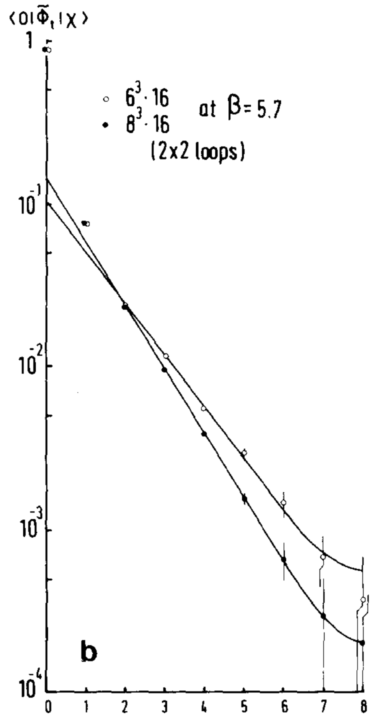
where $\lambda$ is the (dimensionful) triple-glueball coupling constant. With our $6^3 \cdot 16$ and $8^3 \cdot 16$ results we can determine $G = (3/16\pi)[\lambda/m_g(\infty)]^2$. We obtain

$$G = 155 \pm 45\,, \qquad (9)$$

which indicates that glueballs interact very strongly. A similar value has been obtained recently by Münster [13] from strong coupling expansions. Applying this correction to our data in table 1 we obtain the glueball masses in the infinite volume limit as in table 3. The corrected masses are also plotted in fig. 2.

We see (for the first time) that finite size effects on the glueball mass are substantial. However, they are not visible at two lattice spacings because the correla-



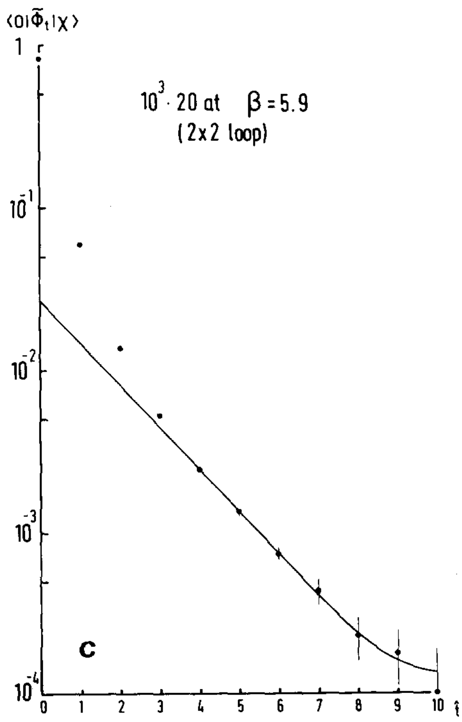




Table 1

| $\beta$ | $L_s^3 \cdot L_t$ | Fitted range | $m_g a$ |
|---|---|---|---|
| 5.5 | $6^3 \cdot 12$ | $2 \leqslant \hat{t} \leqslant 10$ | 1.07 ± 0.03 |
| 5.7 | $6^3 \cdot 16$ | $3 \leqslant \hat{t} \leqslant 13$ | 0.66 ± 0.04 |
| | $8^3 \cdot 16$ | $3 \leqslant \hat{t} \leqslant 13$ | 0.86 ± 0.04 |
| 5.9 | $10^3 \cdot 20$ | $4 \leqslant \hat{t} \leqslant 16$ | 0.62 ± 0.04 |

tion function has not yet reached its asymptotic exponential fall-off. (Remark: to ensure that only the asymptotic volume correction (8) is relevant we need "measurements" on more lattice sizes.)

*5. (Asymptotic) scaling?* In fig. 2 we plot the $\beta$ dependence of $\sqrt{K}(\infty)$ [2] which remains when we express $\sqrt{K}$ in units of $\Lambda_L$ as for the glueball mass. We conclude:

Fig. 1. $\langle 0|\tilde{\phi}_t|\chi\rangle$ as a function of $\hat{t} = t/a$ for (a) the $1 \times 2$ loop on the $6^3 \cdot 12$ lattice at $\beta = 5.5$, (b) the $2 \times 2$ loop on $6^3 \cdot 16$ and $8^3 \cdot 16$ lattices at $\beta = 5.7$ and (c) the $2 \times 2$ loop on the $10^3 \cdot 20$ lattice at $\beta = 5.9$.

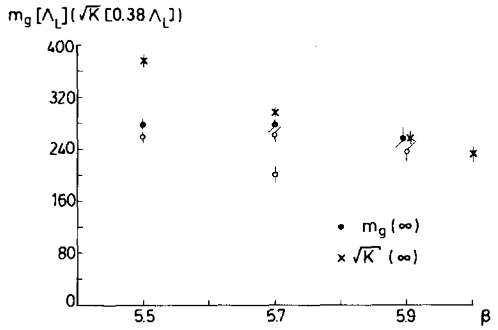



Fig. 2. $m_g(\infty)$ ($\sqrt{K}(\infty)$ [2]) in units of $\Lambda_L$ (0.38 $\Lambda_L$) as a function of $\beta$. Also shown are the "raw" values: the open circles correspond to the results obtained on the $6^3 \cdot 12$, $8^3 \cdot 16$, $10^3 \cdot 20$ lattices at $\beta = 5.5, 5.7, 5.9$; the open square corresponds to the result obtained on the $6^3 \cdot 16$ lattice at $\beta = 5.7$.

Table 2

| $\beta$ | $L_s^3 \cdot L_t$ | Loop | $\langle \phi \rangle$ |
|---|---|---|---|
| 5.5 | $6^3 \cdot 12$ | 1 × 1 | 0.49628 ± 0.00016 |
| | | 1 × 2 | 0.25910 ± 0.00020 |
| 5.7 | $6^3 \cdot 16$ | 1 × 1 | 0.54892 ± 0.00025 |
| | | 1 × 2 | 0.32432 ± 0.00034 |
| | | 2 × 2 | 0.13208 ± 0.00025 |
| | $8^3 \cdot 16$ | 1 × 1 | 0.54914 ± 0.00007 |
| | | 1 × 2 | 0.32470 ± 0.00005 |
| | | 2 × 2 | 0.13222 ± 0.00006 |
| 5.9 | $10^3 \cdot 20$ | 1 × 1 | 0.58177 ± 0.00002 |
| | | 1 × 2 | 0.36778 ± 0.00005 |
| | | 2 × 2 | 0.17412 ± 0.00005 |

Table 3

| $\beta$ | $m_g(\infty)\, a$ |
|---|---|
| 5.5 | 1.14 ± 0.03 |
| 5.7 | 0.90 ± 0.04 |
| 5.9 | 0.67 ± 0.04 |

(i) Taken over the whole range $5.5 \leqslant \beta \leqslant 5.9$ the $0^{++}$ glueball mass adheres remarkably accurately to asymptotic scaling.

(ii) The behaviour of $\sqrt{K}$ is quite different over this $\beta$ range. There is no universal $\beta$-function that can accommodate both $\sqrt{K}$ and $m_g$ down to $\beta = 5.5$.

(iii) If we restrict our attention to $\beta = 5.7$ and $\beta = 5.9$ then we find that $m_g(\infty)$ is within $\approx 1\sigma$ of asymptotic scaling on the one hand and within $\approx 1\sigma$ of $m_g(\infty)/\sqrt{K}(\infty)$ being constant on the other.

We hope to reduce the error on our $\beta = 5.9$ data sufficiently and to be able to calculate at higher values of $\beta$ by using a more refined source so as to resolve the two possibilities in (iii).

*6. Other states, other sources.* In the same way as one constructs wave-functionals of differing $J^{PC}$ [6,7] and momentum [8] in the variational approach, one can, almost trivially, construct sources that will project on $\boldsymbol{p} \neq 0$ or onto other quantum numbers $J^{PC}$. We have performed $2^{++}$ source calculations with smaller statistics. The source we have used was, among others,

$$U_{n,\mu=1,3} = 1\,, \quad U_{n,\mu=2} = -1\,, \quad \text{at } t = 0\,. \tag{10}$$

(It does not matter that $U_{n,\mu=2} = -1$ is not an SU(3) matrix.) On this basis we estimate that we needed $\simeq 50\,000$ sweeps on the $8^3 \cdot 16$ lattice at $\beta = 5.7$ and $\simeq 100\,000$ sweeps on the $10^3 \cdot 20$ lattice at $\beta = 5.9$ for a "good" $2^{++}$ calculation which is not impractical.

We have also been investigating (and still are) different $0^{++}$ sources. Here we mention a couple of examples (for more details see ref. [11]). We have compared our present source on a $4^3 \cdot 16$ lattice at $\beta = 2.3$ in SU(2) with a source where the $U_{n,\mu=1,2,3}$ at $t = 0$ are randomly chosen (i.e. characteristic of $\beta = 0$). The loops now approach their asymptotic values from below (i.e. $C$ in eq. (7) is negative), but we find no significant difference in terms of signal/error or projection onto $|g\rangle$ between the two types of source.

For a very different source we go back to eqs. (3), (4) and take $\tilde{U}_{n,\mu=1,2,3} = 1$, $\gamma_J = 1$ at $t = 0$ and $\gamma_J = 0$ otherwise. With this source all the link variables in the lattice are updated, except that now the space-like link variables at $t = 0$ interact with the source as well as with each other. We have compared this source with the source used in this paper on a $4^3 \cdot 16$ lattice at $\beta = 2.3$ in SU(2) and on a $6^3 \cdot 16$ lattice at $\beta = 6.0$ in SU(3). With this source we obtain a definite improvement: the correlation function is flatter when we are close to the source, and we estimate that we gain about a lattice spacing in the onset of the asymptotic exponential decay. These results certainly demonstrate that one can and should improve the source used herein. A systematic "variational" source method can be based on using as new sources the field configurations several lattice spacings away from the frozen source. On these time-slices the transfer matrix will have filtered out most of the higher mass excitations.

The numerical calculations have been done on the Cray X-MP at Jülich. We are grateful to J. Speth for the hospitality of the Institut für Kernphysik during the course of this work and to F. Hossfeld for his support. We are also indebted to E. Brökel and J. Chiavaut for valuable computational assistance. One of us (M.T.) thanks F. Gutbrod for the hospitality of the DESY Theory Group during part of this work and acknowledges financial support from DRET (contract No. 82/1252/DRET/DS/SR). Two of us (G.S. and M.T.) thank the Aspen Center of Physics for its hospitality where this work has been completed and L. Yaffe for discussions.

*References*

[1] F. Gutbrod, P. Hasenfratz, Z. Kunszt and I. Montvay, Phys. Lett. 128B (1983) 415;
A. Hasenfratz, P. Hasenfratz, U. Heller and F. Karsch, CERN preprint TH-3842 (1984);
D. Barkai, K.J.M. Moriarty and C. Rebbi, Brookhaven preprint BNL-34462 (1984);
S. Otto and J.D. Stack, Caltech preprint CALT-68-1113 (1984).

[2] Ph. de Forcrand, G. Schierholz, H. Schneider and M. Teper, DESY preprint in print.

[3] A.D. Kennedy, J. Kuti, S. Meyer and B.J. Pendleton, Santa Barbara preprint NSF-ITP-84-61 (1984).

[4] M. Falcioni et al., Phys. Lett. 110B (1982) 295;
K. Ishikawa, G. Schierholz and M. Teper, Phys. Lett. 110B (1982) 399;
B. Berg, A. Billoire and C. Rebbi, Ann. Phys. 142 (1982) 185.

[5] A. König, K.H. Mütter, J. Paech and K. Schilling, Wuppertal preprint WU B 83-26 (1983), and earlier work referenced therein;
M. Falcioni et al., Phys. Lett. 110B (1982) 295;
C. Michael and I. Teasdale, Nucl. Phys. B215 (1983) 433.

[6] K. Ishikawa, A. Sato, G. Schierholz and M. Teper, Z. Phys. C21 (1983) 167.

[7] B. Berg and A. Billoire, Nucl. Phys. B226 (1983) 405.

[8] G. Schierholz and M. Teper, Phys. Lett. 136B (1984) 64.

[9] Ph. de Forcrand, Polytechnique preprint A615.0784 (1984).

[10] M. Falcioni, M.L. Paciello, G. Parisi and B. Taglienti, Rome preprint (1984).

[11] Ph. de Forcrand, G. Schierholz, H. Schneider and M. Teper, in progress.

[12] M. Lüscher, DESY preprint 83-116 (1983).

[13] G. Münster, DESY preprint 84-088 (1984).