# THE STRING AND ITS TENSION IN SU(3) LATTICE GAUGE THEORY: TOWARDS DEFINITIVE RESULTS

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We consider the correlation of Polyakov loops It is shown that the ground-state energy receives a universal contribution $-\pi/3A$ ($A$ being the length of the loop) arising from long wavelength fluctuations of the string. The correlation function is calculated for $\beta = 5.5$, 5 7, 5 9 and 6 0 on lattices ranging in size from $6^3 \times 12$ to $12^3 \times 24$ using the source method. The calculation is accurate enough to identify the asymptotic exponential decay which makes sure that we are extracting the ground state energy and hence the (asymptotic) linearly rising piece of the potential We find the Coulomb-like contribution to be remarkably consistent with $-\pi/3A$ The string tension violates asymptotic scaling by $\approx 60\%$, taken over the whole range of $\beta$ This stands in sharp contrast to the $\beta$ dependence of the mass gap which is consistent with asymptotic scaling.

Monte Carlo simulations of the (physically interesting) SU(3) lattice gauge theory have provided a powerful and successful tool for extracting non-perturbative physical quantities such as the string tension and hadron masses. The crucial and ambitious question now is whether we can see precise continuum physics at the values of bare couplings presently accessible to us. To answer this question requires sufficiently small statistical errors and, what is the real difficulty, one must correspondingly control the systematic errors which arise if we extract our string tension and masses at a point too early along a correlation function. The cure for this is to calculate *accurate* correlation functions far enough to see an (asymptotic) exponential decay over several lattice spacings. In this letter we present the first results of a calculation of this kind for the string tension, for the long distance fluctuations of the string, and in ref. [1] for the mass gap.

To perform such a calculation with the standard method, which rests on computing averages of large Wilson loops, would be very expensive. The calculations in this paper use the source method which disturbs the vacuum and "measures" the long distance correlations of Polyakov loops with distance from the source.

*Polyakov loops and string picture.* Before we turn to the method, let us consider the case without the source. The potential between heavy quarks is defined by the average of the product of two Polyakov loops of length $A$ in the periodic direction separated by a distance $B$ via

$$\langle L(A,B)L(A,0)\rangle \underset{A \gg B}{\cong} \exp[-V,(B)A]. \quad (1)$$

As the separation $B$ becomes large the potential approaches

$$V(B) = KB + m - \pi/12B, \tag{2}$$

where $K$ is the string tension which we are interested in, and $m$ is the (divergent) self mass of heavy quarks. The Coulomb-like term $-\pi/12B$ arises from long wavelength fluctuations of the string. It is a universal term that one expects in any string model [2], and the quantitative verification of its presence clearly would give evidence of the underlying dynamics of strings.

In this paper we are interested in the large $B$ decay of the correlation function (1). The dominant term we already know [from eqs. (1), (2)]. To compute the subdominant terms we take the gaussian string action

$$S = \tfrac{1}{2}K\int_0^A \mathrm{d}\tau \int_0^B \mathrm{d}\rho \nabla x_\mu \nabla x_\mu, \tag{3}$$

which, for the long wavelength modes that (as it will turn out) provide the leading correction to the string tension, is the universal nonrelativistic limit of a whole class of actions. By making a definite choice of parameters, $x_4 = \tau$ and $x_1 = \rho$, this can be written

$$S = KAB + \tfrac{1}{2}K\int_0^A \mathrm{d}\tau \int_0^B \mathrm{d}\rho \nabla x_\perp \nabla x_\perp . \tag{4}$$

The correlation function then becomes

$$\langle L(A,B)L(A,0)\rangle = \exp(-KAB)\int \prod_{\tau,\rho} \mathrm{d}^2 x_\perp(\tau,\rho) \exp\left(-\tfrac{1}{2}K\int_0^A \mathrm{d}\tau \int_0^B \mathrm{d}\rho \nabla x_\perp \nabla x_\perp\right), \tag{5}$$

with periodic boundary conditions at $\tau = 0,\ A$ and zero boundary conditions $x_\perp = 0$ at $\rho = 0,\ B$. The result of doing the path integral is formally

$$\langle L(A,B)L(A,0)\rangle = \exp(-KAB)\left[\det\left(-\tfrac{1}{2}K\nabla^2\right)\right]^{-1}, \tag{6}$$

which leaves us to define and evaluate the determinant. We shall use the lattice regularisation. The eigenvalues of the laplacean $-\nabla^2$ (with periodic and zero boundary conditions, respectively) then are

$$\lambda^2_{m,n} = 4 - 2\cos(2\pi m/\hat{A}) - 2\cos(\pi n/\hat{B}), \quad -\tfrac{1}{2}\hat{A} < m \leqslant \tfrac{1}{2}\hat{A}, \quad 1 \leqslant n \leqslant \hat{B} - 1, \tag{7}$$

where $A = \hat{A}a$, $B = \hat{B}a$ and $a$ is the lattice spacing, which gives

$$\left[\det\left(-\tfrac{1}{2}K\nabla^2\right)\right]^{-1} = \exp\left(-\sum_{m,n} \ln\left[\tfrac{1}{2}Ka^2\lambda^2_{m,n}\right]\right). \tag{8}$$

To evaluate the determinant we write

$$\sum_{m,n} \ln\left[\tfrac{1}{2}Ka^2\lambda^2_{m,n}\right] = \hat{A}(\hat{B}-1)\ln\left(\tfrac{1}{2}Ka^2\right) + 2\sum_{m,n=1}^{\hat{A}/2-1,\hat{B}-1} \ln\left[4 - 2\cos(2\pi m/\hat{A}) - 2\cos(\pi n/\hat{B})\right]$$
$$+ \sum_{n=1}^{\hat{B}-1} \left\{\ln\left[2 - 2\cos(\pi n/\hat{B})\right] + \ln\left[6 - 2\cos(\pi n/\hat{B})\right]\right\}. \tag{9}$$

Using (repeatedly) Euler's sum formula (see e.g. ref. [3]) and

$$\sum_{i=1}^{I-1} \ln\left[2 - 2\cos(\pi i/I)\right] = \ln I, \tag{10}$$

(which follows from 1.396, ref. [4]) we obtain for the first sum

$$2\sum_{m,n=1}^{\hat{A}/2-1,\hat{B}-1} \ln[4-2\cos(2\pi m/\hat{A})-2\cos(\pi n/\hat{B})]$$

$$=\frac{\hat{A}\hat{B}}{\pi^2}\int_0^\pi \mathrm{d}x\int_0^\pi \mathrm{d}y \ln(4-2\cos x-2\cos y)-(\hat{B}+\hat{A}/2)\ln(3+\sqrt{8})-(\hat{B}/\hat{A})2\pi B_2-\ln(\hat{A}/2)+\ln\sqrt{32}$$
$$+\mathrm{O}(\hat{B}\hat{A}^{-2}), \tag{11}$$

while for the second sum we find

$$\sum_{n=1}^{\hat{B}-1}\{\ln[2-2\cos(\pi n/\hat{B})]+\ln[6-2\cos(\pi n/\hat{B})]\}=\ln\hat{B}+\hat{B}\ln(3+\sqrt{8})-\ln\sqrt{32}+\mathrm{O}(\hat{B}^{-1}). \tag{12}$$

Altogether this gives ($B_2=1/6$)

$$\sum_{m,n}\ln\left[\tfrac{1}{2}Ka^2\lambda^2_{m,n}\right]$$

$$=\hat{A}(\hat{B}-1)\ln\left(\tfrac{1}{2}Ka^2\right)+\frac{\hat{A}\hat{B}}{\pi^2}\int_0^\pi \mathrm{d}x\int_0^\pi \mathrm{d}y\ln(4-2\cos x-2\cos y)-(\hat{B}/\hat{A})\pi/3+\ln\hat{B}-\tfrac{1}{2}\hat{A}\ln(3+\sqrt{8})$$
$$-\ln(\hat{A}/2)+\mathrm{O}(\hat{B}\hat{A}^{-2})+\mathrm{O}(\hat{B}^{-1}). \tag{13}$$

The first two terms renormalise the string tension, so that the final result is

$$\langle L(A,B)L(A,0)\rangle \underset{B\gg A}{=} \exp\left\{-\left[KAB-\tfrac{1}{3}\pi B/A+\ln B+\mathrm{O}(BA^{-2})+\mathrm{O}(A)\right]\right\}. \tag{14}$$

The universal Coulomb-like term $-\pi/12B$ in eq. (2) is now replaced by the (universal) term $-\pi/3A$. This is four times as big, which strongly favours studying the dynamics of strings through the large $B$ correlation of Polyakov loops over any other method. It should also be noted that the (true) Coulomb term does not contribute in this limit and neither does the self mass of the lines [cf. eq. (2)], which makes the assignment of an eventually observed $1/A$ contribution unambiguous. The logarithmic term $\ln B$ is (basically) of kinematic origin as can easily be traced from eq. (9). (See also the following discussion.) The presence of a Coulomb-like term $-\pi/3A$ has been conjectured before by Rakow on the basis of duality type arguments [5].

Physically speaking the Polyakov loops communicate by the exchange of flux line configurations which wind around the lattice. The energies of zero total momentum, $E_i$, of these configurations are given by

$$\sum_{x_\perp}\left\langle L\left(A,\left(z^2+x_\perp^2\right)^{1/2}\right)L(A,0)\right\rangle=\sum_i c_i\exp(-E_i z). \tag{15}$$

Assuming that Lorentz symmetry is restored this can be written

$$\int \mathrm{d}^2x_\perp\left\langle L\left(A,\left(z^2+x_\perp^2\right)^{1/2}\right)L(A,0)\right\rangle=2\pi\int_Z \mathrm{d}B\,B\langle L(A,B)L(A,0)\rangle, \tag{16}$$

which, by inserting (14), leads to

$$\sum_{x_\perp}\left\langle L\left(A,\left(z^2+x_\perp^2\right)^{1/2}\right)L(A,0)\right\rangle=2\pi\int_Z \mathrm{d}B\,B\exp\left\{-\left[KAB-\tfrac{1}{3}\pi B/A+\ln B+\mathrm{O}(BA^{-2})+\mathrm{O}(A)\right]\right\}$$

$$=2\pi\int_Z \mathrm{d}B\exp\left\{-\left[KA-\pi/3A+\mathrm{O}(A^{-2})+\mathrm{O}(AB^{-1})\right]B\right\}\cong c_0\exp\{-[KA-\pi/3A]z\}. \tag{17}$$

Thus the ground state energy is

$$E_0 = KA - \pi/3A. \tag{18}$$

This expression could have also been obtained from eq. (14).

*Source method.* We work on (periodic) $L_s^3 \times L_t$ lattices. The Polyakov loops will be in the spatial direction, i.e. $A = L_s$, at times $t = 0$ and $t$. The source that we shall use (see ref. [1] and references therein) consists of setting all space-like links at $t = 0$ to unity so that the transfer matrix of the undisturbed theory is retained. The state at $t = 0$, $|\chi\rangle$, is then the unit eigenstate of the field operators $U_{n,\mu=1,2,3}$. It has zero total momentum and $L(aL_s, 0)|\chi\rangle = |\chi\rangle$ [i.e. $L(aL_s, 0)$ does not fluctuate]. For a periodic lattice we then expect the asymptotic behaviour

$$P(L_s, t) \equiv \sum_{x_\perp} \langle 0 | L\left(aL_s, \left(t^2 + x_\perp^2\right)^{1/2}\right) | \chi \rangle \underset{t\ \mathrm{large}}{=} c \cosh\left[(\hat{t} - L_t/2)\left(Ka^2 L_s - \pi/3L_s\right)\right], \tag{19}$$

where we have set $t = \hat{t}a$. To verify the universal Coulomb-like term, which bears important new physics and has been overlooked in previous calculations [6], one has to compute $P(L_s, t)$ on various sized lattices. For a further discussion of the source method, including other sources, the reader is referred to ref. [1].

*Calculation and results.* We work with the standard Wilson action and compute $P(L_s, t)$ on a $6^3 \times 12$ lattice at $\beta = 5.5$, on $6^3 \times 16$ and $8^3 \times 16$ lattices at $\beta = 5.7$, on a $10^3 \times 20$ lattice at $\beta = 5.9$ and on a $12^3 \times 24$ lattice at $\beta = 6.0$. We use about 1000 sweeps to reach equilibrium and about 20000 sweeps for "measurements". As we get further from the source we perform in most cases more sweeps. For details see ref. [7]. The "measurements" come typically in 3 sequences from independent starting configurations. In ref. [1] we have used the same gauge field configurations to extract the $0^{++}$ glueball mass which allows us to compare the continuum (?) behaviour of both the string tension and $m_g$ without any bias.

$P(L_s, t)$ is fitted by the functional form

$$P(L_s, t) = c \cosh\left[(\hat{t} - L_t/2) E_0 a\right], \tag{20}$$

first for $1 \leqslant \hat{t} \leqslant L_t - 1$, then $2 \leqslant \hat{t} \leqslant L_t - 2$ and so on until a $\chi^2$ is achieved that is acceptable and a stable mean value of $E_0 a$ is obtained. The error on $E_0 a$ is determined by varying $E_0 a$ until it drops below a 20% confidence level. The data is presented in fig. 1a–d. The results for $E_0 a$ are listed in table 1.

The entry in the last column, $\sqrt{K}a$, is the string tension that follows if one (naively) assumes $E_0 a = Ka^2 L_s$, i.e. ignoring the Coulomb-like term.

Fig. 1 displays that the calculation is accurate enough to identify (for the first time) the asymptotic exponential decay (20) of the correlation function. Moreover, it is quite impressive to see that the source method allows to follow the decay over e.g. 12 lattice spacings at $\beta = 6.0$ (fig. 1d) which provides a new standard. The statistical errors on $E_0 a$ are on the level of 3% for $\beta \leqslant 5.9$ and 6% at $\beta = 6.0$, which are hard to beat by the standard, Wilson-loop type calculations [even in SU(2)].

To verify the presence of the universal $1/L_s$ term and to properly extract (the infinite volume limit) of the string tension, we have performed our calculation at $\beta = 5.7$ on two different lattice sizes, $L_s = 6$ and 8 (fig. 1b). If we fit $E_0 a$ now by the form

$$E_0 a = Ka^2 L_s + r/L_s, \tag{21}$$

we obtain

$$r = -1.09 \pm 0.39, \tag{22}$$

in good agreement (magnitude *and* sign) with the expected

$$r = -\pi/3 = -1.05. \tag{23}$$

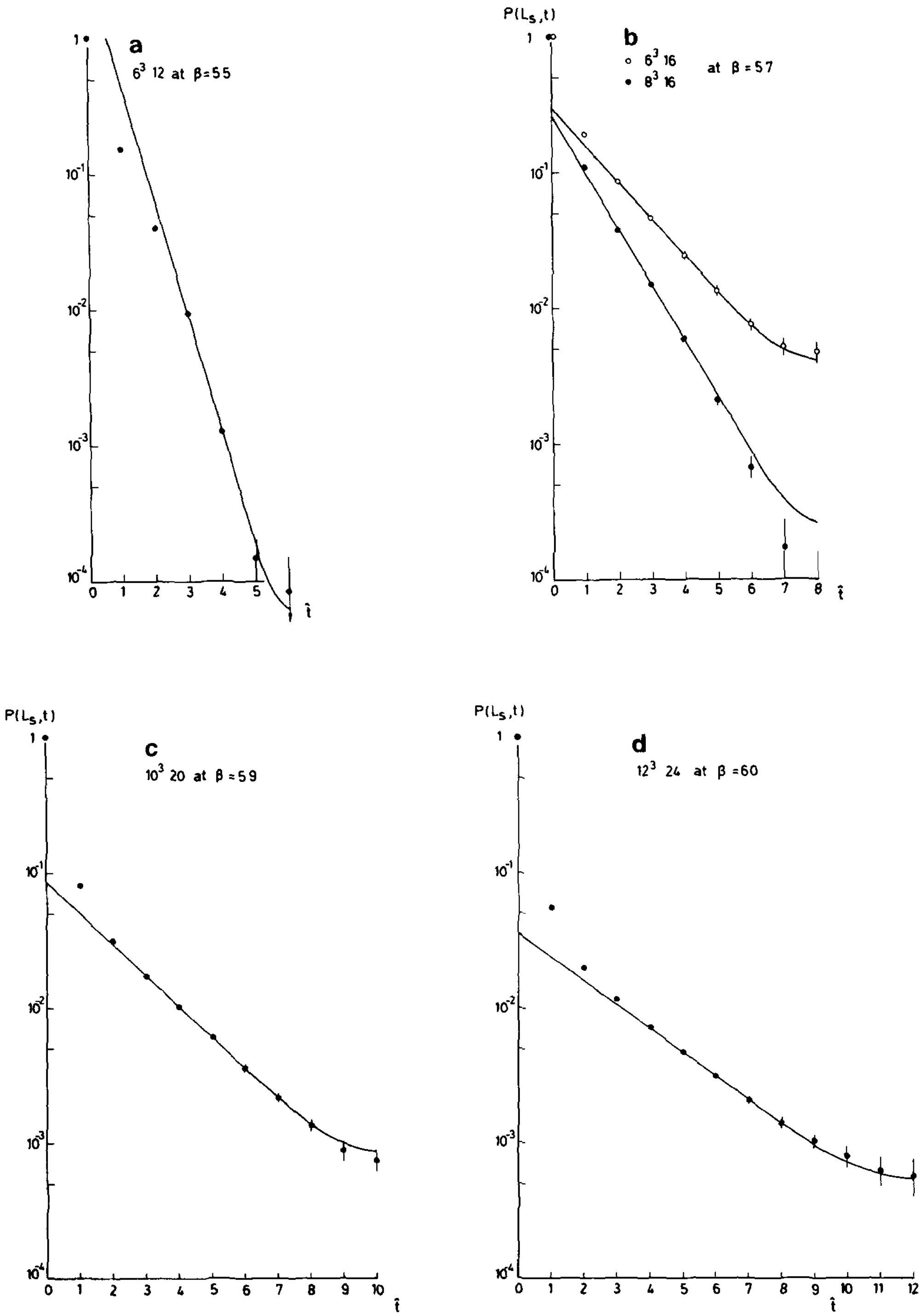


Fig 1 $P(L_s, t)$ as a function of $\hat{t} = t/a$ on (a) the $6^3 \times 12$ lattice at $\beta = 5\ 5$, (b) $6^3 \times 16$ and $8^3 \times 16$ lattices at $\beta = 5\ 7$, (c) the $10^3 \times 20$ lattice at $\beta = 5\ 9$ and (d) the $12^3 \times 24$ lattice at $\beta = 6\ 0$

Table 1

| $\beta$ | $L_s^3 \times L_t$ | Fitted range | $E_0 a$ | $\sqrt{K} a$ |
|---|---|---|---|---|
| 5 5 | $6^3 \times 12$ | $3 \leq \hat{t} \leq 9$ | 1 86 $\pm$0 06 | 0 557 $\pm$ 0 009 |
| 5 7 | $6^3 \times 16$ | $2 \leq \hat{t} \leq 14$ | 0.626 $\pm$ 0 016 | 0 323 $\pm$ 0 004 |
| | $8^3 \times 16$ | $2 \leq \hat{t} \leq 14$ | 0.94 $\pm$0 03 | 0 343 $\pm$ 0 006 |
| 5 9 | $10^3 \times 20$ | $3 \leq \hat{t} \leq 17$ | 0.529 $\pm$ 0 015 | 0 230 $\pm$ 0 003 |
| 6 0 | $12^3 \times 24$ | $4 \leq \hat{t} \leq 20$ | 0.413 $\pm$ 0.025 | 0 186 $\pm$ 0 006 |

Table 2

| $\beta$ | $\sqrt{K}(\infty) a$ |
|---|---|
| 5 5 | 0 583 $\pm$ 0 013 |
| 5 7 | 0 367 $\pm$ 0 007 |
| 5 9 | 0 253 $\pm$ 0 009 |
| 6 0 | 0 205 $\pm$ 0 008 |

This result nicely supports the existence of strings, which so far was only supported by the calculation of the string tension. (To consolidate this result it is important now to ensure that only the asymptotic volume correction is relevant. That is to say, we need "measurements" on more lattice sizes.)

Applying this correction now to our data from table 1, i.e. writing $E_0 a = K a^2 L_s - \pi/3L_s$, we obtain the string tension as in table 2. The corrections are of the order of 10% even for these large loops (of length ~ 10).

*Scaling (?) and comparison.* In fig. 2 we plot the $\beta$ dependence of $\sqrt{K}$ which remains when we express $\sqrt{K}$ in units of $\Lambda_L$ using the two-loop formula

$$a(\beta) = \Lambda_L^{-1}\left(\tfrac{8}{33}\pi^2\beta\right)^{51/121} \exp\left(-4\pi^2\beta/33\right). \tag{24}$$

Also shown are the uncorrected values (from table 1) and some recent results of other groups [6,8–11].

We conclude:

(i) Taken over the whole range $5.5 \leqslant \beta \leqslant 6.0$ (where the mass gap adheres remarkably accurately to asymptotic scaling [1]), the string tension is off from asymptotic scaling by $\approx 60\%$, and there is little sign that it levels off at larger $\beta$. In particular, we find no indication that asymptotic scaling has been reached at $\beta = 6.0$.

(ii) There are substantial deviations between the results of the various groups and methods, in particular at $\beta = 6.0$, which have to be clarified before one can draw any conclusions as to whether one sees asymptotic scaling beyond $\beta = 6.0$.

We are confident that the source method will also work for $\beta > 6$ with the same source, and we hope to address this region in a future publication.

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*Note added.* If the reader wishes to see how the source method compares to conventional Polyakov loop calculations, he is referred to ref. [13]: there the calculation of ref. [6] is repeated at the same coupling, $\beta = 6.0$, and on the same lattice, $10^3 \times 20$, but using in addition a source. Unfortunately the work of ref.

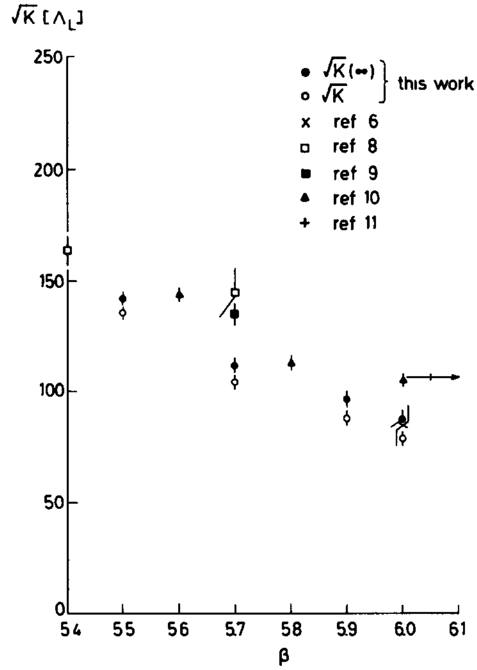


Fig. 2. $\sqrt{K}(\infty)$ in units of $\Lambda_L$ as a function of $\beta$ (solid circles), together with the "raw" values $\sqrt{K}$ (open circles) which correspond to the results obtained on the $6^3 \times 12$, $8^3 \times 16$, $10^3 \times 20$ and $12^3 \times 24$ lattices at $\beta$ = 5 5, 5 7, 5 9 and 6 0 assuming $E_0 a = K a^2 L_s$ (i e the quoted values in table 1) Also shown are the results obtained in refs [6,8–11]

[13] is not (statistically) accurate enough to be used in conjunction with our $\beta = 6.0$ results on a $12^3 \times 24$ lattice, to test the universal Coulomb-like term.

*References*

[1] Ph. de Forcrand, G Schierholz, H Schneider and M Teper, DESY preprint 84–107 (1984)
[2] M Lüscher, K Symanzik and P Weisz, Nucl Phys B173 (1980) 365;
see also: J Ambjorn, P Olesen and C Peterson, Lund preprint LU TP 84–7 (1984)
[3] M. Lüscher, Nucl. Phys B180 (1981) 317
[4] I S Gradshteyn and I M Ryzhik, Tables of integrals, series and products (Academic Press, New York, 1965)
[5] P E.L Rakow, Liverpool preprint (1984)
[6] G Parisi, R. Petronzio and F Rapuano, Phys Lett. 128B (1983) 418
[7] Ph de Forcrand, G Schierholz, H Schneider and M Teper, in progress
[8] F. Gutbrod, P. Hasenfratz, Z Kunszt and I Montvay, Phys Lett 128B (1983) 415
[9] A Hasenfratz, P Hasenfratz, U Heller and F Karsch, CERN preprint TH3842 (1984)
[10] D. Barkai, K J M Moriarty and C Rebbi, Brookhaven preprint BNL-34462 (1984).
[11] S Otto and J D Stack, Caltech preprint CALT-68-1113 (1984)
[12] Ph de Forcrand, D Lellouch and C Roiesnel, Ecole Polytechnique preprint A601 0384 (March 1984)