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MONOPOLE-FERMION AND DYON-FERMION BOUND STATES (III). Monopole-fermion system with $j = |q| - \frac{1}{2}$ and large $\kappa|q|$

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In the first part of the paper, we give analytic, approximate results for the Kazama-Yang monopole-fermion binding energies and wave functions, valid for large values of $A = \frac{1}{2}Z|eg|\kappa$, where κ is the extra magnetic moment. In the second part, more general results are obtained for the same problem that are valid when *either* A is large *or* the binding is weak. Numerical results for the binding energy are tabulated and compared.

1. Introduction

The monopole-fermion bound states were first investigated by Kazama and Yang [1]. Later, additional results were given [2, 3]. The monopole is assumed to be infinitely heavy, and the hamiltonian for the fermion of spin $\frac{1}{2}$ is [1]

$$H = \boldsymbol{\alpha}\cdot(\boldsymbol{p} - Ze\boldsymbol{A}) + \beta M - \kappa q\beta\boldsymbol{\sigma}\cdot\boldsymbol{r}/(2Mr^3), \tag{1.1}$$

where the notation is that of refs. [1–3].

While the direct numerical approach of paper I [3] leads to very accurate results, it is also useful to have approximate, more explicit formulas. So far, the limiting case that has received most attention is that of weak binding:

$$M - E \ll M \tag{1.2}$$

for the lowest angular momentum $j = |q| - \frac{1}{2}$. Results in this case have been given

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in ref. [2] for the monopole and in paper II [4] generalized to the dyon. In paper II, it is assumed that A is neither large nor small, where

$$A = \tfrac{1}{2}\kappa|q| . \tag{1.3}$$

A second limiting case of interest is

$$A \gg 1 . \tag{1.4}$$

This case is of possible relevance for a description of monopole-nucleus interactions at large distances. In the first part of this paper, consisting of sects. 3 and 4, this case (1.4) is treated by the WKB approximation for bound states of the lowest angular momentum.

It is then natural to raise the question whether the two approximations, the one in ref. [2] for the case (1.2) and the one in sects. 3 and 4 for (1.4), can be combined. In other words, is it possible to find an approximation that covers both cases, i.e., is valid whenever *either* (1.2) *or* (1.4) holds? This turns out to be the case, and in the second part of this paper, consisting of sects. 5–7, such an approximation is presented, together with numerical results in sect. 8, for comparison with those from sects. 3 and 4 and refs. [1, 2].

2. Eigenvalue problem

With the standard decomposition for bound states of the lowest angular momentum [1–4], the eigenvalue problem

$$H\psi = E\psi \tag{2.1}$$

reduces to two coupled ordinary differential equations:

$$\frac{\mathrm{d}G}{\mathrm{d}\rho} = \left(A - B - \frac{1}{\rho^2}\right)F,$$

$$\frac{\mathrm{d}F}{\mathrm{d}\rho} = \left(A + B - \frac{1}{\rho^2}\right)G, \tag{2.2}$$

where ρ, B, F and G are defined in refs. [1–4]. Let $S(\rho)$ and $T(\rho)$ be the sum and difference of F and G:

$$S = F + G,$$

$$T = F - G, \tag{2.3}$$

then

$$\left(\frac{d}{d\rho} - A + \frac{1}{\rho^2}\right)S = -BT,$$

$$\left(\frac{d}{d\rho} + A - \frac{1}{\rho^2}\right)T = BS. \tag{2.4}$$

Thus $S(\rho)$ and $T(\rho)$ each satisfy a second-order ordinary differential equation with irregular singularities at 0 and ∞:

$$\left[\frac{d^2}{d\rho^2} - \left(A^2 - B^2 - \frac{2A}{\rho^2} + \frac{2}{\rho^3} + \frac{1}{\rho^4}\right)\right]S = 0, \tag{2.5}$$

$$\left[\frac{d^2}{d\rho^2} - \left(A^2 - B^2 - \frac{2A}{\rho^2} - \frac{2}{\rho^3} + \frac{1}{\rho^4}\right)\right]T = 0. \tag{2.6}$$

These are the equations to be treated here.

3. WKB approximation: wave function between turning points

Under the assumption (1.4) of large A, (2.5) and (2.6) may be solved by the WKB approximation. By (2.4), it is sufficient to solve one of these two equations; we choose to solve (2.6).

It is clear that the important scale for ρ is $A^{-1/2}$. Accordingly, let

$$\rho = \tau A^{-1/2}. \tag{3.1}$$

In terms of the variable τ, the differential equation (2.6) is

$$\frac{d^2T}{d\tau^2} + \left[A\left(-1 + \frac{B^2}{A^2} + \frac{2}{\tau^2} - \frac{1}{\tau^4}\right) + A^{1/2}\frac{2}{\tau^3}\right]T = 0. \tag{3.2}$$

It is the presence of the last term that makes it necessary to modify slightly the standard WKB procedure. The function T is oscillatory in the range

$$-1 + \frac{B^2}{A^2} + \frac{2}{\tau^2} - \frac{1}{\tau^4} > 0, \tag{3.3}$$

or

$$\left(1 + \frac{B}{A}\right)^{-1/2} < \tau < \left(1 - \frac{B}{A}\right)^{-1/2}, \tag{3.4}$$

where we have chosen, without loss of generality, B to be positive. (From (2.2), negative values of B can be covered by interchanging F and G.) In this range (3.4), the WKB approximation to T is

$$T \simeq T_0(\tau)\mathrm{e}^{iA^{1/2}\phi(\tau)} + \text{c.c.}, \tag{3.5}$$

where

$$\phi(\tau) = \int \mathrm{d}\tau \left(-1 + \frac{B^2}{A^2} + \frac{2}{\tau^2} - \frac{1}{\tau^4}\right)^{1/2}. \tag{3.6}$$

In order to determine $T_0(\tau)$, (3.5) is substituted into (3.2), and the sum of the resulting terms of order $A^{1/2}$ is equated to zero. This gives the first-order equation for $T_0(\tau)$:

$$\phi''(\tau)T_0(\tau) + 2\phi'(\tau)T_0'(\tau) - \frac{2i}{\tau^3}T_0(\tau) = 0. \tag{3.7}$$

Therefore

$$\begin{aligned}T_0(\tau) &= [\phi'(\tau)]^{-1/2}\exp\left[i\int\frac{\mathrm{d}\tau}{\tau^3\phi'(\tau)}\right]\\
&= \text{const}[\phi'(\tau)]^{-1/2}\left\{\left[-\left(1-\frac{B}{A}-\frac{1}{\tau^2}\right)\right]^{1/2} + i\left[1+\frac{B}{A}-\frac{1}{\tau^2}\right]^{1/2}\right\}\\
&= \text{const}\left\{\left[-\left(1-\frac{B}{A}-\frac{1}{\tau^2}\right)\right]^{1/4}\left[1+\frac{B}{A}-\frac{1}{\tau^2}\right]^{-1/4}\right.\\
&\qquad\left. + i\left[-\left(1-\frac{B}{A}-\frac{1}{\tau^2}\right)\right]^{-1/4}\left[1+\frac{B}{A}-\frac{1}{\tau^2}\right]^{1/4}\right\}.\end{aligned} \tag{3.8}$$

The required WKB solution then follows from (2.3), (2.4) and (3.8):

$$\begin{aligned}F &\simeq C\left[-\left(A-B-\frac{1}{\rho^2}\right)\right]^{-1/4}\left[A+B-\frac{1}{\rho^2}\right]^{1/4}\\
&\quad\times\exp\left[iA^{1/2}\int_{(1-B^2/A^2)^{-1/4}}^{A^{1/2}\rho}\mathrm{d}\tau\left(-1+\frac{B^2}{A^2}+\frac{2}{\tau^2}-\frac{1}{\tau^4}\right)^{1/2}\right] + \text{c.c.},\\
G &\simeq iC\left[-\left(A-B-\frac{1}{\rho^2}\right)\right]^{1/4}\left[A+B-\frac{1}{\rho^2}\right]^{-1/4}\\
&\quad\times\exp\left[iA^{1/2}\int_{(1-B^2/A^2)^{-1/4}}^{A^{1/2}\rho}\mathrm{d}\tau\left(-1+\frac{B^2}{A^2}+\frac{2}{\tau^2}-\frac{1}{\tau^4}\right)^{1/2}\right] + \text{c.c.}\end{aligned} \tag{3.9}$$

Here C is a complex coefficient, and we have chosen the phase of F to be zero at the geometric average of the two turning points.

The phase of this complex coefficient can be determined by the following symmetry [2]:

$$\rho \to (A^2 - B^2)^{-1/2} \rho^{-1},$$

$$F \to \pm \left(\frac{A+B}{A-B} \right)^{1/4} G,$$

$$G \to \pm \left(\frac{A-B}{A+B} \right)^{1/4} F. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$iC = \pm C^*, \tag{3.11}$$

or

$$C = N e^{\mp i\pi/4}, \tag{3.12}$$

where N is the real normalization constant.

The normalization is determined by

$$\int_0^\infty \mathrm{d}\rho \,(F^2 + G^2) = M/A, \tag{3.13}$$

which may be approximated by

$$\int_{(A+B)^{-1/2}}^{(A-B)^{-1/2}} \mathrm{d}\rho \,(F^2 + G^2) = M/A. \tag{3.14}$$

The substitution of (3.9) into (3.14) then gives

$$2N^2 \int_{(A+B)^{-1/2}}^{(A-B)^{-1/2}} \mathrm{d}\rho \left\{ \left[-\left(A - B - \frac{1}{\rho^2} \right) \right]^{-1/2} \left[A + B - \frac{1}{\rho^2} \right]^{1/2} + \left[-\left(A - B - \frac{1}{\rho^2} \right) \right]^{1/2} \left[A + B - \frac{1}{\rho^2} \right]^{-1/2} \right\} = M/A, \tag{3.15}$$

or

$$N = \tfrac{1}{2} \left[\frac{AB}{M} \int_{(A+B)^{-1/2}}^{(A-B)^{-1/2}} \mathrm{d}\rho\, \rho^2 \left\{ \left[1 - (A-B)\rho^2 \right] \left[(A+B)\rho^2 - 1 \right] \right\}^{-1/2} \right]^{-1/2}. \tag{3.16}$$

As we shall see in sect. 4, the integral can be expressed by an elliptic integral of the second kind.

4. WKB approximation: Wilson-Sommerfeld quantization

When A is large, the binding energy can be determined approximately by the Wilson-Sommerfeld quantization condition [5]

$$A^{1/2}\left\{\phi\left[(1-y)^{-1/2}\right]-\phi\left[(1+y)^{-1/2}\right]\right\}=n\pi, \tag{4.1}$$

where ϕ is defined by (3.6) and

$$y=B/A. \tag{4.2}$$

It should be noted that the right-hand side here is $n\pi$, *not* the usual $(n+\frac{1}{2})\pi$. This shift of a half is related to the presence of the sub-asymptotic terms $2/\rho^3$ in eqs. (2.5) and (2.6), and to the boundary conditions. We can see this by considering the WKB approximations to eqs. (2.5) and (2.6):

$$A^{1/2}\left[\phi_S(\tau_{S1})-\phi_S(\tau_{S2})\right]=\left(n_S+\tfrac{1}{2}\right)\pi, \tag{4.3}$$

$$A^{1/2}\left[\phi_T(\tau_{T1})-\phi_T(\tau_{T2})\right]=\left(n_T+\tfrac{1}{2}\right)\pi, \tag{4.4}$$

where

$$\phi_S(\tau)=\int d\tau\left(-1+\frac{B^2}{A^2}+\frac{2}{\tau^2}-\frac{2}{\tau^3A^{1/2}}-\frac{1}{\tau^4}\right)^{1/2}, \tag{4.5}$$

$$\phi_T(\tau)=\int d\tau\left(-1+\frac{B^2}{A^2}+\frac{2}{\tau^2}+\frac{2}{\tau^3A^{1/2}}-\frac{1}{\tau^4}\right)^{1/2}. \tag{4.6}$$

In (4.3) and (4.4), τ_{S1} and τ_{S2} are the larger and smaller positive zeroes of the integrand in (4.5), while τ_{T1} and τ_{T2} are those in (4.6). Since $2/\tau^3A^{1/2}$ is a higher-order term in the integrands, ϕ is given to leading order by

$$\phi(\tau)=\tfrac{1}{2}\left[\phi_S(\tau)+\phi_T(\tau)\right]. \tag{4.7}$$

We recall that the presence of the half on the right-hand sides of (4.3) and (4.4) is due to the fact that $[\phi_S'(\tau)]^2$ and $[\phi_T'(\tau)]^2$ vanish linearly at the turning points.

The point now is that

$$n_T=n_S+1 \quad (\text{modulo } 2), \tag{4.8}$$

as can be seen from the boundary conditions: an analysis of eq. (2.4) for small ρ

shows that

$$\frac{S}{T} \underset{\rho\to 0}{\sim} -\tfrac{1}{2}B\rho^2, \tag{4.9}$$

i.e., with $B > 0$, S and T have for small ρ opposite signs. A corresponding analysis of eq. (2.4) for large ρ shows that

$$\frac{S}{T} \underset{\rho\to\infty}{\sim} \frac{1}{B}\left[A - \sqrt{A^2 - B^2}\right]. \tag{4.10}$$

Thus, with A and B positive, S and T will for large ρ have the same sign. It follows from these results (4.9) and (4.10) that if S has an even number of zeros, then T has an odd number of zeros and vice versa. This proves eq. (4.8).

Eqs. (4.7) and (4.8) give immediately the desired result (4.1), with no half on the right-hand side.

By (3.6), (4.1) is

$$A^{1/2}I(y) = n\pi, \tag{4.11}$$

where

$$I(y) = \int_{(1+y)^{-1/2}}^{(1-y)^{-1/2}} \mathrm{d}\tau \left[-(1-y-\tau^{-2})(1+y-\tau^{-2})\right]^{1/2}. \tag{4.12}$$

The task here is to evaluate this integral.

This integral can be expressed in terms of complete elliptic integrals of the first and second kinds. The answer can be obtained in a number of different ways. For example, one way is to recognize that the right-hand side of (4.12) is a special case of the hypergeometric function, and then to reduce this special case to elliptic integrals. Here we prefer to follow a more elementary procedure.

The first step is to take out a factor τ^{-2} so that the square root is that of a fourth-order polynomial in τ. Integrating $\int \mathrm{d}\tau\tau^{-2}$ by parts so that the square root appears in the denominator, we get

$$I(y) = 2(1+y)^{-1/2}K(k) + I_1(y), \tag{4.13}$$

where

$$k = \left(\frac{2y}{1+y}\right)^{1/2}, \tag{4.14}$$

$K(k)$ is the complete elliptic integral of the first kind, and

$$I_1(y) = -(1-y^2)^{1/2}\int_C \mathrm{d}\tau\tau^2\left\{\left[(1-y)^{-1} - \tau^2\right]\left[\tau^2 - (1+y)^{-1}\right]\right\}^{-1/2}, \tag{4.15}$$

with the contour C around the branch cut from $(1+y)^{-1/2}$ to $(1-y)^{-1/2}$ in the clockwise direction. Deformation of this contour to the imaginary axis gives an alternative form for I_1:

$$I_1(y)=(1-y^2)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}t\left\{t^2\left[(1-y)^{-1}+t^2\right]^{-1/2}\left[(1+y)^{-1}+t^2\right]^{-1/2}-1\right\}, \tag{4.16}$$

where the last term comes from the semicircle at large distances in the complex plane. The change of variable

$$t=(1+y)^{-1/2}\tan\theta \tag{4.17}$$

then yields

$$I_1(y)=-2(1-y)(1+y)^{-1/2}K(k)+I_2(y), \tag{4.18}$$

where

$$I_2(y)=(1-y)^{1/2}\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\sec^2\theta\left[\left(\frac{1-y}{1+y}\right)^{1/2}(1-k^2\sin^2\theta)^{-1/2}-1\right]. \tag{4.19}$$

Another integration by parts gives

$$I_2(y)=-2y(1-y)(1+y)^{-3/2}\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\sin^2\theta(1-k^2\sin^2\theta)^{-3/2}. \tag{4.20}$$

This integral on the right-hand side of (4.20) is recognized as the derivative of $K(k)$. Since this derivative can be expressed in terms of the complete elliptic integrals $K(k)$ and $E(k)$, the final answer for $I(y)$ is a linear combination of these two integrals:

$$I(y)=2(1+y)^{-1/2}[K(k)-(1+y)E(k)]. \tag{4.21}$$

Therefore, for large positive A, the energy E_n of the $n+1$ bound state is given approximately by the transcendental equation

$$2\sqrt{A}\sqrt{\frac{M}{M+E_n}}\left[K\left(\sqrt{\frac{2E_n}{M+E_n}}\right)-\left(\frac{M+E_n}{M}\right)E\left(\sqrt{\frac{2E_n}{M+E_n}}\right)\right]=n\pi. \tag{4.22}$$

This result can be rewritten in a number of equivalent ways using the transformation properties of the complete elliptic integrals.

It also follows from (3.16), (4.12), (4.13), (4.15) and (4.21) that the normalization N is given by

$$N=\tfrac{1}{2}\left[A^{1/2}(1+y)^{-1/2}y(1-y)^{-1}M^{-1}E(k)\right]^{-1/2}, \tag{4.23}$$

where y and k are given by (4.2) and (4.14).

5. Covering approximation: wave function

In this second part of the paper, we present an approximation that covers both the weak-binding case [2] and the WKB case of the first part. In the absence of an obvious name, we shall call it the covering approximation. However, it should be immediately obvious that the procedure is far from being unique, and many similar but distinct methods can be devised.

Because of the inversion symmetry [2] of the differential equations (2.2), it is sufficient to consider the region

$$\rho \geqslant (A^2 - B^2)^{-1/4}. \tag{5.1}$$

In the limit of weak binding with A not too large

$$\frac{A-B}{A} \ll 1, \qquad A = \mathrm{O}(1), \tag{5.2}$$

(2.2) can be approximated by

$$\frac{\mathrm{d}g}{\mathrm{d}\eta} = \left(A - B - \frac{1}{\eta^2}\right) f,$$

$$\frac{\mathrm{d}f}{\mathrm{d}\eta} = 2Bg. \tag{5.3}$$

The reason for using the coefficient $2B$ in the second equation, instead of $A + B$ or $2A$, is the desire to keep this coefficient the same as that for G in (2.2) at the turning point $\rho = (A - B)^{-1/2}$. In the weak-binding approximation

$$\rho = \eta, \tag{5.4}$$

$$F(\rho) = f(\eta), \tag{5.5a}$$

$$G(\rho) = g(\eta). \tag{5.5b}$$

The covering approximation consists of modifying (5.4) and (5.5) such that they are valid when either (5.2) or (1.4) holds.

A most straightforward way of achieving this modification is to compare the WKB approximations to (2.2) and (5.3). For this purpose, (3.9) is not convenient. Instead, let ϑ_1 and ϑ_2 be the WKB phases (see the appendix)

$$\vartheta_1(\rho) = \int_\rho^{(A-B)^{-1/2}} \mathrm{d}\rho' \left[(-A + B + \rho'^{-2})(A + B - \rho'^{-2})\right]^{1/2}, \tag{5.6}$$

$$\vartheta_2(\eta) = \int_\eta^{(A-B)^{-1/2}} \mathrm{d}\eta' \left[(-A + B + \eta'^{-2})(2B)\right]^{1/2}. \tag{5.7}$$

Let us relate ρ and η by

$$\mathcal{G}_1(\rho) = \mathcal{G}_2(\eta). \tag{5.8}$$

This is the generalization of (5.4). With (5.8), it is seen from (3.9) that the two WKB approximations differ only by the amplitude factors. Therefore the covering approximation is, omitting an overall constant factor,

$$F(\rho) = \frac{F_0(\rho)}{f_0(\eta)} f(\eta),$$

$$G(\rho) = \frac{G_0(\rho)}{g_0(\eta)} g(\eta), \tag{5.9}$$

where

$$F_0(\rho) = \left(\frac{A + B - \rho^{-2}}{-A + B + \rho^{-2}} \right)^{1/4},$$

$$f_0(\eta) = \left(\frac{2B}{-A + B + \eta^{-2}} \right)^{1/4},$$

$$G_0(\rho) = iF_0(\rho)^{-1},$$

$$g_0(\eta) = if_0(\eta)^{-1}. \tag{5.10}$$

This approximation (5.9) reduces to (5.5) in the weak-binding limit, and to the WKB solution when A is large.

For completeness we write down explicitly $f(\eta)$ and $g(\eta)$ from (5.3):

$$f(\eta) = z^{1/2} K_{ip}(z),$$

$$g(\eta) = \left(\frac{A - B}{2B} \right)^{1/2} \frac{\mathrm{d}}{\mathrm{d}z} \left[z^{1/2} K_{ip}(z) \right], \tag{5.11}$$

where

$$z = [2B(A - B)]^{1/2} \eta, \tag{5.12}$$

$$p = (2B - \tfrac{1}{4})^{1/2}. \tag{5.13}$$

Note that

$$F(\rho) = f(\eta), \qquad G(\rho) = g(\eta), \tag{5.14}$$

at the turning point $\rho = \eta = (A - B)^{-1/2}$.

6. Covering approximation: Bessel function

In studying the energy levels in the weak-binding approximation, the series expansion of $K_{ip}(z)$ is used [2,4]. Since the present approximation holds when either (5.2) or (1.4) applies, it is necessary to use an appropriate formula for $K_{ip}(z)$ that holds when

$$z \ll 1, \tag{6.1}$$

and when

$$p > z \gg 1. \tag{6.2}$$

In this section, we obtain this required formula since it does not seem to be available in the literature.

When (6.1) holds, then [6]

$$K_{ip}(z) \simeq \frac{-i\pi}{2\sinh(\pi p)}\left[\frac{(\frac{1}{2}z)^{-ip}}{\Gamma(1-ip)} - \text{c.c.}\right], \tag{6.3}$$

whereas when (6.2) holds, one has [6]

$$K_{ip}(z) \simeq \sqrt{2\pi}\, e^{-\frac{1}{2}p\pi}(p^2-z^2)^{-1/4}\sin\left[p\cosh^{-1}\left(\frac{p}{z}\right) - (p^2-z^2)^{1/2} + \tfrac{1}{4}\pi\right]. \tag{6.4}$$

For the present purpose, however, this form (6.4) is not convenient. The reason is that, in the WKB approximation, as seen from (5.7) or more explicitly (A.7), it is necessary to use

$$p' = (2B)^{1/2} = (p^2 + \tfrac{1}{4})^{1/2} \tag{6.5}$$

instead of p. Therefore, instead of (6.4), we shall use the alternative, equivalent asymptotic expansion

$$K_{ip}(z) \simeq \sqrt{2\pi}\, e^{-\frac{1}{2}p\pi}(p'^2-z^2)^{-1/4}\sin\left[p'\cosh^{-1}\left(\frac{p'}{z}\right) - (p'^2-z^2)^{1/2} + \tfrac{1}{4}\pi\right]. \tag{6.6}$$

Note the similarity between the amplitude $(p'^2 - z^2)^{-1/4}$ and the $f_0(\eta)$ of (5.10).

In the covering approximation, a formula is needed for the Bessel function $K_{ip}(z)$ that reduces to (6.3) when (6.1) is satisfied, and to (6.6) when (6.2) holds. This is accomplished by the ansatz

$$K_{ip}(z) \simeq \mathcal{Q}_K(p'^2 - z^2)^{-1/4}\sin\left[\frac{p}{p'}\mathcal{G}_K(p', z) + \Phi_K\right], \tag{6.7}$$

where

$$\mathcal{G}_K(p', z) = p'\cosh^{-1}\left(\frac{p'}{z}\right) - (p'^2 - z^2)^{1/2}. \tag{6.8}$$

It will be seen in sect. 7 that (6.7) has precisely the required form.

Clearly, $\mathcal{Q}_K$ has to satisfy

$$\mathcal{Q}_K \simeq \sqrt{2\pi}\, e^{-\frac{1}{2}p\pi} \quad \text{for } p \gg 1. \tag{6.9}$$

It is uniquely determined by a comparison with eq. (6.3), the right-hand side of which we rewrite as

$$\frac{\pi}{\sinh(\pi p)} \frac{1}{|\Gamma(1-ip)|} \sin\left[-p\ln(\tfrac{1}{2}z) + \arg\Gamma(1+ip)\right]. \tag{6.10}$$

Evaluating now (6.7) for $z \ll p$, and comparing with (6.10), we find

$$\mathcal{Q}_K = \left(\frac{p'}{p}\right)^{1/2} \left[\frac{\pi}{\sinh(\pi p)}\right]^{1/2}, \tag{6.11}$$

where we have also used

$$|\Gamma(1-ip)| = \left[\frac{\pi p}{\sinh(\pi p)}\right]^{1/2}. \tag{6.12}$$

The expression (6.11) is seen to be consistent with (6.9).

In order to determine the phase Φ_K, we expand the argument of the sine in (6.7) for $z \ll p$. This gives

$$p\ln\frac{2p'}{z} - p + \Phi_K. \tag{6.13}$$

Comparing with (6.10), we find

$$\Phi_K = \arg\Gamma(1+ip) - p\ln p' + p. \tag{6.14}$$

As $p \to \infty$, this approaches $\frac{1}{4}\pi$, in agreement with (6.6). The desired approximation to the Bessel function is thus

$$K_{ip}(z) \simeq \left(\frac{p'}{p}\right)^{1/2} \left[\frac{\pi}{\sinh(\pi p)}\right]^{1/2} (p'^2 - z^2)^{-1/4}$$
$$\times \sin\left[p\cosh^{-1}\left(\frac{p'}{z}\right) - \frac{p}{p'}(p'^2 - z^2)^{1/2} + \arg\Gamma(1+ip) - p\ln p' + p\right], \tag{6.15}$$

valid when either (6.1) or (6.2) is satisfied.

7. Covering approximation: energy levels

The energy eigenvalues can for example be determined using the inversion symmetry [2]

$$F(\bar{\rho}) = (-1)^{n+1}\left(\frac{A+B}{A-B}\right)^{1/4} G(\rho), \tag{7.1}$$

where ρ and $\bar{\rho}$ are two points whose geometric mean is the symmetry point, ρ_s:

$$\rho\bar{\rho} = \rho_s^2 = (A^2 - B^2)^{-1/2}, \tag{7.2}$$

and with n labelling the levels.

The way we determine the energy is to compare two expressions for $F(\rho)$, one obtained directly from (5.9), and one from (7.1). From eqs. (5.9)–(5.11) and (6.7), we have

$$F(\rho) \simeq F_0(\rho)\mathscr{A}_K\left(\frac{A-B}{2B}\right)^{1/4} \sin\left[\frac{p}{p'}\mathscr{G}_K(p', [2B(A-B)]^{1/2}\eta) + \Phi_K\right]. \tag{7.3}$$

We note that this expression, because of the approximation used for the Bessel function, is not valid around the turning point, $\rho = (A-B)^{-1/2}$. On the other hand, we shall see that it is crucial for an accurate determination of the energy that the radial variable ρ (or η) appears only in $F_0(\rho)$ and in the argument of the sine. This has been achieved through the introduction of p' in sect. 6.

We next turn to the evaluation of $G(\rho)$ in the matching region. From (5.9)–(5.11) and (6.7), we have

$$G(\rho) = \frac{f_0(\eta)}{F_0(\rho)}\left(\frac{A-B}{2B}\right)^{1/2}\mathscr{A}_K \times \frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{1/2}(p'^2 - z^2)^{-1/4}\sin\left[\frac{p}{p'}\mathscr{G}_K(p', z) + \Phi_K\right]\right\}\Bigg|_{z = [2B(A-B)]^{1/2}\eta}. \tag{7.4}$$

The differentiation yields

$$z^{-1/2}(p'^2 - z^2)^{-1/4}\left\{\tfrac{1}{2} + \frac{z^2}{2(p'^2 - z^2)} - \frac{ip}{p'}\sqrt{p'^2 - z^2}\right\} \times \frac{1}{2i}\exp\left[\frac{p}{p'}\mathscr{G}_K(p', z) + \Phi_K\right] + \text{c.c.} \tag{7.5}$$

The second term, which arises from differentiating the factor $(p'^2 - z^2)^{-1/4}$, is small in either limit, (5.2) or (1.4). We therefore neglect it. The expression (7.5) can then be rewritten as

$$z^{-1/2}(p'^2 - z^2)^{1/4} \frac{1}{p'} \left\{ \frac{1}{2} \frac{p'}{\sqrt{p'^2 - z^2}} - ip \right\} \frac{1}{2i} \exp\left[\frac{p}{p'} \mathcal{G}_K(p', z) + \Phi_K \right] + \text{c.c.} \tag{7.6}$$

Here, the factor $p'/\sqrt{p'^2 - z^2}$ that appears in the curly bracket can be approximated by one. In the limit of weak binding, $z^2 \ll p'^2$, whereas in the limit (1.4), the second term, $-ip$, dominates anyway. With*

$$e^{-i\psi_B} \equiv \frac{\frac{1}{2} - ip}{p'}, \tag{7.7}$$

the expression (7.5) can then be written

$$z^{-1/2}(p'^2 - z^2)^{1/4} \sin\left[\frac{p}{p'} \mathcal{G}_K(p', z) + \Phi_K - \psi_B \right], \tag{7.8}$$

and thus

$$G(\rho) \simeq \frac{1}{F_0(\rho)} \mathcal{A}_K \left(\frac{A - B}{2B} \right)^{1/4} \sin\left[\frac{p}{p'} \mathcal{G}_K(p', [2B(A - B)]^{1/2} \eta) + \Phi_K - \psi_B \right]. \tag{7.9}$$

By the inversion symmetry, eq. (7.1), we have

$$F(\bar{\rho}) \simeq (-1)^{n+1} \frac{1}{F_0(\rho)} \mathcal{A}_K \left(\frac{A + B}{2B} \right)^{1/4} \sin\left[\frac{p}{p'} \mathcal{G}_K(p', z) + \Phi_K - \psi_B \right], \tag{7.10}$$

whereas evaluating (7.3) at $\bar{\rho}$, we get

$$F(\bar{\rho}) \simeq F_0(\bar{\rho}) \mathcal{A}_K \left(\frac{A - B}{2B} \right)^{1/4} \sin\left[\frac{p}{p'} \mathcal{G}_K(p', \bar{z}) + \Phi_K \right], \tag{7.11}$$

where

$$z = [2B(A - B)]^{1/2} \eta, \qquad \bar{z} = [2B(A - B)]^{1/2} \bar{\eta}, \tag{7.12}$$

and with $\bar{\eta}$ related to $\bar{\rho}$ through (5.8). Comparing the right-hand sides of eqs. (7.10) and (7.11) we shall obtain an equation for the eigenvalue B.

* Note that ψ_B is real because of (6.5).

Since (cf. eqs. (5.10) and (7.2))

$$F_0(\rho)F_0(\bar{\rho})=\left(\frac{A+B}{A-B}\right)^{1/4}, \tag{7.13}$$

we are left with the phase matching condition

$$(-1)^{n+1}\sin\left[\frac{p}{p'}\vartheta_K(p',z)+\Phi_K-\psi_B\right]=\sin\left[\frac{p}{p'}\vartheta_K(p',\bar{z})+\Phi_K\right], \tag{7.14}$$

which we rewrite as

$$\frac{p}{p'}\left[\vartheta_K(p',z)+\vartheta_K(p',\bar{z})\right]+2\Phi_K-\psi_B=n\pi. \tag{7.15}$$

By construction, the phase function $\vartheta_K(p',z)$ is related to $\vartheta_2(\eta)$ (cf. eqs. (5.12), (6.8) and (A.7)):

$$\vartheta_K(p',z)=\vartheta_2(\eta). \tag{7.16}$$

Further, using eqs. (5.8) and (A.13), we get

$$\vartheta_K(p',z)+\vartheta_K(p',\bar{z})=\vartheta_1(\rho)+\vartheta_1(\bar{\rho})=\Phi_{\mathrm{WKB}}, \tag{7.17}$$

and (7.15) can be written as

$$\frac{p}{p'}\Phi_{\mathrm{WKB}}+2\Phi_K-\psi_B=n\pi. \tag{7.18}$$

The matching condition is thus independent of ρ. This has been achieved through the introduction of p' in the approximation to the Bessel function in sect. 6, and through the approximations involved in getting from (7.5) to (7.8).

Eq. (7.18) determines the energy in the covering approximation. All quantities on the left-hand side depend on B. Written out explicitly, B is determined by

$$\begin{aligned}&2\left(\frac{2B-\frac{1}{4}}{2B}\right)^{1/2}A^{1/2}\left(\frac{A}{A+B}\right)^{1/2}\left[K\left(\sqrt{\frac{2B}{A+B}}\right)-\frac{A+B}{A}E\left(\sqrt{\frac{2B}{A+B}}\right)\right]\\&\quad+2\left[\arg\Gamma\left(1+i\sqrt{2B-\tfrac{1}{4}}\right)-\sqrt{2B-\tfrac{1}{4}}\ln\sqrt{2B}+\sqrt{2B-\tfrac{1}{4}}\right]\\&\quad-\tan^{-1}\left[2\sqrt{2B-\tfrac{1}{4}}\right]=n\pi.\end{aligned} \tag{7.19}$$

When A becomes large, the left-hand side approaches Φ_{WKB}, and (7.18) or (7.19) reduces to (4.22). Similarly, when B is close to A, we can expand the elliptic integrals and thus recover the weak-binding approximation of ref. [2].

TABLE 1
Binding energies $(E-M)/M$ versus $A=\frac{1}{2}|q|\kappa$

A	Method	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	Exact	$4.8697 \cdot 10^{-4}$	$3.4447 \cdot 10^{-7}$	$2.4337 \cdot 10^{-10}$		
	WBA	$4.8757 \cdot 10^{-4}$	$3.4447 \cdot 10^{-7}$	$2.4336 \cdot 10^{-10}$	$1.7193 \cdot 10^{-13}$	$1.2147 \cdot 10^{-16}$
	WKB	$1.0939 \cdot 10^{-3}$	$2.0439 \cdot 10^{-6}$	$3.8169 \cdot 10^{-9}$		
	CA	$4.8715 \cdot 10^{-4}$	$3.4447 \cdot 10^{-7}$	$2.4336 \cdot 10^{-10}$		
1.0	Exact	$5.4882 \cdot 10^{-3}$	$4.7757 \cdot 10^{-5}$	$4.1337 \cdot 10^{-7}$	$3.5777 \cdot 10^{-9}$	$3.0964 \cdot 10^{-11}$
	WBA	$5.5186 \cdot 10^{-3}$	$4.7762 \cdot 10^{-5}$	$4.1337 \cdot 10^{-7}$	$3.5777 \cdot 10^{-9}$	$3.0964 \cdot 10^{-11}$
	WKB	$6.8793 \cdot 10^{-3}$	$8.1078 \cdot 10^{-5}$	$9.5370 \cdot 10^{-7}$	$1.1217 \cdot 10^{-8}$	
	CA	$5.4994 \cdot 10^{-3}$	$4.7759 \cdot 10^{-5}$	$4.1337 \cdot 10^{-7}$	$3.5777 \cdot 10^{-9}$	
2.0	Exact	$2.3610 \cdot 10^{-2}$	$9.2814 \cdot 10^{-4}$	$3.6214 \cdot 10^{-5}$	$1.4118 \cdot 10^{-6}$	$5.5038 \cdot 10^{-8}$
	WBA	$2.3831 \cdot 10^{-2}$	$9.2902 \cdot 10^{-4}$	$3.6216 \cdot 10^{-5}$	$1.4118 \cdot 10^{-6}$	$5.5038 \cdot 10^{-8}$
	WKB	$2.5239 \cdot 10^{-2}$	$1.0939 \cdot 10^{-3}$	$4.7296 \cdot 10^{-5}$	$2.0439 \cdot 10^{-6}$	$8.83 \quad \cdot 10^{-8}$
	CA	$2.3698 \cdot 10^{-2}$	$9.2837 \cdot 10^{-4}$	$3.6215 \cdot 10^{-5}$	$1.4118 \cdot 10^{-6}$	$5.5038 \cdot 10^{-8}$
5.0	Exact	$7.9322 \cdot 10^{-2}$	$1.0610 \cdot 10^{-2}$	$1.4227 \cdot 10^{-3}$	$1.9036 \cdot 10^{-4}$	$2.5453 \cdot 10^{-5}$
	WBA	$7.9680 \cdot 10^{-2}$	$1.0652 \cdot 10^{-2}$	$1.4241 \cdot 10^{-3}$	$1.9039 \cdot 10^{-4}$	$2.5454 \cdot 10^{-5}$
	WKB	$8.0454 \cdot 10^{-2}$	$1.0990 \cdot 10^{-2}$	$1.5098 \cdot 10^{-3}$	$2.0715 \cdot 10^{-4}$	$2.8407 \cdot 10^{-5}$
	CA	$7.9638 \cdot 10^{-2}$	$1.0619 \cdot 10^{-2}$	$1.4230 \cdot 10^{-3}$	$1.9036 \cdot 10^{-4}$	$2.5453 \cdot 10^{-5}$
10.0	Exact	$1.4522 \cdot 10^{-1}$	$3.4786 \cdot 10^{-2}$	$8.4778 \cdot 10^{-3}$	$2.0656 \cdot 10^{-3}$	$5.0275 \cdot 10^{-4}$
	WBA	$1.4374 \cdot 10^{-1}$	$3.4960 \cdot 10^{-2}$	$8.5026 \cdot 10^{-3}$	$2.0679 \cdot 10^{-3}$	$5.0294 \cdot 10^{-4}$
	WKB	$1.4589 \cdot 10^{-1}$	$3.5169 \cdot 10^{-2}$	$8.6384 \cdot 10^{-3}$	$2.1227 \cdot 10^{-3}$	$5.2119 \cdot 10^{-4}$
	CA	$1.4569 \cdot 10^{-1}$	$3.4824 \cdot 10^{-2}$	$8.4809 \cdot 10^{-3}$	$2.0658 \cdot 10^{-3}$	$5.0277 \cdot 10^{-4}$
20.0	Exact	$2.2492 \cdot 10^{-1}$	$8.0174 \cdot 10^{-2}$	$2.9480 \cdot 10^{-2}$	$1.0895 \cdot 10^{-2}$	$4.0274 \cdot 10^{-3}$
	WBA	$2.1726 \cdot 10^{-1}$	$8.0199 \cdot 10^{-2}$	$2.9605 \cdot 10^{-2}$	$1.0928 \cdot 10^{-2}$	$4.0341 \cdot 10^{-3}$
	WKB	$2.2527 \cdot 10^{-1}$	$8.0454 \cdot 10^{-2}$	$2.9654 \cdot 10^{-2}$	$1.0990 \cdot 10^{-2}$	$4.0743 \cdot 10^{-3}$
	CA	$2.2545 \cdot 10^{-1}$	$8.0256 \cdot 10^{-2}$	$2.9494 \cdot 10^{-2}$	$1.0898 \cdot 10^{-2}$	$4.0278 \cdot 10^{-3}$
50.0	Exact	$3.3857 \cdot 10^{-1}$	$1.7016 \cdot 10^{-1}$	$8.9098 \cdot 10^{-2}$	$4.7258 \cdot 10^{-2}$	$2.5175 \cdot 10^{-2}$
	WBA	$3.1295 \cdot 10^{-1}$	$1.6683 \cdot 10^{-1}$	$8.8929 \cdot 10^{-2}$	$4.7405 \cdot 10^{-2}$	$2.5270 \cdot 10^{-2}$
	WKB	$3.3871 \cdot 10^{-1}$	$1.7030 \cdot 10^{-1}$	$8.9214 \cdot 10^{-2}$	$4.7347 \cdot 10^{-2}$	$2.5239 \cdot 10^{-2}$
	CA	$3.3905 \cdot 10^{-1}$	$1.7029 \cdot 10^{-1}$	$8.9138 \cdot 10^{-2}$	$4.7271 \cdot 10^{-2}$	$2.5180 \cdot 10^{-2}$
100.0	Exact	$4.2299 \cdot 10^{-1}$	$2.5244 \cdot 10^{-1}$	$1.5718 \cdot 10^{-1}$	$9.9489 \cdot 10^{-2}$	$6.3440 \cdot 10^{-2}$
	WBA	$3.7606 \cdot 10^{-1}$	$2.4110 \cdot 10^{-1}$	$1.5457 \cdot 10^{-1}$	$9.9093 \cdot 10^{-2}$	$6.3529 \cdot 10^{-2}$
	WKB	$4.2305 \cdot 10^{-1}$	$2.5251 \cdot 10^{-1}$	$1.5725 \cdot 10^{-1}$	$9.9550 \cdot 10^{-2}$	$6.3490 \cdot 10^{-2}$
	CA	$4.2338 \cdot 10^{-1}$	$2.5257 \cdot 10^{-1}$	$1.5724 \cdot 10^{-1}$	$9.9514 \cdot 10^{-2}$	$6.3451 \cdot 10^{-2}$

Exact: numerical results of paper I [3]; WBA: weak-binding approximation of ref. [2]; WKB: sects. 3 and 4 of this paper; CA: covering approximation, sects. 5–7 of this paper.

(Some entries are left blank because of limited computer accuracy.)

8. Numerical results and discussion

Some numerical results for the binding energy are given in table 1. For a set of A-values ranging from 0.5 to 100, we compare the accurate results of paper I [3] and those of the weak-binding approximation [2] with those of the WKB method (first part of this paper) and with those of the covering approximation (second part of this paper). Five levels are considered, $n = 1$ to 5.

For large values of A, where the WKB method applies, it gives excellent results. They are best for the most strongly bound states. (It is amusing to note that the zero-energy level ($n = 0$) is exactly given by the WKB result.) We note that our analytic result for the WKB limit differs from that of ref. [1].

The results of the covering approximation are generally excellent for all A. When the binding is weak, they are comparable with, or even better than the results of the weak-binding approximation. Likewise, when A is large, they are comparable with the results of the WKB method.

The basic idea of the covering approximation is to obtain a result that is valid under two or more distinct circumstances. In spirit it is related to the uniform approximation of Langer [7]. It may have important applications in many branches of physics, and should be explored systematically.

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Appendix

THE PHASE INTEGRALS $\mathcal{I}_1(\rho)$ AND $\mathcal{I}_2(\eta)$

For an evaluation of the wave function in the covering approximation, one needs the explicit phase integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ defined by eqs. (5.6) and (5.7).

Integrating by parts, we find

$$\mathcal{I}_1(\rho) = \frac{1}{\rho}\left\{\left[-(A-B)\rho^2+1\right]\left[(A+B)\rho^2-1\right]\right\}^{1/2}$$
$$+2A\int_\rho^{(A-B)^{-1/2}} \frac{d\rho'}{\left\{\left[-(A-B)\rho'^2+1\right]\left[(A+B)\rho'^2-1\right]\right\}^{1/2}}$$
$$-2(A^2-B^2)\int_\rho^{(A-B)^{-1/2}} \frac{\rho'^2\, d\rho'}{\left\{\left[-(A-B)\rho'^2+1\right]\left[(A+B)\rho'^2-1\right]\right\}^{1/2}}. \tag{A.1}$$

The two remaining integrals can be expressed in terms of incomplete elliptic integrals. With

$$k=\left(\frac{2B}{A+B}\right)^{1/2}, \tag{A.2}$$

$$\sin^2\theta=\frac{1}{k^2}\left[1-(A-B)\rho^2\right], \tag{A.3}$$

we find [8]

$$\begin{aligned}\mathcal{I}_1(\rho)=&\frac{1}{\rho}\{[-(A-B)\rho^2+1][(A+B)\rho^2-1]\}^{1/2}\\&+2\sqrt{A}\left[\left(\frac{A}{A+B}\right)^{1/2}F(\theta,k)-\left(\frac{A+B}{A}\right)^{1/2}E(\theta,k)\right],\end{aligned} \tag{A.4}$$

where F and E are incomplete elliptic integrals of the first and second kind, respectively.

The integral $\mathcal{I}_2$ is elementary. With a suitable change of variable, $t=[1-(A-B)\eta'^2]^{1/2}$, we immediately get

$$\mathcal{I}_2(\eta)=(2B)^{1/2}\left[\tfrac{1}{2}\ln\frac{1+t_{\max}}{1-t_{\max}}-t_{\max}\right], \tag{A.5}$$

where

$$t_{\max}=\left[1-(A-B)\eta^2\right]^{1/2}. \tag{A.6}$$

Alternatively, we can write (A.5) as

$$\mathcal{I}_2(\eta)=(2B)^{1/2}\left\{\cosh^{-1}\left(\frac{1}{\sqrt{A-B}\,\eta}\right)-\left[1-(A-B)\eta^2\right]^{1/2}\right\}. \tag{A.7}$$

Expanding the elliptic integrals for small values of θ, we find

$$\mathcal{I}_1=\tfrac{4}{3}\sqrt{B}\,(A-B)^{3/4}x^{3/2}+O(x^{5/2}), \tag{A.8}$$

where

$$x=(A-B)^{-1/2}-\rho>0. \tag{A.9}$$

Similarly,

$$\mathcal{I}_2=\tfrac{4}{3}\sqrt{B}\,(A-B)^{3/4}y^{3/2}+O(y^{5/2}), \tag{A.10}$$

with

$$y=(A-B)^{-1/2}-\eta>0. \tag{A.11}$$

Near the turning point, the condition (5.8) thus reduces to (5.4), as it should.

Finally, we quote two useful properties of ϑ_1 that can be shown using properties of the elliptic integrals.

(i) At the symmetry point, $\rho_s = (A^2 - B^2)^{-1/4}$,

$$\vartheta_1(\rho_s) = A^{1/2}\left[\left(\frac{A}{A+B}\right)^{1/2} K(k) - \left(\frac{A+B}{A}\right)^{1/2} E(k)\right], \tag{A.12}$$

where K and E are complete elliptic integrals of the first and second kind, respectively.

(ii) The sum of the phase functions ϑ_1 at two points ρ and $\bar{\rho}$ related by the inversion symmetry (7.2) is a constant:

$$\vartheta_1(\rho) + \vartheta_1(\bar{\rho}) = 2\vartheta_1(\rho_s) = A^{1/2} I(B/A) \equiv \Phi_{\mathrm{WKB}}, \tag{A.13}$$

where I is defined by eq. (4.21).

References

[1] Y. Kazama and C.N. Yang, Phys. Rev. D15 (1977) 2300

[2] K. Olaussen, H.A. Olsen, P. Osland and I. Øverbø, Nucl. Phys. B228 (1983) 567

[3] P. Osland and T.T. Wu, Nucl. Phys. B247 (1984) 421

[4] P. Osland and T.T. Wu, Nucl. Phys. B247 (1984) 450

[5] W. Wilson, Phil. Mag. 29 (1915) 795;
A. Sommerfeld, Ann. der Phys. 51 (1916) 1

[6] Bateman Manuscript Project, Higher transcendental functions, vol. 2, ed. A. Erdélyi (McGraw-Hill, New York, 1953) chap. 7

[7] R.E. Langer, Trans. Amer. Math. Soc. 33 (1931) 23

[8] P.F. Byrd and M.D. Friedman, Handbook of elliptic integrals for engineers and scientists, 2nd ed. (Springer, Berlin, 1971)