Nuclear Physics B263 (1985) 155-172

NON-LOCAL CHARGES IN LOCAL QUANTUM FIELD THEORY

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Received 20 May 1985

Non-local charges [1, 2] are studied in the general setting of local quantum field theory. It is shown, that these charges can be represented as polynomials in the incoming respectively outgoing fields with coefficients (kernels) which are subject to specific constraints. For the restricted class of models of a scalar, massive, self-interacting particle in four dimensions, a more detailed analysis shows that all non-local charges of the generic type (genus 2) are products of generators of the Poincaré group. This analysis, which is based on the macroscopic causality properties of the S-matrix, seems to indicate that less trivial examples of non-local charges can only exist in two dimensions.

1. Introduction

We present in this article a systematic investigation of non-local charges in the general setting of local quantum field theory. It was argued in [2] that a few basic properties of field-theoretic models, such as locality, covariance and the existence of non-trivial scattering should be sufficient to determine these charges explicitly (similar to the case of the standard charges, cf. [3-5]). Our present results are only another step towards a solution of this problem. But they reveal the strong constraints imposed on non-local charges by the fundamental principles of quantum field theory.

The prototypes of non-local charges have been discovered in the quantum non-linear σ-model in two dimensions [1]. They can formally be represented by

$$Q^{ab} = \sum_c \int \mathrm{d}x \int \mathrm{d}y\, \varepsilon(x-y) j_0^{ac}(x) j_0^{cb}(y) - Z \int \mathrm{d}x\, j_1^{ab}(x)\,, \tag{1.1}$$

where j_μ^{ab}, $a, b = 1, \ldots, n$ are the Noether currents corresponding to the $\mathrm{O}(n)$ symmetry of the model, and Z is a renormalization constant. The operators Q^{ab} are distinguished by the fact that they commute with the hamiltonian, i.e. they are constants of motion.

As this example illustrates, the non-local charges are typically obtained by multiple integration of expressions involving products of local fields. As a consequence, they have properties which are not shared by the standard charges. One can show, for example, that Q^{ab} does not commute with the S-matrix. In fact, the restrictions arising from the existence of Q^{ab} essentially fix the S-matrix of the σ-model [1]. In view of this result it is an important question whether non-local charges can also exist in physical space-time.

In a general analysis of non-local charges one is, at the very beginning, faced with the problem of giving a proper definition of these quantities. As a matter of fact, in the concrete example given above one must go through a detailed analysis of the short-distance behaviour of the currents j_μ^{ab} in order to see that the charges Q^{ab} are well defined [1]. Therefore it seems hopeless to base a general analysis of non-local charges on an explicit representation of these quantities in terms of local fields, such as in (1.1).

It has therefore been proposed in [2] to characterize the non-local charges by a few general properties which can be extracted from the known examples. We recall these properties in sect. 2. We will then discuss (sect. 3) how the non-local charges act on collision states. Our main result in this context, which was already quoted in [2], says that these charges can always be expanded in terms of a *finite* number of asymptotic creation and annihilation operators. The remaining problem is then to determine the form of the kernels in this expansion. Since there exists an abundance of non-local charges in free field theories we will concentrate in this part of our analysis on theories with non-trivial scattering.

In order to avoid complications arising in the presence of internal symmetries we consider only models of a single, scalar self interacting particle. Moreover, we restrict our attention to the simplest non-trivial types of non-local charges (the charges of genus 1 and 2 according to the terminology of sect. 2). We will show in sects. 4 and 5 that, in four space-time dimensions, the only charges of genus 1 are the generators P^μ and $M^{\mu\nu}$ of the Poincaré transformations, and that the charges of genus 2 are bilinear in these operators. Similar results hold also in two dimensions if there is non-trivial multi-particle scattering and particle-production in the model (cf. sect. 6).

Although we have studied only a very restricted class of models, our results seem to indicate that non-local charges are polynomials in the generators of the space-time and internal symmetries also in general. As will become clear, the obstruction to more interesting examples are the clustering properties of the S-matrix. It is only in two dimensions, where these clustering properties need not hold, that non-local charges of the type found in the σ-model can exist.

2. Assumptions and notations

We are dealing with the Wightman theory of a single massive, scalar particle subject to the standard assumptions, such as locality, covariance, and relativistic spectrum condition. In particular, we assume that the mass shell of the particle is isolated from the rest of the spectrum, and that there is a Wightman field $\phi(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$ connecting the vacuum Ω and the single particle states. The collision states can then be constructed in the usual manner (cf., for example, [6]), and we assume that they form a dense set of vectors in the physical Hilbert space (asymptotic completeness).

Let us now turn to the characterization of the non-local charges Q (cf. [2]). These charges are in general unbounded operators; the information about their domain which is needed here is contained in our first postulate.

(i) Q is a closed, hermitian operator whose domain $\mathscr{D}(Q)$ contains the Wightman domain $\mathscr{D}_0$, i.e. the set of vectors which are generated from the vacuum Ω by applying polynomials in the Wightman fields.

In view of the difficulties arising in the explicit construction of non-local charges (cf. the remarks in the introduction) one might wish to consider also situations in which Q is only defined in the sense of sesquilinear forms on $\mathscr{D}_0 \times \mathscr{D}_0$. But then it is unclear whether there exists an unambiguous extension of Q to the collision states.

Our second assumption imposes certain continuity properties on Q.

(ii) Given $n \in \mathbb{N}$ there exists a Schwartz-norm $\| \cdot \|_\alpha$ on $\mathscr{S}(\mathbb{R}^4)$ such that

$$\| Q\phi(f_1) \cdots \phi(f_n)\Omega \| \leq c \|f_1\|_\alpha \cdots \|f_n\|_\alpha$$

for all $f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^4)$ and some constant c.

Finally, we come to the most important property of non-local charges, which reflects the fact that they are obtained by integrating products of local fields over some space-like plane (compare e.g. (1.1)). Due to this construction and locality, all multiple commutators involving a non-local charge Q and sufficiently many local field operators have to vanish if the fields are localized at different points of that plane. Moreover, since Q is a constant of motion, these commutators also vanish if one replaces Q by $Q(x) = U(x)QU(x)^{-1}$, where $U(x)$ is any space-time translation. In fact, we will only use these commutation properties of $Q(x)$, thereby allowing also for charges Q which are not invariant under space-time translations (as, e.g., the generators of the Poincaré transformations). Hence our last postulate reads as follows:

(iii) There exists a number N such that (in the weak sense on $\mathscr{D}_0 \times \mathscr{D}_0$)

$$[\cdots[Q(x), \phi(f_1)], \cdots \phi(f_{N+1})] = 0$$

for all translations x and test functions f_i, $i = 1, \ldots, N+1$ having support in $N+1$ disjoint double-cones with compact base in some fixed space-like plane. The minimal number N for which this relation holds is called the *genus* of Q.

It is clear that this characterization of non-local charges extends to theories in any number of space-time dimensions with an arbitrary particle spectrum. It also applies to spinorial (super-) charges. However, there one must admit in postulate (iii) commutators as well as anticommutators, depending on the Bose or Fermi character of the quantities involved.

Charges of genus 0 are clearly multiples of the identity. The standard charges, which are obtained by integrating current densities over all space, are of genus 1 and trivial examples of non-local charges of arbitrary genus N can be obtained by taking N-fold products of charges of genus 1. The non-local charges (1.1) in the σ-model provide non-trivial examples of charges of genus 2 in two dimensions.

As we will demonstrate in the subsequent sections, the characteristic properties of non-local charges summarized in the above postulates contain sufficient information for their detailed analysis.

3. Action of non-local charges on collision states

In this section we shall show how postulates (i) to (iii) can be used to determine the action of non-local charges on collision states. We begin by recalling some basic facts from collision theory.

Given n single-particle wave functions $f_i \in \mathcal{S}(\mathbb{R}^3)$ one obtains the corresponding incoming respectively outgoing n-particle collision state by the Haag-Ruelle construction [6], taking the limits

$$s-\lim_{t\to t^{\mathrm{ex}}} \phi(f_{1,t}) \cdots \phi(f_{n,t})\Omega = \Psi^{\mathrm{ex}}(f_1, \cdots, f_n)\,. \tag{3.1}$$

Here "ex" stands for "outgoing" or "incoming" and t^{ex} for $+\infty$ or $-\infty$, respectively. The functions $f_{i,t} \in \mathcal{S}(\mathbb{R}^4)$ are given in momentum space by

$$\tilde{f}_{i,t}(p) = \mathrm{e}^{it(p^0-\omega_p)} f_i(\boldsymbol{p})\Theta(p^0)h(p^2)\,, \tag{3.2}$$

where $\omega_{\boldsymbol{p}} = (\boldsymbol{p}^2 + m^2)^{1/2}$, $p^2 = (p^0)^2 - \boldsymbol{p}^2$; h is a smooth function with support about the isolated point $p^2 = m^2$ in the mass-spectrum and $h(m^2) = 1$. Because of technical reasons we will only work with configurations of wave functions $f_1, \ldots, f_n$ which have compact and mutually disjoint supports. We denote the linear span of the corresponding collision states and the vacuum Ω by $\mathcal{D}_0^{\mathrm{ex}}$; this space is dense in the Fock space $\mathcal{H}_0^{\mathrm{ex}}$ of all collision states. As is well known, one can introduce on $\mathcal{H}_0^{\mathrm{ex}}$ asymptotic creation and annihilation operators $a^*_{\mathrm{ex}}(f)$ and $a_{\mathrm{ex}}(f)$, respectively, satisfying canonical commutation relations. It is our aim to expand the non-local charges in terms of these operators.

In a first step we shall demonstrate that the non-local charges Q are defined on $\mathcal{D}_0^{\mathrm{ex}}$. Although the argument is standard we sketch it here, since similar methods have to be applied at various points of our analysis, where the details will then be omitted. We make use of the fact that the test functions* f_t in the definition (3.1) of the collision states can be approximated by functions $\hat{f}_t$ which have compact support in configuration space. Namely, let Γ be the set of four-velocities

$$\Gamma = \{(1, \boldsymbol{p}/\omega_{\boldsymbol{p}});\ \boldsymbol{p} \in \operatorname{supp} f\}\,, \tag{3.3}$$

and let $\hat{\Gamma}$ be any open bounded neighbourhood of Γ. Then there exist smooth

* To simplify notation we omit the index i for a moment.

functions χ which are equal to 1 on Γ and have support in $\hat{\Gamma}$. Setting

$$\hat{f}_t(x) = \chi(x/t) f_t(x) \tag{3.4}$$

it is obvious that $\hat{f}_t$ has support in the region $t \cdot \hat{\Gamma}$. Moreover, the difference between f_t and $\hat{f}_t$ tends to 0 as $t \to \pm\infty$. For later reference some relevant properties of f_t and $\hat{f}_t$ are listed in the following lemma (cf. for example [6]).

LEMMA 3.1. Let f_t and $\hat{f}_t$ be defined as above and let $\|\cdot\|_\alpha$ be any Schwartz norm on $\mathscr{S}(\mathbb{R}^4)$. If $|t| \geq 1$ then

(a) $f_t, \hat{f}_t \in \mathscr{S}(\mathbb{R}^4)$,

$\|f_t\|_\alpha + \|\hat{f}_t\|_\alpha \leq C_\alpha |t|^{n_\alpha}$ for some constant C_α and some $n_\alpha \in \mathbb{N}$,

$\|f_t - \hat{f}_t\|_\alpha \leq C_n |t|^{-n}$ for all $n \in \mathbb{N}$ and certain constants c_n.

(b) $\operatorname{supp} \hat{f}_t \subset t\hat{\Gamma}$.

(c) the $\mathscr{S}(\mathbb{R}^4)$-valued functions $t \to f_t$ and $t \to \hat{f}_t$ are smooth in the topology of $\mathscr{S}(\mathbb{R}^4)$, and the properties given in (a) and (b) hold analogously for the derivatives $\mathrm{d}f_t/\mathrm{d}t$ and $\mathrm{d}\hat{f}_t/\mathrm{d}t$.

Now given single-particle wave functions $f_1, \ldots, f_n$ with compact and mutually disjoint supports, then the corresponding velocity sets $\Gamma_1, \ldots, \Gamma_n$ are disjoint, hence there exist open neighbourhoods $\hat{\Gamma}_1, \ldots, \hat{\Gamma}_n$ of $\Gamma_1, \ldots, \Gamma_n$, respectively, which are mutually spacelike separated. Using approximating functions $\hat{f}_{1,t}, \ldots, \hat{f}_{n,t}$ of $f_{1,t}, \ldots, f_{n,t}$ which have support in the sets $t \cdot \hat{\Gamma}_1, \ldots, t \cdot \hat{\Gamma}_n$ as well as the space-like commutation properties of the field ϕ we can prove

PROPOSITION 3.2. $\mathscr{D}_0^{\mathrm{ex}}$ is contained in the domain $\mathscr{D}(Q)$ of any non-local charge Q.

Proof: Since Q is closed and the vectors $\phi(f_{1,t}) \cdots \phi(f_{n,t})\Omega$ are elements of $\mathscr{D}(Q)$ we must only verify that the vectors $Q\phi(f_{1,t}) \cdots \phi(f_{n,t})\Omega$ converge strongly as $t \to t^{\mathrm{ex}}$. To this end we apply the familiar trick and take the derivative of these vectors with respect to t. That this may be done follows from the smoothness of the functions $t \to f_{i,t}$ and the continuity properties of Q given in postulate (ii). Taking into account that $(\mathrm{d}/\mathrm{d}t)\phi(f_{i,t})\Omega = 0$ we get

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} Q\phi(f_{1,t}) \cdots \phi(f_{n,t})\Omega \right\| \leq \sum_{1 \leq k < l \leq n} \left\| Q\phi(f_{1,t}) \cdots \left[\phi\left(\frac{\mathrm{d}}{\mathrm{d}t} f_{k,t}\right), \phi(f_{l,t}) \right] \cdots \phi(f_{n,t})\Omega \right\|$$

and using postulate (ii) as well as the above lemma it is clear that we can replace the test functions $\mathrm{d}f_{k,t}/\mathrm{d}t$ and $f_{l,t}$ in the commutators by $\mathrm{d}\hat{f}_{k,t}/\mathrm{d}t$ and $\hat{f}_{l,t}$, respectively, the difference being rapidly decreasing in $|t|$. Yet since the functions $\mathrm{d}\hat{f}_{k,t}/\mathrm{d}t$ and $\hat{f}_{l,t}$ have support in the space-like separated regions $t \cdot \hat{\Gamma}_k$ and $t \cdot \hat{\Gamma}_l$, all commutators $[\phi(\mathrm{d}\hat{f}_{k,t}/\mathrm{d}t), \phi(\hat{f}_{l,t})]$ are equal to 0. So we get the bound

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} Q\phi(f_{1,t}) \cdots \phi(f_{n,t})\Omega \right\| \leq c_j |t|^{-j}$$

for any $j \in \mathbb{N}$. The desired result then follows by integration.

Let us remark that the matrix elements of

$$(\Psi^{\mathrm{ex}}(g_1,\ldots,g_m),\ Q\Psi^{\mathrm{ex}}(f_1,\ldots,f_n)) \tag{3.5}$$

regarded as linear functionals on the test functions $\bar{g}_1,\ldots,\bar{g}_m, f_1,\ldots,f_n$ are (restrictions of) distributions in $\mathscr{S}'(\mathbb{R}^{3(m+n)})$*. We call these distributions kernels and denote them by**

$$^{\mathrm{ex}}\langle \boldsymbol{q}_1,\ldots,\boldsymbol{q}_m|Q|\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\rangle^{\mathrm{ex}}\,. \tag{3.6}$$

For the expansion of the non-local charges in terms of asymptotic creation and annihilation operators it is more convenient to work with the truncated kernels

$$^{\mathrm{ex}}\langle \boldsymbol{q}_1,\ldots,\boldsymbol{q}_m|Q|\boldsymbol{p}_1,\ldots \boldsymbol{p}_n\rangle^{\mathrm{ex}}_{\mathrm{T}} \tag{3.7}$$

which are given by the recursive relation

$$^{\mathrm{ex}}\langle \boldsymbol{q}_1,\ldots,\boldsymbol{q}_m|Q|\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\rangle^{\mathrm{ex}}$$

$$\sum_{\mathrm{I,J}} {}^{\mathrm{ex}}\langle \boldsymbol{q}_{i_1},\ldots,\boldsymbol{q}_{i_k}|Q|\boldsymbol{p}_{j_1},\ldots,\boldsymbol{p}_{j_l}\rangle^{\mathrm{ex}}_{\mathrm{T}}\cdot {}^{\mathrm{ex}}\langle \boldsymbol{q}_{i_{k+1}},\ldots,\boldsymbol{q}_{i_m}|\boldsymbol{p}_{i_{l+1}},\ldots,\boldsymbol{p}_{i_n}\rangle^{\mathrm{ex}}\,. \tag{3.8}$$

Here the sum extends over all ordered subsets $I=(i_1,\ldots,i_k)\subseteq(1,\ldots,m)$ and $J=(j_1,\ldots,j_l)\subseteq(1,\ldots,n)$ as well as $I=\varnothing$ and $J=\varnothing$, where we adopt the convention that $\varnothing$ labels the vacuum. It is a simple consequence of the above proposition that one can represent Q in terms of asymptotic creation and annihilation operators according to

$$Q=\sum_{m,n=0}^{\infty}\frac{1}{m!\,n!}\int\frac{\mathrm{d}^3q_1}{2\omega_{q_1}}\cdots\int\frac{\mathrm{d}^3q_m}{2\omega_{q_m}}\int\frac{\mathrm{d}^3p_1}{2\omega_{p_1}}\cdots\int\frac{\mathrm{d}^3p_n}{2\omega_{p_n}}$$
$$\times a^*_{\mathrm{ex}}(\boldsymbol{q}_1)\cdots a^*_{\mathrm{ex}}(\boldsymbol{q}_m)\,{}^{\mathrm{ex}}\langle \boldsymbol{q}_1,\ldots,\boldsymbol{q}_m|Q|\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\rangle^{\mathrm{ex}}_{\mathrm{T}}\,a_{\mathrm{ex}}(\boldsymbol{p}_1)\cdots a_{\mathrm{ex}}(\boldsymbol{p}_n) \tag{3.9}$$

in the sense of sesquilinear forms on $\mathscr{D}_0^{\mathrm{ex}}\times\mathscr{D}_0^{\mathrm{ex}}$.

Up to this point we did not make use of the characteristic properties of non-local charges, given in postulate (iii). Using this input, we will now establish certain specific support properties of the corresponding truncated kernels (3.7).

PROPOSITION 3.3. Let Q be a non-local charge of genus N. Then

$$^{\mathrm{ex}}\langle \boldsymbol{q}_1,\ldots,\boldsymbol{q}_m|Q|\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\rangle^{\mathrm{ex}}_{\mathrm{T}}=0$$

in the sense of distributions on the (open) set of configurations $\{\boldsymbol{q}_1,\ldots,\boldsymbol{q}_m,\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\}$ consisting of more than N different momenta. If, in particular $m>N$ or $n>N$, then the truncated kernels vanish identically.

* To verify this one has to evaluate the dependence of the functions f_t and $\hat{f}_t$ on the underlying single-particle wave functions f, see for example [7].

** Note that these kernels are unambiguously defined only for non-coinciding momenta $\boldsymbol{q}_1,\ldots,\boldsymbol{q}_m$ and $\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n$, respectively. We will therefore restrict our attention to such configurations.

Proof: Let $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ be two sets of smooth single-particle wave functions with compact and (for each set separately) mutually disjoint supports. It is actually sufficient here to consider functions whose supports are contained in small balls such that the support of any function f_i $i=1, \ldots, n$ has a trivial intersection with the support of at most one of the functions g_j, $j=1, \ldots, m$. Using eq. (3.9) and the fact that the operators $\phi(f_t)$ and $\phi(g_t)^*$ converge for $t \to t^{\mathrm{ex}}$ to asymptotic creation and annihilation operators, respectively, it follows by an argument as in the proof of the preceding proposition that the smeared truncated kernels can be represented in the form

$$^{\mathrm{ex}}\langle g_1, \ldots, g_m | Q | f_1, \ldots, f_n \rangle_{\mathrm{T}}^{\mathrm{ex}}$$
$$= \lim_{t \to t^{\mathrm{ex}}} (-1)^n \cdot (\Omega, [\cdots [Q, \phi(f_{1,t})], \ldots, \phi(f_{n,t})], \phi(g_{1,t})^*], \ldots, \phi(g_{m,t})^*]\Omega)$$

in an obvious notation. Moreover, one can permute on the right-hand side of this equation the testfunctions $f_{i,t}$ $i=1, \ldots, n$, respectively $g_{j,t}$ $j=1, \ldots, m$, without changing the limit. Now in order to prove the statement we must show that this limit is zero if there are $N+1$ functions amongst $f_1, \ldots, f_n, g_1, \ldots, g_m$ with mutually disjoint supports; without restriction of generality we may assume that $f_1, \ldots, f_n, g_1, \ldots, g_{N+1-n}$ (if $n \leq N$) are these functions. According to lemma 3.1 we can approximate the corresponding testfunctions $f_{1,t}, \ldots, g_{N+1-n,t}$ by functions $\hat{f}_{1,t}, \ldots, \hat{g}_{N+1-n,t}$ having support in $N+1$ disjoint double cones with compact base in the space-like plane $x_0 = t$, $\boldsymbol{x} \in \mathbb{R}^3$. It then follows from postulate (iii) that the above multiple commutator vanishes in the limit of asymptotic times t. If n or m are larger than N, then the same argument shows that the regularized truncated kernels vanish for any possible choice of the single-particle wave functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$, respectively.

It is an immediate consequence of this proposition that the expansion (3.9) of Q in terms of incoming as well as outgoing fields terminates at a finite sum. Using the preceding results we now can give more explicit representations of the kernels of charges Q of genus 1 and 2.

Genus 1: If Q is a charge of genus 1 it is completely determined on $\mathscr{D}_0^{\mathrm{ex}}$ by the kernels

$$\langle \Omega | Q | \Omega \rangle, \qquad \langle \Omega | Q | \boldsymbol{p} \rangle, \qquad \langle \boldsymbol{q} | Q | \Omega \rangle, \qquad \langle \boldsymbol{q} | Q | \boldsymbol{p} \rangle_{\mathrm{T}}$$

involving only the vacuum and single-particle states (cf. eq. (3.9) and proposition 3.3). It thus follows that Q commutes with the S-matrix (weakly on $\mathscr{D}_0^{\mathrm{ex}} \times \mathscr{D}_0^{\mathrm{ex}}$). Since Q is a hermitian operator and since Ω is in the domain of Q it is clear that $\langle \Omega | Q | \Omega \rangle$ is a real number and that $\langle \Omega | Q | \boldsymbol{p} \rangle = \overline{\langle \boldsymbol{p} | Q | \Omega \rangle}$ is a square-integrable function with respect to $\mathrm{d}^3 p / 2\omega_p$. The remaining distribution $\langle \boldsymbol{q} | Q | \boldsymbol{p} \rangle_{\mathrm{T}}$ vanishes if $\boldsymbol{q} \neq \boldsymbol{p}$ (cf. proposition 3.3). Moreover, taking into account that this distribution is the kernel of an operator which contains in its domain the one-particle subspace of $\mathscr{D}_0^{ex}$ we find that*

$$\langle \boldsymbol{q}|Q|\boldsymbol{p}\rangle_{\mathrm{T}} = \sum_{m=0}^{M} A_{(m)}(\boldsymbol{q})\partial_{\boldsymbol{q}}^{(m)}\delta(\boldsymbol{q}-\boldsymbol{p}) , \tag{3.10}$$

where $A_{(m)}$ are locally square-integrable functions. From the hermiticity of Q there follow further obvious properties of these functions, which we do not need to give here.

Genus 2: Similarly to the case considered above one can see that a charge of genus 2 is determined on $\mathscr{D}_0^{\mathrm{ex}}$ by the kernels

$$\langle \Omega|Q|\Omega\rangle , \quad \langle \boldsymbol{q}|Q|\Omega\rangle , \quad {}^{\mathrm{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\Omega\rangle , \quad \langle \boldsymbol{q}|Q|\boldsymbol{p}\rangle_{\mathrm{T}} ,$$
$$ {}^{\mathrm{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\boldsymbol{p}\rangle_{\mathrm{T}} , \quad {}^{\mathrm{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{\mathrm{T}}^{\mathrm{ex}}$$

and their complex conjugates.

Besides the obvious properties of these kernels which follow from the fact that $\mathscr{D}_0^{\mathrm{ex}}$ is in the domain of Q we have further information from proposition 3.3 on the last two kernels. Namely,

$${}^{\mathrm{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\boldsymbol{p}\rangle_{\mathrm{T}} = \sum_{m=0}^{M} [C_{(m)}^{\mathrm{ex}}(\boldsymbol{q}_1, \boldsymbol{q}_2)\partial_{\boldsymbol{q}_2}^{(m)}\delta(\boldsymbol{q}_2-\boldsymbol{p})+(\boldsymbol{q}_1\leftrightarrow \boldsymbol{q}_2)] , \tag{3.11}$$

$${}^{\mathrm{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{\mathrm{T}}^{\mathrm{ex}} = 2 \sum_{m+m'=0}^{M} [B_{(m)(m')}^{\mathrm{ex}}(\boldsymbol{q}_1, \boldsymbol{q}_2)\partial_{\boldsymbol{q}_1}^{(m)}\delta(\boldsymbol{q}_1-\boldsymbol{p}_1)\partial_{\boldsymbol{q}_2}^{(m')}\delta(\boldsymbol{q}_2-\boldsymbol{p}_2)$$
$$+(\boldsymbol{q}_1\leftrightarrow \boldsymbol{q}_2)] , \tag{3.12}$$

where $B_{(m)(m')}^{\mathrm{ex}}(\boldsymbol{q}_1, \boldsymbol{q}_2) = B_{(m')(m)}^{\mathrm{ex}}(\boldsymbol{q}_2, \boldsymbol{q}_1)$ because of the symmetry properties of two-particle states.

In the case of a free field theory every operator Q of the above form is of finite genus, simply because free fields have c-number commutation relations. So the general results obtained so far are in a certain sense optimal. Yet, if there is non-trivial scattering in the model much more can be said. This will be exemplified in the case of charges of genus 1 and 2 in the subsequent sections.

4. Charges of genus 1

In this section we study charges Q of genus 1 in interacting theories. Assuming that there is non-trivial elastic scattering and particle production we will arrive at the result that the set of these charges consists of the generators of the Poincaré transformations (proposition 4.1).

* We use the notation (m) for any multiindex of the form $(i_1, \ldots, i_m)$ with $i_k = 1, 2, 3$. The symbol $\boldsymbol{a}^{(m)}$, where $\boldsymbol{a}$ is a vector, stands for $a_{i_1}, \ldots a_{i_m} = (-1)^m a^{i_1} \cdots a^{i_m}$. The following summation convention is used for symmetric tensors of rank m: $\boldsymbol{a}^{(m)}T_{(m)} = \sum_{i_1 \cdots i_m = 1}^{3} a_{i_1} \cdots a_{i_m} T_{i_1 \cdots i_m}$.

From sect. 3 we already know the general form of Q on $\mathscr{D}_0^{\text{ex}}$. We want to show now that Q cannot change the asymptotic particle number, i.e. that

$$\langle \boldsymbol{q}|Q|\Omega\rangle = 0\,. \tag{4.1}$$

Recalling the fact that Q commutes with the S-matrix, it is clear that the vectors $U(a, \Lambda)Q\Psi(f)$, where $\Psi(f)$ is any single-particle state and (a, Λ) any Poincaré transformation, describe states which do not scatter. Now, if the component of any one of these vectors in the two-particle subspace of $\mathscr{H}_0^{\text{ex}}$ would be different from zero, we would obtain, by varying f and (a, Λ), a total set of two-particle collision states which do not scatter. Since this would be in contradiction to our assumptions, eq. (4.1) follows.

In order to obtain more information about $\langle \boldsymbol{q}|Q|\boldsymbol{p}\rangle_{\text{T}}$ we apply Q to collision states having a large spatial separation from the origin. On these states the terms with the highest number of derivatives in the representation (3.10) of $\langle \boldsymbol{q}|Q|\boldsymbol{p}\rangle_{\text{T}}$ give the dominant contributions. This fact will simplify the analysis of the corresponding coefficients $A_{(M)}$. More specifically, let g_1, g_2 and f_1, f_2 be smooth functions with compact and mutually disjoint supports and let

$$g_{k,\lambda}(\boldsymbol{q}) = \mathrm{e}^{i\lambda \boldsymbol{a}\cdot\boldsymbol{q}} g_k(\boldsymbol{q})\,, \qquad k = 1, 2\,, \tag{4.2}$$

where $\lambda > 0$ and $\boldsymbol{a}$ is a unit vector. The functions $f_{k,\lambda}$, $k = 1, 2$ are defined analogously. Then

$$\begin{aligned}\lim_{\lambda\to+\infty} \frac{1}{(i\lambda)^M} &(\Psi^{\text{out}}(g_{1,\lambda}, g_{2,\lambda}), Q\Psi^{\text{in}}(f_{1,\lambda}, f_{2,\lambda}))\\ &= (\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(\boldsymbol{a}^{(M)}A_{(M)}f_1, f_2))\\ &\quad + (\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(f_1, \boldsymbol{a}^{(M)}A_{(M)}f_2))\end{aligned}$$

and a similar relation holds if "in" and "out" are interchanged. Therefore, using the hermiticity of Q and the fact that $\boldsymbol{a}$ is arbitrary we arrive at the relation

$$\begin{aligned}(\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(A_{(M)}f_1, f_2)) + (\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(f_1, A_{(M)}f_2))\\ = (\Psi^{\text{out}}(\bar{A}_{(M)}g_1, g_2), \Psi^{\text{in}}(f_1, f_2)) + (\Psi^{\text{out}}(g_1, \bar{A}_{(M)}g_2), \Psi^{\text{in}}(f_1, f_2))\,.\end{aligned} \tag{4.3}$$

Hence $A_{(M)}$ satisfies the functional equation

$$A_{(M)}(\boldsymbol{p}_1) + A_{(M)}(\boldsymbol{p}_2) = A_{(M)}(\boldsymbol{q}_1) + A_{(M)}(\boldsymbol{q}_2) \tag{4.4}$$

whenever $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$, $\boldsymbol{p}_1 \neq \boldsymbol{p}_2$ and $(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2)$ belongs to the support of ${}^{\text{out}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle^{\text{in}}$. We note that in order to make the step from (4.3) to (4.4) rigorous one must actually regularize $A_{(M)}$ by averaging $A_{(M)}(\Lambda \cdot \boldsymbol{p})$ over the Lorentz transformations Λ. The resulting function, for which eq. (4.4) still holds, is then smooth in $\boldsymbol{p}$.

It is well known that the only solutions of (4.4) are (apart from constants) linear combinations of energy and momentum. The weakest condition under which this result has been derived is the assumption that eq. (4.4) holds in all Lorentz-systems for at least one non-trivial elastic two-particle scattering configuration [8]. Because of the Lorentz invariance of the S-matrix, this condition is given, whenever there is non-trivial elastic two-particle scattering in the model, i.e. scattering in some non-forward direction.

Arguing now as in [3] (cf. also the proof of proposition 5.4) one can show that $A_{(M)}=0$ if $M \geq 2$ and that the remaining terms in (3.10) are proportional to the kernels of the generators of the Poincaré transformations. We remark that a term of the form $\delta(\boldsymbol{q}-\boldsymbol{p})$ cannot appear in (3.10) because it would correspond to the kernel of the particle number operator, which does not commute with the S-matrix since there is particle production [9]. So we arrive at

PROPOSITION 1. Let Q be a non-local charge of genus 1 in a theory with non-trivial elastic two-particle scattering. Then

$$Q = \alpha \cdot \mathbb{1} + \alpha_\mu P^\mu + \alpha_{\mu\nu} M^{\mu\nu} \qquad \text{on } \mathscr{D}_0^{\mathrm{ex}},$$

where α, α_μ, $\alpha_{\mu\nu}$ are real numbers.

5. Charges of genus 2

We turn now to the more interesting charges Q of genus 2, where the analysis is much more complicated. We proceed as follows. In a first step we decompose Q into a sum of operators with definite energy-momentum transfer. Then, using the non-triviality of the S-matrix and its clustering properties, we will find that charge operators with non-zero energy-momentum transfer do not exist (lemma 5.1, 5.2 and 5.3). With this information at hand we shall finally show that non-local charges of genus 2 can only be polynomials of second degree in the generators of the Poincaré transformations (proposition 5.4).

So let Q be any non-local charge of genus 2 and let $\chi \in \mathscr{S}(\mathbb{R}^4)$ be any real-valued smooth function. Then the expression

$$Q(\chi) = \int \mathrm{d}^4 a\, \chi(a) U(a) Q U(a)^{-1}$$

is a well-defined hermitian operator on $\mathscr{D}_0$, and it is straightforward to show that its operator closure, which we also denote by $Q(\chi)$, is again a non-local charge of genus 2. Its kernels are

$$(2\pi)^2 \tilde{\chi}(q_1 + \cdots + q_m - p_1 - \cdots - p_n) \cdot {}^{\mathrm{ex}}\langle \boldsymbol{q}_1, \ldots, \boldsymbol{q}_m | Q | \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n \rangle^{\mathrm{ex}}, \qquad (5.1)$$

where we set $q_j^0 = \omega_{\boldsymbol{q}_j}$, $p_i^0 = \omega_{\boldsymbol{p}_i}$ for $j = 1, \ldots, m$, $i = 1, \ldots, n$; $\tilde{\chi}$ denotes the Fourier transform of χ.

Now we choose χ in a particular way. Namely we assume that $\tilde{\chi}$ has compact support and satisfies one of the following conditions:

(1) $\quad \operatorname{supp} \tilde{\chi} \subset \{q \in \mathbb{R}^4;\ \varepsilon < q^2 < 4m^2 - \varepsilon\}$,

(2) $\quad \text{supp}\,\tilde{\chi} \subset \{q \in \mathbb{R}^4;\ m^2 + \varepsilon < q^2\}$,

(3) $\quad \text{supp}\,\tilde{\chi} \subset \{q \in \mathbb{R}^4;\ q^2 < m^2 - \varepsilon\} \backslash \{0\}$,

(4) $\quad \text{supp}\,\tilde{\chi} \subset \{q \in \mathbb{R}^4;\ |q| < \varepsilon\}$,

where $q^2 = (q^0)^2 - \boldsymbol{q}^2$, $|q|^2 = (q^0)^2 + \boldsymbol{q}^2$ and $0 < \varepsilon < \frac{1}{2}m^2$. Since the sets in (1)-(4) form an open covering of $\mathbb{R}^4$, it is clear that by varying χ in $Q(\chi)$ one can rediscover Q on $\mathscr{D}_0^{\text{ex}}$. Moreover, recalling the general form of Q on $\mathscr{D}_0^{\text{ex}}$ (cf. sect. 3), it is easy to check, that only the following kernels of $Q(\chi)$ (and their complex conjugates) can be different from zero if χ satisfies one of the conditions (1)-(4).

(1) $\quad \langle \boldsymbol{q}|Q(\chi)|\Omega\rangle, \qquad {}^{\text{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q(\chi)|\boldsymbol{p}\rangle_{\text{T}}$,

(2) $\quad {}^{\text{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q(\chi)|\Omega\rangle$,

(3) $\quad \langle \boldsymbol{q}|Q(\chi)|\boldsymbol{p}\rangle_{\text{T}}$,

(4) $\quad \langle \Omega|Q(\chi)|\Omega\rangle, \qquad \langle \boldsymbol{q}|Q(\chi)|\boldsymbol{p}\rangle_{\text{T}}, \qquad {}^{\text{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q(\chi)|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{\text{T}}^{\text{ex}}$.

We denote the corresponding operators by Q_1, Q_2, Q_3 and Q_4, respectively. Assuming that there is non-trivial elastic two-particle scattering, we obtain the following lemmas.

LEMMA 5.1. $Q_1 = 0$.

Proof: Let us first assume that $M \geq 1$ in (3.11) and calculate matrix elements of Q * between suitable one- and two-particle states with large angular momentum. (Compare the argument leading to proposition 4.1.) Due to the M-fold derivatives in the kernel (3.11) we get for large λ the asymptotic expansion

$$\frac{1}{(i\lambda)^M}(\Psi^{\text{out}}(g_1, g_{2,\lambda}), Q\Psi(f_\lambda)) = (\Psi^{\text{out}}(g_1, g_{2,\lambda}), \Psi^{\text{in}}_{2,\lambda}) + O\left(\frac{1}{\lambda}\right),$$

where $g_{2,\lambda}$, f_λ are defined as in (4.2), and $\Psi^{\text{in}}_{2,\lambda}$ is an incoming two-particle state with wave function

$$\text{sym}\,[\boldsymbol{a}^{(M)} C^{\text{in}}_{(M)}(\boldsymbol{p}_1, \boldsymbol{p}_2) f(\boldsymbol{p}_2)\, \mathrm{e}^{i\lambda \boldsymbol{a}\cdot \boldsymbol{p}_2}].$$

Here "sym" denotes symmetrization with respect to the momenta. Now, using the clustering properties of the S-matrix** (see [6]) we obtain from the above expansion in the limit $\lambda \to +\infty$,

$$\lim_{\lambda\to+\infty} \frac{1}{(i\lambda)^M}(\Psi^{\text{out}}(g_1, g_{2,\lambda}), Q\Psi(f_\lambda))$$
$$= \int \frac{\mathrm{d}^3 q_1}{2\omega_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}^3 q_2}{2\omega_{\boldsymbol{q}_2}} \overline{g_1(\boldsymbol{q}_1) g_2(\boldsymbol{q}_2)}\, \boldsymbol{a}^{(M)} C^{\text{in}}_{(M)}(\boldsymbol{q}_1, \boldsymbol{q}_2) f(\boldsymbol{q}_2).$$

* To simplify notation we omit in the proof the index 1.

** Hepp's cluster theorem does not only hold for test functions but for arbitrary square-integrable wave functions. This follows from the fact that the S-matrix is a bounded operator and that the translation operators are uniformly bounded. We have to use this fact, because non-local charges may transform smooth wave functions into functions which do not belong to the space of test functions.

On the other hand, using (3.11), we have the asymptotic expansion

$$\frac{1}{(i\lambda)^M}(Q\Psi^{\text{out}}(g_1, g_{2,\lambda}), \Psi(f_\lambda)) = (\Psi_{1,\lambda}, \Psi(f_\lambda)) + O\left(\frac{1}{\lambda}\right),$$

where $\Psi_{1,\lambda}$ is a single-particle state with the wave function

$$2\int \frac{d^3q_1}{2\omega_{q_1}} \operatorname{sym}[g_1(\boldsymbol{q}_1)g_2(\boldsymbol{q}_2)\, e^{i\lambda \boldsymbol{a}\cdot\boldsymbol{q}_2}]\boldsymbol{a}^{(M)}\overline{C^{\text{out}}_{(M)}(\boldsymbol{q}_1, \boldsymbol{q}_2)}.$$

Hence, by the Riemann–Lebesgue lemma, we get

$$\lim_{\lambda\to+\infty} \frac{1}{(i\lambda)^M}(Q\Psi^{\text{out}}(g_1, g_{2,\lambda}), \Psi(f_\lambda))$$

$$= \int \frac{d^3q_1}{2\omega_{q_1}}\int \frac{d^3q_2}{2\omega_{q_2}} \overline{g_1(\boldsymbol{q}_1)g_2(\boldsymbol{q}_2)}\boldsymbol{a}^{(M)}C^{\text{out}}_{(M)}(\boldsymbol{q}_1, \boldsymbol{q}_2)f(\boldsymbol{q}_2).$$

Because of the hermiticity of Q, this limit coincides with the limit obtained above. So we arrive at the conclusion that the coefficients $C^{\text{ex}}_{(M)}$ in (3.11) are independent of "ex". The same result can be derived for $M=0$ if one takes into account that the kernel $\langle \boldsymbol{q}|Q|\Omega\rangle$ is independent of "ex". Using the identity

$$\frac{1}{\lambda^M}(\Psi^{\text{out}}(g_1, g_2, g_{3,\lambda}), Q\Psi^{\text{in}}(f_1, f_{2,\lambda})) = \frac{1}{\lambda^M}(Q\Psi^{\text{out}}(g_1, g_2, g_{3,\lambda}), \Psi^{\text{in}}(f_1, f_{2,\lambda}))$$

we obtain in a similar way the following relations:

(i) If $M \geq 1$, then

$$(\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(c_{(M)}, f_1)) = (\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{out}}(c_{(M)}, f_1)), \tag{5.2}$$

where

$$c_{(M)}(\boldsymbol{p}) = \int \frac{d^3q}{2\omega_q} C_{(M)}(\boldsymbol{p}, \boldsymbol{q})f_2(\boldsymbol{q})\overline{g_3(\boldsymbol{q})}$$

is a square integrable function with respect to $d^3p/2\omega_p$. This result means that the states $\Psi^{\text{ex}}(c_{(M)}, f_1)$ do not scatter. Hence we can proceed along the lines of the proof of eq. (4.1) in sect. 4 with the result $C_{(M)} = 0$.

(ii) If $M=0$, then

$$(\Psi^{\text{out}}(g_1, g_2), \Psi_2^{\text{in}}) = (\Psi^{\text{out}}(g_1, g_2), \Psi_2^{\text{out}}), \tag{5.3}$$

where Ψ_2^{ex} is an asymptotic two-particle state with the wave function

$$2\operatorname{sym}[(\langle \boldsymbol{q}_1|Q|\Omega\rangle + C_{(0)}(\boldsymbol{q}_1, \boldsymbol{q}_2))(g_3, f_2) + c_{(0)}(\boldsymbol{q}_1)]f_1(\boldsymbol{q}_2).$$

Here $c_{(0)}(\boldsymbol{p})$ is defined as in (i), and (g_3, f_2) denotes the scalar product of the wave functions g_3 and f_2.

Now from the fact that the states Ψ_2^{ex} and $Q\Psi(f_1)$ do not scatter, it follows that $\Psi^{\text{ex}}(c_{(0)}, f_1)$ does not scatter either. Therefore, by a similar argument as in sect. 4 we obtain $C_{(0)} = 0$ and $\langle \boldsymbol{q}|Q|\Omega\rangle = 0$, which proves the lemma.

LEMMA 5.2. $Q_2 = 0$.

Proof: Let us consider the operator* $\Delta_a = i[Q(a), Q]$, where $a \in \mathbb{R}^4$ and $Q(a) = U(a)QU(a)^{-1}$. (Note that this operator is well defined on $\mathscr{D}_0^{\text{ex}}$ because of the square-integrability of the kernel ${}^{\text{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\Omega\rangle$.) If we can show that $\Delta_a = 0$ for all $a \in \mathbb{R}^4$ the statement of the lemma follows, as can be seen as follows: the relation $\Delta_a = 0$ implies that

$$(Q\Omega, U(a)Q\Omega) = (Q\Omega, U(-a)Q\Omega),$$

hence by the spectrum condition and the uniqueness of the vacuum we obtain

$$(Q\Omega, U(a)Q\Omega) = |(\Omega, Q\Omega)|^2 = 0$$

for all $a \in \mathbb{R}^4$. But this means that $Q\Omega = 0$ or, equivalently, that ${}^{\text{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\Omega\rangle = 0$.

Now, using the explicit form of Q, it is easy to see that

$$\Delta_a = \langle\Omega|\Delta_a|\Omega\rangle \cdot \mathbb{1} + \int \frac{d^3p}{2\omega_{\boldsymbol{p}}} \int \frac{d^3q}{2\omega_{\boldsymbol{q}}}\, a^*_{\text{ex}}(\boldsymbol{q})\langle \boldsymbol{q}|\Delta_a|\boldsymbol{p}\rangle_{\text{T}} a_{\text{ex}}(\boldsymbol{p})$$

in the sense of sesquilinear forms on $\mathscr{D}_0^{\text{ex}} \times \mathscr{D}_0^{\text{ex}}$; moreover, if $\langle \boldsymbol{q}|\Delta_a|\boldsymbol{p}\rangle_{\text{T}} = 0$ then $\langle\Omega|\Delta_a|\Omega\rangle = 0$. Therefore, in order to show that $\Delta_a = 0$ we must only verify that $\langle \boldsymbol{q}|\Delta_a|\boldsymbol{p}\rangle_T$ vanishes. But this is an immediate consequence of the following facts: first, $\langle \boldsymbol{q}|\Delta_a|\boldsymbol{p}\rangle_{\text{T}}$ is a square-integrable function with respect to $(d^3q/2\omega_{\boldsymbol{q}}) \cdot (d^3p/2\omega_{\boldsymbol{p}})$ (because of the square-integrability of ${}^{\text{ex}}\langle q_1, q_2|Q|\Omega\rangle$) and, second, the support of $\langle \boldsymbol{q}|\Delta_a|\boldsymbol{p}\rangle_{\text{T}}$ is confined to the plane $\boldsymbol{q} = \boldsymbol{p}$. The latter statement is a consequence of the subsequent lemma, if one takes into account that the operator Δ_a has, after integration with a function χ of type (3), the same structure as the charges Q_3.

LEMMA 5.3. $Q_3 = 0$.

A proof of this lemma can be found in the literature (cf. for example [4]). We mention as an aside that the existence of elastic two-particle scattering in some non-forward direction is sufficient for the derivation of this result. Now we can prove the main result of this section.

PROPOSITION 5.4. Let Q be a non-local charge of genus 2 in a theory with non-trivial elastic two-particle scattering. Then

$$Q = \alpha\mathbb{1} + \alpha_\mu P^\mu + \alpha_{\mu\nu}M^{\mu\nu} + \beta_{\mu\nu}P^\mu P^\nu + \alpha_{\mu\nu\rho}(P^\mu M^{\nu\rho} + M^{\nu\rho}P^\mu)$$
$$+ \alpha_{\mu\nu\rho\tau}(M^{\mu\nu}M^{\rho\tau} + M^{\rho\tau}M^{\mu\nu}) \qquad \text{on } \mathscr{D}_0^{\text{ex}},$$

where α, α_μ, $\alpha_{\mu\nu}$, $\beta_{\mu\nu}$, $\alpha_{\mu\nu\rho}$, $\alpha_{\mu\nu\rho\tau}$ are real numbers.

Proof: Summing up the results obtained so far, we see that Q has zero energy-momentum transfer. Hence the only non-vanishing kernels of Q can be

$$\langle\Omega|Q|\Omega\rangle, \qquad \langle \boldsymbol{q}|Q|\boldsymbol{p}\rangle_{\text{T}}, \qquad {}^{\text{ex}}\langle \boldsymbol{q}_1, \boldsymbol{q}_2|Q|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{\text{T}}^{\text{ex}}.$$

* We omit the index 2 of Q_2 in the following.

Moreover, for the latter two kernels the representations (3.10) and (3.12), respectively, are valid. For our subsequent analysis of these kernels the following identities are basic (in the limit $\lambda \to +\infty$):

$$\frac{1}{\lambda^M}(\Psi^{\text{out}}(g_{1,\lambda}, g_{2,\lambda}), Q\Psi^{\text{in}}(f_{1,\lambda}, f_{2,\lambda})) = \frac{1}{\lambda^M}(Q\Psi^{\text{out}}(g_{1,\lambda}, g_{2,\lambda}), \Psi^{\text{in}}(f_{1,\lambda}, f_{2,\lambda})), \tag{5.4}$$

$$\frac{1}{\lambda^M}(\Psi^{\text{out}}(g_{1,\lambda}, g_{2,\lambda}, g_{3,\lambda}), Q\Psi^{\text{in}}(f_{1,\lambda}, f_{2,\lambda}, f_{3,\lambda}))$$
$$= \frac{1}{\lambda^M}(Q\Psi^{\text{out}}(g_{1,\lambda}, g_{2,\lambda}, g_{3,\lambda}), \Psi^{\text{in}}(f_{1,\lambda}, f_{2,\lambda}, f_{3,\lambda})). \tag{5.5}$$

Here

$$g_{k,\lambda}(\boldsymbol{q}) = e^{i\lambda \boldsymbol{a}_k \cdot \boldsymbol{q}} g_k(\boldsymbol{q}), \qquad k = 1, 2, 3,$$

where g_k, $k = 1, 2, 3$ are smooth functions with compact and mutually disjoint supports, $\boldsymbol{a}_k$, $k = 1, 2, 3$ are unit vectors, and $\lambda > 0$. The functions $f_{k,\lambda}$, $k = 1, 2, 3$ are analogously defined.

We proceed now in the following five steps. (Since the arguments are very similar to those given in Lemma 4.1 we can be very brief.)

(i) From (5.4) with $\boldsymbol{a}_1 \neq \boldsymbol{a}_2$ we obtain in the limit $\lambda \to +\infty$ (due to the clustering properties of the S-matrix) that $B^{\text{ex}}_{(m)(m')}$ in eq. (3.12) is independent of "ex" if $m + m' = M$.

(ii) Similarly, we get from (5.5) with $\boldsymbol{a}_1 = \boldsymbol{a}_2 \neq \boldsymbol{a}_3$ and (5.4) with $\boldsymbol{a}_1 = \boldsymbol{a}_2$ in the limit $\lambda \to +\infty$,

$$(\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(b_{(m)(m')}f_1, f_2)) + (\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(f_1, b_{(m)(m')}f_2))$$
$$= (\Psi^{\text{out}}(\bar{b}_{(m)(m')}g_1, g_2), \Psi^{\text{in}}(f_1, f_2)) + (\Psi^{\text{out}}(g_1, \bar{b}_{(m)(m')}g_2), \Psi^{\text{in}}(f_1, f_2)) \tag{5.6}$$

if $m + m' = M$. Here

$$b_{(m)(m')}(\boldsymbol{p}) = \int \frac{d^3q}{2\omega_{\boldsymbol{q}}} B_{(m)(m')}(\boldsymbol{p}, \boldsymbol{q}) \overline{g_3(\boldsymbol{q})} f_3(\boldsymbol{q}).$$

Thus, it follows by the arguments of sect. 4 (cf. eq. (4.3)) that

$$B_{(m)(m')}(\boldsymbol{p}, \boldsymbol{q}) = a_{(m)(m')\mu\nu} p^\mu q^\nu + b_{(m)(m')\mu} p^\mu + b_{(m')(m)\mu} q^\mu + c_{(m)(m')} \tag{5.7}$$

if $m + m' = M$; here $a_{(m)(m')\mu\nu}$, $b_{(m)(m')\mu}$, $c_{(m)(m')}$ are constants and $p^0 = \omega_{\boldsymbol{p}}$, $q^0 = \omega_{\boldsymbol{q}}$.

(iii) Combining (5.4) ($\boldsymbol{a}_1 = \boldsymbol{a}_2$) and (5.7) we conclude that

$$(\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(A'_{(M)}f_1, f_2)) + (\Psi^{\text{out}}(g_1, g_2), \Psi^{\text{in}}(f_1, A'_{(M)}f_2))$$
$$= (\Psi^{\text{out}}(\bar{A}'_{(M)}g_1, g_2), \Psi^{\text{in}}(f_1, f_2)) + (\Psi^{\text{out}}(g_1, \bar{A}'_{(M)}g_2), \Psi^{\text{in}}(f_1, f_2)), \tag{5.8}$$

where

$$A'_{(M)}(\boldsymbol{p}) = A_{(M)}(\boldsymbol{p}) - a_{(M)\mu\nu}p^\mu p^\nu ,$$

$$a_{(M)\mu\nu} = \sum_{m+m'=M} a_{(m)(m')\mu\nu} .$$

So again we can apply the arguments of sect. 4, giving

$$A_{(M)}(\boldsymbol{p}) = a_{(M)\mu\nu}p^\mu p^\nu + d_{(M)\mu}p^\mu + e_{(M)} , \tag{5.9}$$

where $d_{(M)\mu}$, $e_{(M)}$ are constants. If in eqs. (3.10) and (3.12) $M=0$, then the information on the kernels obtained so far implies that

$$Q = \alpha \cdot \mathbb{1} + \alpha_\mu P^\mu + \alpha_{\mu\nu}P^\mu P^\nu \qquad \text{on } \mathscr{D}_0^{\text{ex}} ,$$

so there is nothing more to prove. (Note that $b_{(0)(0)\mu} = e_{(0)} = 0$ due to the presence of inelastic scattering [9].)

(iv) If $M \geq 1$ we define on $\mathscr{D}_0^{\text{ex}}$ the operator $\hat{Q} = i[Q, P^0]$, and similarly $\hat{\hat{Q}} = i[\hat{Q}, P^0]$ if $M \geq 2$. It is obvious that $\hat{Q}$ ($\hat{\hat{Q}}$) is a hermitian operator with kernels of the form (3.10) and (3.12); but now there appear at most $M-1$, $(M-2)$ derivatives in these expressions. By applying the results of the previous steps to $\hat{Q}$ and $\hat{\hat{Q}}$ it follows that the corresponding coefficients $\hat{A}_{(M-1)}$, $\hat{B}_{(m)(m')}$ and $\hat{\hat{A}}_{(M-2)}$, $\hat{\hat{B}}_{(m)(m')}$ have the form given in eqs. (5.9) and (5.7), respectively.

(v) From the very definition of $\hat{Q}$ in the previous step we obtain the following equations*:

$$ip^0\hat{A}_{(M-1)}(\boldsymbol{p}) = \boldsymbol{p}^{(1)}A_{(1,M-1)}(\boldsymbol{p}) , \tag{5.10}$$

$$ip^0q^0\hat{B}_{(m)(m')}(\boldsymbol{p}, \boldsymbol{q}) = (m+1)q^0\boldsymbol{p}^{(1)}B_{(1,m)(m')}(\boldsymbol{p}, \boldsymbol{q}) + (m'+1)p^0\boldsymbol{q}^{(1)}B_{(m)(1,m')}(\boldsymbol{p}, \boldsymbol{q}) ,$$

if $m+m' = M-1$, $\quad M \geq 1$.

Clearly, analogous equations hold also for $\hat{\hat{A}}_{(M-2)}$, $\hat{A}_{(M-1)}$ and $\hat{\hat{B}}_{(m)(m')}$, $\hat{B}_{(1,m)(m')}$, $\hat{B}_{(m)(1,m')}$ if $m+m' = M-2$, $M \geq 2$.

The proof is now accomplished by solving these equations, taking into account the constraints on $A_{(M)}$, $B_{(m)(m')}$, $\hat{A}_{(M-1)}$, $\hat{B}_{(m)(m')}$ and $\hat{\hat{A}}_{(M-2)}$, $\hat{\hat{B}}_{(m)(m')}$ resulting from eqs. (5.7) and (5.9). It turns out that $A_{(M)} = B_{(m)(m')} = 0$ if $m+m' = M > 2$. This means that the number of derivatives appearing in the kernels (3.10) and (3.12) is less or equal to 2. If $M \leq 2$ then the above equations do have non-zero solutions, and it is straightforward to show that the corresponding charge-operators are of the form given in the statement of the proposition.

These results have been obtained by a rather tedious computation. A simpler solution of this problem would be important for the analysis of non-local charges of higher genus and of theories with several particles.

* We use the notation $\boldsymbol{a}^{(k)}T_{(k,l)} = \sum_{i_1\cdots i_k=1}^{3} a_{i_1}\cdots a_{i_k}T_{i_1\cdots i_k j_1\cdots j_l}$ for contractions. Here $T_{(k,l)}$ is any symmetric tensor of rank $k+l$ and $\boldsymbol{a}$ any vector (cf. also footnote on page 162).

6. Conclusions

From the "axiomatic" point of view adopted in this paper, the non-local charges appear as a quite natural generalization of the standard charges. It is another virtue of this general approach that it is based only on a few intrinsic properties of non-local charges, thereby avoiding all difficulties arising in an explicit construction of these quantities in terms of local fields.

Although we have confined our attention to a restricted class of models, it is clear that many of our arguments carry over to more general situations. For example, the analysis in sect. 3 can be performed in models with an arbitrary number of massive particles and in any number of space-time dimensions. Up to some minor notational complications, the results are the same.

Only a little more effort than in sect. 4 is needed for the analysis of non-local charges of genus 1 in models with an arbitrary particle spectrum. Using the methods outlined in [4] one finds that the set of these charges consists of the generators of space-time and internal symmetries. In the case of spinorial charges it follows from the arguments in [10] that such charges are generators of supersymmetries.

The calculation of non-local charges of higher genus is, however, fairly complicated. As was demonstrated in sect. 5, an important tool for the analysis of these charges are the clustering properties of the S-matrix. For the class of models considered here, it is this property of "macroscopic causality" which admits as non-local charges of genus 2 only polynomials of second degree in the generators of the Poincaré transformations.

The clustering properties of the S-matrix are also useful in the analysis of non-local charges of arbitrary genus in models with any number of particles. They lead to a set of constraint equations for the kernels of these charges, similar to eqs. (5.2), (5.3), (5.6), (5.8) and (5.10). Unfortunately, we have not been able to find the general solution of these equations. Our partial results seem to indicate that there cannot exist solutions other than those corresponding to polynomials in the generators of space-time, internal and supersymmetries. But since we cannot definitely exclude solutions which are more interesting, this matter should be settled completely.

Let us finally comment on the particular situation in two space-time dimensions, where the arguments of sects. 4 and 5 are not sufficient. Here again we restrict our attention to models of a single massive particle.

First, in two dimensions the functional equation

$$A(p_1)+A(p_2)=A(q_1)+A(q_2)$$

on the two-particle scattering manifold (compare eq. (4.4)) does not impose any constraints on the function A, because on the scattering manifold $p_1=q_1$, $p_2=q_2$ or $p_1=q_2$, $p_2=q_1$. But if, for example,

$$A(p_1)+A(p_2)+A(p_3)=A(q_1)+A(q_2)+A(q_3)$$

on the three-particle scattering manifold, then one can show that A must be linear in energy and momentum. Taking this fact into account one obtains for the charges of genus 1 the same results as in sect. 4, provided there is particle production and non-trivial three- (or many-) particle scattering in the model, i.e. scattering for an open set of momenta on the scattering manifold. (This is for example the case in the $P(\phi)_2$-models.)

The second peculiarity in two dimensions is the lack of clustering properties of the S-matrix. So the methods of sect. 5 cannot be used for the analysis of non-local charges of genus 2. But there is a more direct approach to this problem.

If, for example, Q is a constant of motion it follows from the analysis of sect. 3 that Q cannot change the asymptotic particle number. So the only non-trivial kernels of Q are of the form given in eqs. (3.10) and (3.12). It is obvious then that the $(M+1)$-fold commutator of Q with P^μ vanishes. Now, in contrast to the situation in four dimensions, there cannot appear any derivatives in the kernels of non-local charges which commute with P^μ. Using this fact, the general analysis of Q essentially boils down to the study of the special cases, where $M=0$ in eqs. (3.10) and (3.12).

Now, if one evaluates matrix-elements of such a charge Q between incoming and outgoing three-particle collision states, say, one obtains for the kernels of Q a functional equation of the form

$$B^{\text{out}}(p_1, p_2) + B^{\text{out}}(p_2, p_3) + B^{\text{out}}(p_3, p_1) = B^{\text{in}}(q_1, q_2) + B^{\text{in}}(q_2, q_3) + B^{\text{in}}(q_3, q_1)$$

for all momenta on the three-particle scattering manifold, for which non-trivial scattering occurs. It follows that $B^{\text{out}} = B^{\text{in}}$ which implies that Q commutes with the S-matrix. If there is only elastic two-particle scattering in the model nothing more can be said. But if there occur also three- (or many-) body collisions, then one can show that the only solutions of the above equation are polynomials of second degree in energy and momentum (cf. eq. (5.7)). With this information at hand it is easy to show that Q has the form given in proposition 5.4.

In models with more than one type of particles the S-matrix elements do not simply factor out of the constraint-equations for the kernels of Q, and consequently the evaluation of these equations is more complicated. As the example of the σ-model shows, one can in general no longer conclude that non-local charges of genus 2 must commute with the S-matrix, in contrast to the restricted class of models considered above. But, it is clear from the discussion of these special models that the existence of multi-particle scattering and particle production imposes strong constraints on the kernels of non-local charges. Infering from these partial results it seems that interesting examples of non-local charges can only exist in models where such collision-processes do not occur.

We would like to thank Rodolf Haag for his kind support of our work and Klaus Fredenhagen as well as Eyvind H. Wichmann for important remarks. We also gratefully acknowledge the hospitality and support extended to us at the University of California, Berkeley (D.B.), the University of Hamburg (J.L. and Sz.R.) and the

University of Wrocław (D.B. and Sz.R.). Without the help of these institutions this collaboration would not have been possible.

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