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# TUMBLING AND COMPLEMENTARITY IN A CHIRAL GAUGE THEORY

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We consider in detail a chiral SU($N$) gauge theory which undergoes multiple tumbling. An extension of the notion of complementarity is used which allows us to deduce the set of massless fermions, in the confining phase of the theory, which we needed for anomaly matching. The likelihood of this confining phase ever being realized in practice is discussed.

An important and very interesting question has emerged in composite models of quarks and leptons, namely, do there exist confining theories with massless spin-$\frac{1}{2}$ fermions in the bound state spectrum? In models which possess a chiral symmetry at the preon level such massless composite fermions can arise whenever the chiral symmetry remains unbroken at the bound state level. ’t Hooft [1] has given a precise necessary condition for the preservation of chiral symmetry in the binding: there must be a matching of the value of the chiral anomaly, computed at the preon level, with the value obtained at the bound state level by computing this anomaly in terms of all the spin-$\frac{1}{2}$ massless states. To be more precise consider a set of conserved chiral currents

$$J^a_\mu = \sum_{i,j} \bar\psi_i \gamma_\mu (1-\gamma_5) t^a_{ij} \psi_j , \qquad (1)$$

where $i$, $j$ run over the color and flavor indices of the fermions. The ’t Hooft matching condition can then be written as

$$\left(\mathrm{Tr}\, t^a\{t^b, t^c\}\right)_{\text{preons}} = \left(\mathrm{Tr}\, t^a\{t^b, t^c\}\right)_{\text{massless composite fermions}} . \qquad (2)$$

This condition is only a necessary condition for chirality to be preserved in the binding. Actually massless Goldstone bosons can also reproduce the anomalies at the composite level, thus signalling spontaneous breakdown of the chiral symmetry.

This latter possibility is realized in QCD and probably in most vector gauge theories (i.e. theories with vector couplings of gauge fields to fermions) as can be argued by considering mass inequalities between bound states [2] or by using large-$N$ considerations [3], for example. The behavior of chiral gauge theories, i.e. gauge theories with fermions transforming in such a way under the gauge group that no mass terms can be formed at the preon level, should, on the other hand, be quite different. For some special chiral gauge theories, formally allowing a large-$N$ limit, it actually can be shown that massless spin-$\frac{1}{2}$ bound states must be present in the spectrum at $N=\infty$ [4], provided some mild assumptions about the behavior of the large-$N$ limit hold. Chiral gauge theories therefore hold some promise of being candidates for bound state models of quarks and leptons, and therefore deserve continuing investigation.

In general the algebraic constraints imposed by eq. (2) are difficult to satisfy: one has to solve Diophantine equations which become particularly intricate when the subgroup H that the flavor group G at the preon level is broken to has to be determined at the same time. However, there exists a way to systematically generate solutions in a large number of chiral gauge theories [5]. It is based on complementarity, i.e. the hypothesis that a gauge theory spontaneously broken by a scalar condensate in the fundamental representation of the gauge group is in a phase which analytically continues into the confining phase of the theory [6]. In particular, one expects the spectrum of massless states to be identical in the Higgs and the confinement phase of the theory [5].

In this paper we shall investigate an interesting, but rather intricate, chiral SU($N$) gauge theory with three species of left-handed fermions:

$$S_{\{i,j\}}, \qquad A^{[i,j]}, \qquad F^{i\alpha}, \qquad i,j=1,\dots,N; \qquad \alpha=1,\dots,8. \tag{3}$$

These fields transform, respectively, under the conjugate of the second-rank symmetric, the second-rank antisymmetric and the fundamental representation of SU($N$). The number 8 of chiral fermions $F^{i\alpha}$ is chosen so that the model is anomaly-free in the gauge sector. It is an easy matter to check that this theory is asymptotically free. This model has been investigated previously by Eichten and Preskill [7], in their general analysis of chiral gauge theories. Some features of this model, notably the fact that it admits a repetition of states (families?) at the bound state level, have been reported by Preskill [8]. Two of us, in collaboration with Eichten and Preskill [4] have also studied some aspects of the model for large $N$. Here we would like to examine the model in a tumbling version of complementarity, which will be more precisely defined below.

At the classical level the flavor symmetry of the model is

$$\mathrm{G_{cl}} = \mathrm{SU}(8)\times\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1), \tag{4}$$

where the U(1) generators can be taken as the fermion number operators $n_{\mathrm{A}}$, $n_{\mathrm{S}}$ and $n_{\mathrm{F}}$. Each of these U(1) symmetries has however a gauge anomaly, so that at the quantum level only two overall U(1) symmetries survive:

$$G_{qu} = SU(8) \times U_1(1) \times U_2(1)\,. \tag{5}$$

The linear combinations of the fermion number operators which are anomaly-free are easily determined and a convenient choice for their generators is:

$$\begin{aligned} Q_1 &= 2n_A - 2n_S + n_F\,, \\ Q_2 &= (N-2)n_S - (N+2)n_A\,. \end{aligned} \tag{6}$$

The generator $Q_1$ has a direct physical meaning in the confining phase of the theory. Obviously, from its definition, $Q_1$ counts the difference between the number of upper and lower SU($N$) indices. Hence it counts $N$ times the number of SU($N$) $\varepsilon$-tensors in any singlet state.

We want to examine this model, to begin with, in the Higgs phase where condensate formation forces the breakdown of the gauge group. For $N$ finite, the most attractive channel (MAC) [9], is one where the fermions S and F condense yielding an effective Higgs field in the fundamental representation of SU($N$):

$$\langle S_{ij} F^{j\alpha} \rangle = \langle \Phi_i^\alpha \rangle \neq 0\,. \tag{7}$$

Because $\alpha = 1, \ldots, 8$ only, the condensate (7) can break in general the gauge symmetry only partially down, to SU($N-8$). Depending on the explicit form of (7) the global symmetry (5) will suffer some breakdown. If we demand that the end result of the breakdown yield as large a global symmetry as possible, then it follows that the condensate (7) must take the form

$$\langle \Phi_i^\alpha \rangle = \Lambda \delta_i^{\alpha+N-8}\,. \tag{8}$$

For non-vanishing $\Lambda$, the condensate (8) forces a breakdown of the gauge × global symmetry of the model: $[SU(N)]_{gauge} \times [SU(8) \times U_1(1) \times U_2(1)]_{global}$, to

$$[SU(N-8)]_{gauge} \times [SU(8) \times U_1'(1) \times U_2'(1)]_{global}\,, \tag{9}$$

that is, the global symmetry is as large as that before the breakdown. Only the gauge symmetry has been reduced to SU($N-8$). The new global symmetry SU(8) is easily seen to be the diagonal part of the old global SU(8) and the SU(8) in $SU(N)_{gauge}$ encompassing the last 8 indices. The new U(1)'s are linear combinations of $U_1(1)$ and $U_2(1)$ and the SU($N$) generator*,

$$Q_N = \begin{pmatrix} I_{N-8} & 0 \\ 0 & \frac{1}{8}(8-N)I_8 \end{pmatrix}. \tag{9}$$

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* Here $I_k$ is the $k$-dimensional unit matrix.

It is clear that (8) leaves both

$$Q_1' = \frac{N-8}{N}\left[Q_1 + \frac{8}{N-8}Q_N\right], \tag{10a}$$

$$Q_2' = \frac{1}{N}\left[12(N-4)Q_1 + NQ_2 - 8(N-6)Q_N\right] \tag{10b}$$

invariant. The complicated combination chosen for $Q_2$, will be explained below.

After the breakdown (8) the theory will contain a number of massless states. These can be easily identified, following the method of Dimopoulos, Raby and Susskind [5]. Namely, one looks at which states in the theory cannot acquire mass from effective $\mathrm{SU}(N) \times \mathrm{G_{qu}}$-invariant four-fermion interactions,

$$\mathcal{L}_{\mathrm{eff}} = \frac{1}{\Lambda^2} F^+_{j\alpha} S^{+jk} S_{kl} F^{l\alpha}, \tag{11}$$

once the condensate (8) forms. It is clear that among the massless fermions one will have all the $A^{ij}$ states, as well as the $S_{ij}$ states for $i, j = 1, \dots, N-8$. Furthermore, it is easy to see that the antisymmetric combination of $F^{j\alpha}$ with $j = N-7, \dots, N$ also acquires no mass. Table 1 summarizes the massless fermions in the theory following the breakdown (8), classified according to their $\mathrm{SU}(N-8)_{\mathrm{gauge}} \times [\mathrm{SU}(8) \times \mathrm{U}_1'(1) \times \mathrm{U}_2'(1)]_{\mathrm{global}}$ transformation properties. As can be seen from the table, there are two gauge singlet states which transform according to the second-rank antisymmetric representation of SU(8) and three states which under $\mathrm{SU}(N-8)_{\mathrm{gauge}} \times [\mathrm{SU}(8) \times \mathrm{U}_1'(1) \times \mathrm{U}_2'(1)]_{\mathrm{global}}$ have *precisely* the same transformation properties that $S_{ij}$, $A^{ij}$ and $F^{i\alpha}$ had under $\mathrm{SU}(N)_{\mathrm{gauge}} \times [\mathrm{SU}(8) \times \mathrm{U}_1(1) \times \mathrm{U}_2(1)]_{\mathrm{global}}$, except that $N \to N-8$. This was the reason for choosing the somewhat complicated form of $Q_2$ in eq. (10b). It guarantees that the eigenvalues of the massless fermions, with gauge quantum numbers, after the breakdown (8) correspond precisely to those of the massless fermions before the breakdown.

TABLE 1
Massless fermions after the breakdown $[\mathrm{SU}(N)] \times [\mathrm{SU}(8) \times \mathrm{U}_1(1) \times \mathrm{U}_2(1)]$
$\to [\mathrm{SU}(N-8)] \times [\mathrm{SU}(8) \times \mathrm{U}_1'(1) \times \mathrm{U}_2'(1)]$

| States | SU($N-8$) | SU(8) | $\mathrm{U}_1'(1)$ | $\mathrm{U}_2'(1)$ |
|---|---|---|---|---|
| $\sqrt{\tfrac{1}{2}}\,(F^{j,\alpha} - F^{\alpha+N-8,\,j-N+8})\Big\}$ $j = N-7, \dots, N$ | 1 | $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | 0 | $N-2$ |
| $A^{j,\alpha+N-8}$ $\Big\}$ $\alpha = 1, \dots, 8$ | 1 | $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | 0 | $N-6$ |
| $A^{ij}$ $i = 1, \dots, N-8;\ j = N-7, \dots, N$ | $\square$ | $\square$ | 1 | 0 |
| $A^{ij}$ $i, j = 1, \dots, N-8$ | $\square$ | 1 | 2 | $-[(N-8)+2]$ |
| $S_{ij}$ $i, j = 1, \dots, N-8$ | $\overline{\square\square}$ | 1 | $-2$ | $+[(N-8)-2]$ |

TABLE 2
Massless fermions after the breakdown $[\mathrm{SU}(8n+k)]\times[\mathrm{SU}(8)\times\mathrm{U}_1(1)\times\mathrm{U}_2(1)]$
$\rightarrow[\mathrm{SU}(k)]\times[\mathrm{SU}(8)\times\mathrm{U}_1(1)^{(n)}\times\mathrm{U}_2(1)^{(n)}]$

| States | SU($k$) | SU(8) | $\mathrm{U}_1(1)^{(n)}$ | $\mathrm{U}_2(1)^{(n)}$ |
|---|---|---|---|---|
| gauge singlets | 1 | □ | 0 | $N-2-4s$<br>$(s=0,1,\ldots,2n-1)$ |
| fundamental $\bar{\mathrm{F}}$ | □ | □ | 1 | 0 |
| antisymmetric $\bar{\mathrm{A}}$ | □ | 1 | 2 | $-(k+2)$ |
| symmetric $\bar{\mathrm{S}}$ | $\overline{□□}$ | 1 | $-2$ | $+(k-2)$ |

The Higgs phase of the model which we have just discussed arose from a condensate $\langle\Phi_i^\alpha\rangle$ which transforms according to the fundamental representation of the gauge group. Thus an application of complementarity would have been warranted. Alas, the situation here is more complicated than that discussed by Dimopoulos, Raby and Susskind [5]. In the examples discussed in ref. [5], after the spontaneous breakdown, the massless fermions left in the theory were either gauge singlets or could be paired under the gauge group, thereby breaking the global symmetry further. In our example, however, the fermions which are non-singlet under SU($N-8$) are chiral. Dimopoulos, Raby and Susskind [5] make use of complementarity in the following way. They argue that the fermionic bound states of zero mass, found in the confining phase of the theory, should agree precisely with the gauge singlet states found in the Higgs phase. Indeed, to support their contention, they show that, in the examples they consider, precisely the set of these massless bound states suffices for anomaly matching. In our case, however, we have additional chiral states which in the Higgs phase have non-trivial gauge quantum numbers. How are we to apply complementarity?

The solution we have found to this conundrum is the following. The spectrum of the SU($N-8$) gauge theory is precisely the same as that of the SU($N$) gauge theory. Hence it is logical that also the SU($N-8$) gauge group will be broken by a condensate analogous to (8). In this way the gauge theory, in the Higgs phase, goes through a tumbling sequence [9], from SU($N$), to SU($N-8$), to SU($N-16$), etc. To be precise, let us write $N=8n+k$, $k=0,\ldots,7$. Then after $n$ steps the remaining symmetry is $[\mathrm{SU}(k)]_{\text{gauge}}\times[\mathrm{SU}(8)\times\mathrm{U}_1(1)^{(n)}\times\mathrm{U}_2(1)^{(n)}]^{\star}$. Similar considerations to those that led us to table 1 yield, for the fermionic massless states at this stage, those indicated in table 2. The $(\mathrm{SU}(k))_{\text{gauge}}$ group finally is totally broken down by the condensate

$$\langle\bar{S}_{ij}\bar{F}^{j\alpha}\rangle=\Lambda'\delta_i^\alpha. \tag{12}$$

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[★] The cases $k=0,1$ are slightly more special. They are discussed in some detail in the appendix.

TABLE 3
Massless fermions after tumbling and their transformation properties under $G_{\text{final}}$

| $SU(k)$ | $SU(8-k)$ | $\bar{U}_1(1)$ | $\bar{U}_2(1)$ | Number of states |
|---|---|---|---|---|
| $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | 1 | $2(8-k)$ | $N-2-4s$ | $s=0,1,\dots,2n+1$ |
| $\square$ | $\square$ | $8-2k$ | $N-2-4s$ | $s=0,1,\dots,2n$ |
| 1 | $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | $-2k$ | $N-2-4s$ | $s=0,1,\dots,2n-1$ |

This condensate also breaks down the global group to

$$G_{\text{final}} = SU(k)\times SU(8-k)\times \bar{U}_1(1)\times \bar{U}_2(1)\,. \tag{13}$$

Here the two remaining conserved U(1)'s are linear combinations of $U_1(1)^{(n)}$, $U_2(1)^{(n)}$ and the SU(8) generator

$$Q_8 = \begin{pmatrix} (k-8)I_k & 0 \\ 0 & kI_{8-k} \end{pmatrix}. \tag{14}$$

A simple calculation shows that the generators $\bar{Q}_1$ and $\bar{Q}_2$ are given by

$$\begin{aligned} \bar{Q}_1 &= (8-k)Q_1^{(n)} - Q_8\,, \\ \bar{Q}_2 &= (k-2)Q_1^{(n)} + Q_2^{(n)}\,. \end{aligned} \tag{15}$$

The spectrum of massless fermions resulting at the end of the tumbling sequence is given in table 3.

At each stage, the successive breakdowns are caused by condensates which are in the fundamental representation of the surviving gauge group. Hence one may hope that there should be *no* phase boundary between the Higgs phase and the confining stage. If this is so, then we may apply complementarity to the final spectrum of fermions since these states, although chiral, carry no gauge quantum numbers. Hence our considerations suggest that a set of massless fermionic bound states with the quantum numbers under $G_{\text{final}}$ shown in table 3 should match the $G_{\text{final}}$ anomalies at the preon level. We remark that this *tumbling complementarity*, although intuitively appealing, is far from obvious if one thinks only of the overall breakdown. One has a theory with an $[SU(8n+k)]_{\text{gauge}}\times[SU(8)\times U_1(1)\times U_2(1)]_{\text{global}}$ symmetry, broken down to a pure $[SU(k)\times SU(8-k)\times \bar{U}_1(1)\times \bar{U}_2(1)]_{\text{global}}$ symmetry. Such a breakdown necessitates condensates which are both in the fundamental *and* the adjoint of the gauge group. From this point of view, there is no reason why the set of fermions in table 3, viewed as bound states of the confining phase, should match anomalies.

To show that tumbling complementarity really works, we must first identify the generators of the global symmetry group $G_{\text{final}}$ in terms of the generators written at the preon level. This is trivial for the non-abelian $SU(k) \times SU(8-k)$ symmetries, but is far from obvious for the $\overline{U}_1(1)$ and $\overline{U}_2(1)$ symmetries. What happened in the tumbling Higgs phase must be reinterpreted entirely in the confining phase, which is difficult. Fortunately, one can bypass these complications by the following observation. The U(1) generators $\overline{Q}_1$ and $\overline{Q}_2$ in the confining phase must be a linear combination of $Q_1$ and $Q_2$ given in eq. (6) and of the SU(8) generator $Q_8$:

$$\overline{Q}_1 = \alpha_{11}Q_1 + \alpha_{12}Q_2 + \alpha_{18}Q_8,$$

$$\overline{Q}_2 = \alpha_{21}Q_1 + \alpha_{22}Q_2 + \alpha_{28}Q_8. \tag{16}$$

The six coefficients $\alpha_{ij}$ in eqs. (16) can be determined by requiring that the chiral anomalies $\overline{Q}_1 SU(k)^2$, $\overline{Q}_1 SU(8-k)^2$, $\overline{Q}_2 SU(k)^2$, $\overline{Q}_2 SU(8-k)^2$ as well as the gravitational anomalies $\mathrm{Tr}\,\overline{Q}_1$, $\mathrm{Tr}\,\overline{Q}_2$ [10] be matched at the preon and bound state levels. These anomalies are linear in the coefficients $\alpha_{ij}$ and the solution of the six linear equations is immediate.

One finds, using table 3 and doing a little algebra, at the bound state level:

$$\mathrm{Tr}\,\overline{Q}_1 SU(k)^2 = \tfrac{1}{2}(8-k)(N+k), \tag{17a}$$

$$\mathrm{Tr}\,\overline{Q}_1 SU(8-k)^2 = \tfrac{1}{2}k(8-k-N), \tag{17b}$$

$$\mathrm{Tr}\,\overline{Q}_1 = 6k(8-k), \tag{17c}$$

$$\mathrm{Tr}\,\overline{Q}_2 SU(k)^2 = \tfrac{3}{8}N^2 - \tfrac{1}{8}k(8-k), \tag{18a}$$

$$\mathrm{Tr}\,\overline{Q}_2 SU(8-k)^2 = \tfrac{3}{8}N^2 - \tfrac{1}{8}k(8-k), \tag{18b}$$

$$\mathrm{Tr}\,\overline{Q}_2 = \tfrac{7}{2}N^2 - \tfrac{3}{2}k(8-k). \tag{18c}$$

On the other hand, at the preon level one has ($i = 1,2$)

$$\mathrm{Tr}\,\overline{Q}_i SU(k)^2 = \tfrac{1}{2}N\left[\alpha_{i1} + \alpha_{i8}(k-8)\right], \tag{19a}$$

$$\mathrm{Tr}\,\overline{Q}_i SU(8-k)^2 = \tfrac{1}{2}N\left[\alpha_{i1} + \alpha_{i8}k\right], \tag{19b}$$

$$\mathrm{Tr}\,\overline{Q}_i = N\left[6\alpha_{i1} - N\alpha_{i2}\right]. \tag{19c}$$

TABLE 4
Transformation of preons under $G_{\text{final}}$

| Preon | SU($N$) | SU($k$) | SU($8-k$) | $\overline{U}_1(1)$ | $\overline{U}_2(1)$ |
|---|---|---|---|---|---|
| F | $\square$ | $\square$ | 1 | $\frac{(8-k)(k+N)}{N}$ | $\frac{3}{4}N-\frac{k(8-k)}{4N}$ |
| F | $\square$ | 1 | $\square$ | $\frac{k(8-k-N)}{N}$ | $\frac{3}{4}N-\frac{k(8-k)}{4N}$ |
| A | $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | 1 | 1 | $\frac{2k(8-k)}{N}$ | $\frac{1}{2}N-2-\frac{k(8-k)}{2N}$ |
| S | $\square\square$ | 1 | 1 | $-\frac{2k(8-k)}{N}$ | $-\frac{1}{2}N-2+\frac{k(8-k)}{2N}$ |

Thus the charges $\overline{Q}_1$ and $\overline{Q}_2$ are seen to be

$$\overline{Q}_1=\frac{k(8-k)}{N}Q_1-Q_8,$$

$$\overline{Q}_2=\left[\tfrac{3}{4}N-\frac{k(8-k)}{4N}\right]Q_1+Q_2. \tag{20}$$

In table 4, we summarize the preon assignments under $G_{\text{final}}$, following from the above identification.

Having fixed all the charges and multiplets both at the preon and bound state level, one still has 6 non-trivial anomaly equations to check, namely SU$(k)^3$, SU$(8-k)^3$, $\overline{Q}_1^3$, $\overline{Q}_1^2\overline{Q}_2$, $\overline{Q}_1\overline{Q}_2^2$ and $\overline{Q}_2^3$. All these conditions are in fact fulfilled. We list the results of the calculation for the preon level, after some simplification, and as an illustration show how the $\overline{Q}_1\overline{Q}_2^2$ anomaly is matched. One finds

$$\operatorname{Tr}\mathrm{SU}(k)^3=N\operatorname{Tr}\square_k^3, \tag{21a}$$

$$\operatorname{Tr}\mathrm{SU}(8-k)^3=N\operatorname{Tr}\square_{8-k}^3, \tag{21b}$$

$$\operatorname{Tr}\overline{Q}_1^3=8k(8-k)[3k(8-k)+N(8-2k)], \tag{21c}$$

$$\operatorname{Tr}\overline{Q}_1^2\overline{Q}_2=6k(8-k)[N^2-k(8-k)], \tag{21d}$$

$$\operatorname{Tr}\overline{Q}_1\overline{Q}_2^2=2k(8-k)[k(8-k)-4], \tag{21e}$$

$$\operatorname{Tr}\overline{Q}_2^3=\tfrac{7}{2}N^2(N^2-8)-\tfrac{3}{4}k(8-k)[k(8-k)-8]. \tag{21f}$$

The anomaly $\overline{Q}_1\overline{Q}_2^2$ at the bound state level is given by

$$\begin{aligned}\operatorname{Tr}\overline{Q}_1\overline{Q}_2^2 = &= \tfrac{1}{2}k(k-1)2(8-k)\sum_{s=0}^{2n+1}(N-2-4s)^2 \\ &+k(8-k)(8-2k)\sum_{s=0}^{2n}(N-2-4s)^2 \\ &+\tfrac{1}{2}(8-k)(7-k)(-2k)\sum_{s=0}^{2n-1}(N-2-4s)^2 .\end{aligned} \tag{22}$$

It is easy to check that the terms proportional to $\Sigma_{s=0}^{2n-1}(N-2-4s)^2$ have in fact zero coefficient. Hence

$$\begin{aligned}\operatorname{Tr}\overline{Q}_1\overline{Q}_2^2 = {}& k(8-k)(8-2k)(N-2-8n)^2 \\ &+k(k-1)(8-k)\left[(N-2-8n)^2+(N-6-8n)^2\right].\end{aligned} \tag{23}$$

Since $N-8n=k$, a little algebra reduces the above equation to (21e). In a similar way one can check that all the other anomalies in eq. (21) are matched at the bound state level. Hence the set of fermions inferred from the tumbling complementarity are precisely the set needed to match the $\mathrm{G}_{\text{final}}$ anomalies.

Several remarks are in order. The massless bound states given in table 3, when constructed in the confining phase, will contain up to $\mathrm{O}(N)$ preons. This is easily seen from the following example. The combination $(FSF)$ carries $\overline{Q}_2$ charge equal to $N-2$ and $\overline{Q}_1$ charge equal to $2(8-k)$, $8-2k$, or $-2k$, depending on the flavor index of F. On the other hand the combination $(AS)$ carries $\overline{Q}_2$ charge equal to $-4$ and no $\overline{Q}_1$ charge. Hence the simplest representation [7,8] of the massless bound states in table 3 is given by the SU($N$) color singlet combination of $(FSF)(AS)^s$, $s=0,1,\ldots 2n+1$. This is a surprising result, because it leads to self-interactions of the bound states growing with $N$ also [11]. We suspect, therefore, that above a certain $N_{\text{critical}}$ the model has really no confining phase at all. Indeed [4] for large enough $N$ the adjoint condensate $\langle S_{ij}A^{ij}\rangle$ is as attractive as the fundamental condensate (7) and the theory may tumble down in a way totally different from the one considered here.

A second remark in the same vein concerns the breakdown of the original U(1) symmetry $Q_1$ connected with the number of $\varepsilon$-tensors in any singlet state of SU($N$). Because in $\mathrm{G}_{\text{final}}$ $Q_1$ is not a good symmetry, the breakdown $\mathrm{SU}(8)\times\mathrm{U}_1(1)\times\mathrm{U}_2(1)\to\mathrm{SU}(k)\times\mathrm{SU}(8-k)\times\overline{\mathrm{U}}_1(1)\times\overline{\mathrm{U}}_2(1)$ must be caused by condensates which carry $\varepsilon$-number, an example being

$$\left\langle (FS)^{2k}\left[(AS)^{k+6}A^{-12}F^{8}\right]^{n}\right\rangle . \tag{24}$$

These condensates will again contain O($N$) fields and it is difficult to see how they would dynamically form.

Notwithstanding these remarks it is worth reemphasizing that tumbling complementarity allows the deduction of the states which match anomalies in a very neat way. Without the guide obtained by these considerations, the deduction of the set of states to use to match the chiral anomalies is much more open and one has to try to use other dynamical guides [7]. From this point of view, the considerations presented in this model may prove useful elsewhere.

## Appendix

In this appendix we consider the case $N \bmod 8 + k = 0, 1$ which requires special care because in the last step of the tumbling sequence one is formally left with the "gauge group" SU(0) or SU(1). Even though an "analytic continuation" of the results derived for $k = 2, \ldots, 7$ to $k = 0, 1$ leads to a consistent anomaly matching, a more careful treatment is worthwhile because it leads to a larger surviving symmetry and correspondingly to a larger number of massless composite fermions.

In order to see this let us write $N = 8n' + k'$ with $k' = 8, 9$. After $n'$ tumbling steps we are left with exactly the massless fermions of table 2 (with $(n, k)$ replaced by $(n', k')$). For the last step of the tumbling sequence we again assume the condensate

$$\langle \bar{S}_{ij} \bar{F}^{j\alpha} \rangle = \Lambda' \delta_i^{\alpha}, \qquad i, \alpha = 1, \ldots, 8 \tag{A.1}$$

to form, which breaks $\mathrm{SU}(k')_{\mathrm{local}} \times [\mathrm{SU}(8) \times \mathrm{U}(1) \times \mathrm{U}(1)]_{\mathrm{global}}$ to $\mathrm{G}_{\mathrm{final}} = [\mathrm{SU}(8) \times \mathrm{U}(1)]_{\mathrm{global}}$ for $k' = 8$ and to $\mathrm{G}_{\mathrm{final}} = [\mathrm{SU}(8) \times \mathrm{U}(1) \times \mathrm{U}(1)]_{\mathrm{global}}$ for $k' = 9$. The new SU(8) generators are a sum of the old SU(8) and the old SU($k'$) generators. Out of the original U(1)'s only one is respected by the condensate (A.1). Its generator can be chosen as

$$\bar{Q}_2 = (k' - 2) Q_1^{(n')} + Q_2^{(n')} \tag{A.2}$$

in terms of the charge assignment of table 2. For $k' = 8$ this is the only U(1) generator commuting with the condensate (A.1), while for $k' = 9$ the remaining diagonal SU(9) generator

$$Q_9 = \begin{pmatrix} I_8 & 0 \\ 0 & -8 I_1 \end{pmatrix} \tag{A.3}$$

combines with $Q_1^{(n')}$ to form the generator

$$\bar{Q}_1 = \tfrac{1}{9}\left( Q_1^{(n')} - Q_9 \right) \tag{A.4}$$

of another conserved U(1).

TABLE 5
Massless fermions after tumbling and their transformation properties under $G_{\text{final}}$ for $k=0$ and $k=1$

| | SU(8) | $\bar{\text{U}}_1(1)$ | $\bar{\text{U}}_2(1)$ | Number of states |
|---|---|---|---|---|
| $k=0$ | $\begin{array}{c}\square\\[-4pt]\square\end{array}$ | | $N-2-4s$ | $s=0,1,\dots,2n-1$ |
| $k=1$ | $\begin{array}{c}\square\\[-4pt]\square\end{array}$ | 0 | $N-2-4s$ | $s=0,1,\dots,2n-1$ |
| | $\square$ | 1 | 3 | 1 |
| | 1 | $-2$ | $-7$ | 1 |

When the condensate (A.1) forms, most of the SU($k'$) non-singlet fermions obtain masses due to residual four-fermion interactions analogous to eq. (11). For $k'=8$, only the $\bar{A}^{ij}$ and the antisymmetric combination $\sqrt{\frac{1}{2}}\,(\bar{F}^{i\alpha}-\bar{F}^{\alpha i})$ $(i,j,\alpha=1,\dots,8)$ remain massless while the case $k'=9$ does not allow mass terms for $\bar{A}^{i9}$ $(i=1,\dots,8)$ and $\bar{S}_{99}$ either. The resulting spectrum of massless fermions and their charge assignment is given in table 5.

When turning to the confinement phase, tumbling complementarity works exactly like in the case of general $k$. Matching the "linear" anomalies $\operatorname{Tr} Q\,\text{SU}(8)^2$ and $\operatorname{Tr} Q$ via the massless fermion multiplets of table 5, now being interpreted as massless bound states, identifies the charge assignment of the preons. We find $(k=0,1)$

$$\bar{Q}_1=\frac{1}{N}Q_1=\frac{1}{N}(2n_{\text{A}}+n_{\text{F}}-2n_{\text{S}}),$$

$$\bar{Q}_2=\frac{3}{4}\frac{N^2+3k}{N}Q_1+Q_2. \tag{A.5}$$

The remaining anomalies, namely $\text{SU}(8)^3$, $\bar{Q}_2^3$ and, for $k=1, \bar{Q}_1^3, \bar{Q}_1^2\bar{Q}_2, \bar{Q}_1\bar{Q}_2^2$, again constitute a non-trivial consistency check on tumbling complementarity. As expected, they all match.

While the $k=0$ solution can be considered a simple extension of the $k=2,\dots,7$ cases, the $k=1$ spectrum is qualitatively different:

(i) the original global symmetry remains unbroken,
(ii) it contains massless states of non-zero SU($N$) ε-number,
(iii) fundamental and singlet representations of SU(8) occur.

It is a simple exercise to find color singlet operators which create the bound states of table 5. For the antisymmetric representation of SU(8) they are again the $(FSF)(AS)^s$ operators encountered previously. The $\bar{Q}_1=-2$ state can be represented by $S[(A^+F)^3(SF)^5A^{+4}]^n$ and (for $n$ even) the $\bar{Q}_1=+1$ state is created by operators like $F[(S^+F^+)^3(AF^+)^5A^4]^{n/2}$. Even though these operators possess the correct quantum numbers, the dynamical question whether they actually do create massless bound state fermions out of the vacuum remains open.

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