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Local Properties of Equilibrium States and the Particle Spectrum in Quantum Field Theory

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Abstract. It is shown that in a quantum field theory describing free, scalar particles with masses m_i, $i \in \mathbb{N}$ there exist locally normal equilibrium states with finite energy density for all temperatures $\beta > 0$ if and only if $\Sigma_{i=1}^{\infty} e^{-\beta m_i} < \infty$. This result lends support to the conjecture that the nuclearity criterion proposed by Buchholz and Wichmann is sensitive to the thermodynamical properties of field-theoretic models.

1. The general framework of local quantum field theory includes many models which do not have a reasonable particle interpretation. Therefore, it has been proposed by Haag and Swieca to amend the fundamental postulates by another condition which is based on the idea that the number of states occupying a finite volume of 'phase space' should be finite [2].

Following Licht [3], one may identify the states which are, at a given time, strictly localized within a bounded region $\mathcal{O}$ of Minkowski space with the set of vectors

$$\{U\Omega \mid U \in \mathfrak{A}(\mathcal{O}), U^*U = 1\} \tag{1}$$

in the physical Hilbert space $\mathcal{H}$. Here $\mathfrak{A}(\mathcal{O})$ denotes the algebra of fields (respectively, observables) which are localized in $\mathcal{O}$, and Ω is the vacuum vector. By cutting off the total energy of these vectors, e.g., with the help of the exponential function $e^{-\beta H}$, $\beta > 0$ of the Hamiltonian, one then obtains a set of states

$$\mathcal{N}(\mathcal{O}, \beta) := \{e^{-\beta H} U\Omega \mid U \in \mathfrak{A}(\mathcal{O}), U^*U = 1\} \tag{2}$$

which, roughly speaking, occupies a finite volume in configuration as well as momentum space. Haag and Swieca argue that the sets $\mathcal{N}(\mathcal{O}, \beta)$, although not finite-dimensional, should be relatively compact in the norm topology in physically acceptable models.

In a recent article [1], it has been pointed out that under quite general conditions the sets $\mathcal{N}(\mathcal{O}, \beta)$ should even be nuclear, i.e., $\mathcal{N}(\mathcal{O}, \beta)$ should be contained in the image of the unit ball $\mathcal{H}_1$ in $\mathcal{H}$ under the action of some trace class operator T,

$$\mathcal{N}(\mathcal{O}, \beta) \subset T \cdot \mathcal{H}_1 . \tag{3}$$

(For a slightly more general definition of nuclearity due to Grothendieck cf. [1].) The argument is based on the following heuristic consideration: disregarding long-range correlations, one can compare the set of vectors (1) with the unit sphere $\mathscr{H}_{V,1}$ in the Hilbert space $\mathscr{H}_V$ of the theory for finite volume V. On the same heuristic basis one can compare $\mathscr{N}(\mathscr{O},\beta)$ with the set of vectors $e^{-\beta H_V}\cdot\mathscr{H}_{V,1}$ where H_V is the finite volume Hamiltonian. Now the crucial input is the assumption that the operators $e^{-\beta H_V}$, describing the Gibbs ensemble, have a finite trace for all $\beta > 0$. This should be true in most models of physical interest, one of the few exceptions being the Hagedorn model [4], for which a maximal temperature exists. It then follows that the sets $e^{-\beta H_V}\cdot\mathscr{H}_{V,1}$ are nuclear, so according to the above picture the same should be true for the sets $\mathscr{N}(\mathscr{O},\beta)$ in the infinite volume theory. A measure for the size of the sets $\mathscr{N}(\mathscr{O},\beta)$ is provided by the nuclear index $\|\mathscr{N}(\mathscr{O},\beta)\|_1$ given by [1]

$$\|\mathscr{N}(\mathscr{O},\beta)\|_1 = \inf_T \operatorname{Tr}|T| . \tag{4}$$

Here the infimum is to be taken over all trace class operators T for which relation (3) holds. So the nuclear index is the analogue of the partition function

$$Z(V,\beta) = \operatorname{Tr} e^{-\beta H_V} . \tag{5}$$

The condition that the partition function (5) exists is equivalent to the requirement that the equilibrium (Gibbs) states in the finite volume theory are normal states on the algebra of all bounded operators on $\mathscr{H}_V$. Proceeding to the thermodynamic limit (if it exists) one may therefore expect that the equilibrium states ω_β of the infinite volume theory are *locally* normal, i.e., the restrictions of ω_β to the local algebras $\mathfrak{A}(\mathscr{O})$ should be represented by density matrices on the Hilbert space $\overline{\mathfrak{A}(\mathscr{O})\Omega} = \mathscr{H}$.

A significant test for the existence of the thermodynamic limit is based on the analysis of thermodynamical quantities, such as the pressure $(V\beta)^{-1}\ln Z(V,\beta)$, in the limit of large volume V. In view of the similarity of the nuclear index $\|\mathscr{N}(\mathscr{O},\beta)\|_1$ to the partition function $Z(V,\beta)$ it should be possible to express these quantities also in terms of $\|\mathscr{N}(\mathscr{O},\beta)\|_1$. From a fundamental point of view, this possibility appears to be very attractive. It would allow one to distinguish, within the general setting of quantum field theory, all models with a reasonable thermodynamic behaviour. This input could then be used to derive more specific properties of these models (cf. [1]).

It is the aim of the present note to substantiate this qualitative picture for the simple class of models of a countable number of free, scalar particles with masses m_i, $i\in\mathbb{N}$. In order to abbreviate the subsequent discussion we will restrict our attention to models with a mass gap, i.e., we assume that $m_i \geqslant m_0 > 0$ for all $i\in\mathbb{N}$. A more significant restriction on the mass spectrum derives from the assumption that the sets $\mathscr{N}(\mathscr{O},\beta)$, $\beta > 0$ are nuclear. It is easy to verify that in this case

$$\sum_{i=1}^{\infty} e^{-\beta m_i} < \infty \quad \text{for } \beta > 0 . \tag{6}$$

Note that the sum in (6) provides a lower bound for the partition function (5). That (5) is also a sufficient condition for the nuclearity of $\mathscr{N}(\mathscr{O},\beta)$ can be taken from the appendix in [1].

As to the thermodynamic properties of these models, we will show that there exist locally normal equilibrium states for all $\beta > 0$ if and only if condition (6) is satisfied. Moreover, we will see that the energy density of these states is finite. In contrast, the energy density of the equilibrium states is infinite for large temperatures whenever the mass spectrum does not comply with (6). These results are in perfect agreement with the above heuristic considerations. We therefore presume that the nuclearity criterion proposed in [1] serves its purposes also in general [5].

2. We begin with a brief description of the model and of our notation. It will be convenient to work with the CCR algebra generated by the 'fields' ϕ at time $t = 0$ and their canonical conjugate 'momenta' π; the respective testfunctions are elements of the complex Hilbert spaces

$$K_\phi = \bigoplus_{i=1}^{\infty} L^2(\mathbb{R}^3) \quad \text{and} \quad K_\pi = \bigoplus_{i=1}^{\infty} L^2(\mathbb{R}^3) . \tag{7}$$

To simplify the notation we also introduce the space $K = K_\phi \oplus K_\pi$ and set, for $F = \mathbf{f} \oplus \mathbf{g} \in K$,

$$A(F) = \phi(\mathbf{f}) + \pi(\mathbf{g}) . \tag{8}$$

The scalar product of $F, G \in K$ is denoted by (F, G) and the antilinear involution on K corresponding to complex conjugation of the functions F in configuration space by Γ. The adjoint of $A(F)$ is given by

$$A^*(F) = A(\Gamma F) \tag{9}$$

and the commutator by

$$[A^*(F), A(G)] = \left(F, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} G\right) \cdot 1 \tag{10}$$

in an obvious notation. The algebra $\mathfrak{P}$ of all polynomials in the fields $A(F)$, $F \in K$ is thus a self-dual CCR algebra [6].

The time translations $t \in \mathbb{R}$ act on $\mathfrak{P}$ by automorphisms α_t according to

$$\alpha_t(A(F)) = A\left(\begin{pmatrix} \cos \boldsymbol{\omega} t & \sin \boldsymbol{\omega} t \\ -\sin \boldsymbol{\omega} t & \cos \boldsymbol{\omega} t \end{pmatrix} \cdot F\right) . \tag{11}$$

Here $\boldsymbol{\omega}$ denotes the operator which is defined on a dense set of vectors $\mathbf{f} = \oplus_{i=1}^{\infty} f_i$ in K_ϕ and K_π, respectively, by

$$\boldsymbol{\omega} \cdot \mathbf{f} = \bigoplus_{i=1}^{\infty} \omega_i f_i . \tag{12}$$

The operators ω_i act on $L^2(\mathbb{R}^3)$ as multiplication operators in momentum space according to

$$(\widetilde{\omega_i f})(\mathbf{p}) = (\mathbf{p}^2 + m_i^2)^{1/2} \cdot \tilde{f}(\mathbf{p}) , \tag{13}$$

where $\tilde{f}$ denotes the Fourier transform of f.

The vacuum state ω_0 is completely fixed by the requirement that it is a ground state for the dynamics (11). It is a quasi-free state with the two-point function

$$S_0(F, G) = \omega_0(A^*(F)A(G)) = \tfrac{1}{2}\left(F, \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} G\right). \tag{14}$$

Similarly, the equilibrium states ω_β, $\beta > 0$ are fixed by the KMS condition [7]. Their two-point function is given by

$$S_\beta(F, G) = \omega_\beta(A^*(F)A(G)) = \tfrac{1}{2}\left(F, \begin{pmatrix} \coth \beta\omega/2 & i \\ -i & \coth \beta\omega/2 \end{pmatrix} G\right). \tag{15}$$

The local algebras $\mathfrak{P}(\mathcal{O})$, assigned to the bounded regions $\mathcal{O} \subset \mathbb{R}^3$, are the subalgebras of $\mathfrak{P}$ generated by the fields $A(F)$, where $F \in K(\mathcal{O}) := K_\phi(\mathcal{O}) \oplus K_\pi(\mathcal{O})$ and

$$K_\phi(\mathcal{O}) = \overline{\omega^{-1/2} \oplus_{i=1}^{\infty} \mathcal{D}(\mathcal{O})}, \qquad K_\pi(\mathcal{O}) = \overline{\omega^{1/2} \oplus_{i=1}^{\infty} \mathcal{D}(\mathcal{O})}. \tag{16}$$

Here $\mathcal{D}(\mathcal{O})$ is the set of all test functions with support in $\mathcal{O}$ and the bar denotes the closure of the respective spaces.

For the analysis of the local properties of ω_β we will make use of a criterion of Araki and Yamagami [6]. To this end we represent the restrictions of the positive forms $S_0(.,.)$ and $S_\beta(.,.)$ to $K(\mathcal{O})$ by positive bounded operators S_0 and S_β, respectively, which are given on K by

$$S_0 = \tfrac{1}{2}E \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} E \quad \text{and} \quad S_\beta = \tfrac{1}{2}E \begin{pmatrix} \coth \beta\omega/2 & i \\ -i & \coth \beta\omega/2 \end{pmatrix} E. \tag{17}$$

Here

$$E = \begin{pmatrix} E_\phi & 0 \\ 0 & E_\pi \end{pmatrix}$$

denotes the orthogonal projection in K onto the closed subspace $K(\mathcal{O})$, and E_ϕ, E_π denote the projections in K_ϕ and K_π, respectively, onto the subspaces $K_\phi(\mathcal{O})$ and $K_\pi(\mathcal{O})$. We now can state the

CRITERION [6]. *The state ω_β is locally normal (i.e., the GNS-representations of $\mathfrak{P}(\mathcal{O})$ induced by ω_β and ω_0, respectively, are quasi equivalent) if and only if the operator $S_\beta^{1/2} - S_0^{1/2}$ is in the Hilbert–Schmidt class.*

3. To begin with we will show that ω_β is *not* locally normal for a given $\beta > 0$ if the mass spectrum of the model is such that

$$\sum_{i=1}^{\infty} e^{-2\beta m_i} = \infty. \tag{18}$$

Because of the identity

$$S_\beta - S_0 = (S_\beta^{1/2} - S_0^{1/2})^2 + S_0^{1/2}\cdot(S_\beta^{1/2} - S_0^{1/2}) + (S_\beta^{1/2} - S_0^{1/2})\cdot S_0^{1/2} \qquad (19)$$

and the fact that the Hilbert–Schmidt operators form an ideal in the algebra of all bounded operators on K it is, according to the above criterion, sufficient to verify that $\Delta S = S_\beta - S_0$ is not in the Hilbert–Schmidt class. Now a crude lower bound on the Hilbert–Schmidt norm of ΔS is given by

$$\mathrm{Tr}_K(\Delta S)^2 \geqslant \sum_{i=1}^{\infty} \lambda^2(m_i), \qquad (20)$$

where

$$\lambda(m_i) = \sup_{g \in \omega_i^{1/2}\mathcal{D}(\mathcal{O})} \frac{\langle g, \frac{1}{2}(\coth(\beta/2)\omega_i - 1)g\rangle}{\langle g, g\rangle} \qquad (21)$$

and $\langle\, .\, ,\, .\,\rangle$ denotes the scalar product in $L^2(\mathbb{R}^3)$. Using the fact that

$$\tfrac{1}{2}(\coth(\beta/2)\omega_i - 1) = (e^{\beta\omega_i} - 1)^{-1} \geqslant e^{-\beta\omega_i} \geqslant e^{-\beta m_i}\, e^{-\beta|\mathbf{P}|}, \qquad (22)$$

where $\mathbf{P}$ is the momentum operator on $L^2(\mathbb{R}^3)$, it is easy to verify that $\lambda(m_i) \geqslant c\cdot e^{-\beta m_i}$ for some constant $c > 0$ depending only on $\mathcal{O}$ and β. From (20) and (18) it then follows that ΔS is not in the Hilbert–Schmidt class.

We mention as an aside that the representations of $\mathfrak{P}(\mathcal{O})$ induced by ω_β and ω_0, respectively, are even disjoint if (18) holds.

Next, we will prove that ω_β is locally normal for a given temperature $\beta > 0$ if

$$\sum_{i=1}^{\infty} e^{-(\beta/2)m_i} < \infty\,. \qquad (23)$$

For the proof it is sufficient to show that the operator $\Delta S = S_\beta - S_0$ has a finite trace, since then $S_\beta^{1/2} - S_0^{1/2}$ is in the Hilbert–Schmidt class (cf., for example, the appendix of [8]). Let χ be the operator which is defined on the elements $\mathbf{f} = \oplus_{i=1}^{\infty} f_i$ of K_ϕ and K_π, respectively, by

$$\chi\cdot\mathbf{f} = \bigoplus_{i=1}^{\infty} \chi\cdot f_i\,. \qquad (24)$$

The operator χ acts on $L^2(\mathbb{R}^3)$ by multiplication (in configuration space) with a test function $\chi(\mathbf{x})$ which has compact support and is equal to 1 on $\mathcal{O}$. According to the very definition of the projections E_ϕ and E_π, we have

$$E_\phi = \boldsymbol{\omega}^{-1/2}\boldsymbol{\chi}\boldsymbol{\omega}^{1/2}\cdot E_\phi\,, \qquad E_\pi = \boldsymbol{\omega}^{1/2}\boldsymbol{\chi}\boldsymbol{\omega}^{-1/2}\cdot E_\pi \qquad (25)$$

and, consequently,

$$\Delta S = E\begin{pmatrix} \boldsymbol{\omega}^{1/2}\boldsymbol{\chi}\boldsymbol{\omega}^{-1}(e^{\beta\boldsymbol{\omega}} - 1)^{-1}\boldsymbol{\chi}\boldsymbol{\omega}^{1/2} & 0 \\ 0 & \boldsymbol{\omega}^{-1/2}\boldsymbol{\chi}\boldsymbol{\omega}(e^{\beta\boldsymbol{\omega}} - 1)^{-1}\boldsymbol{\chi}\boldsymbol{\omega}^{-1/2} \end{pmatrix}E\,. \qquad (26)$$

So it suffices to show that the operators in the diagonal of this matrix have a finite trace in K_ϕ and K_π, respectively. Since

$$0 \leqslant \beta\omega_i (e^{\beta\omega_i} - 1)^{-1} \leqslant 4\, e^{-(\beta/\sqrt{2})\omega_i} \leqslant 4\, e^{-(\beta/2)m_i}\, e^{-(\beta/2)|\mathbf{P}|} \tag{27}$$

and $0 \leqslant \omega_i^{-1} \leqslant m_i^{-1}$, it is obvious that

$$\begin{aligned} &\mathrm{Tr}_{K_\pi} \boldsymbol{\omega}^{-1/2} \chi \boldsymbol{\omega} (e^{\beta\boldsymbol{\omega}} - 1)^{-1} \chi \boldsymbol{\omega}^{-1/2} \\ &\leqslant \frac{4}{\beta m_0} \cdot \sum_{i=1}^{\infty} e^{-(\beta/2)m_i} \cdot \mathrm{Tr}_{L^2(\mathbb{R}^3)} \chi \cdot e^{-(\beta/2)|\mathbf{P}|} \chi , \end{aligned} \tag{28}$$

where m_0 is the minimal mass in the model. A similar argument leads to the estimate

$$\begin{aligned} &\mathrm{Tr}_{K_\phi} \boldsymbol{\omega}^{1/2} \chi \boldsymbol{\omega}^{-1} (e^{\beta\boldsymbol{\omega}} - 1)^{-1} \chi \boldsymbol{\omega}^{1/2} \\ &\leqslant \frac{4}{\beta m_0^2} \cdot \sum_{i=1}^{\infty} e^{-(\beta/2)m_i} \cdot \mathrm{Tr}_{L^2(\mathbb{R}^3)} \omega^{1/2} \chi\, e^{-(\beta/2)|\mathbf{P}|} \chi \omega^{1/2} , \end{aligned} \tag{29}$$

where ω is the operator defined in (13) with mass $m_i = m_0$. Now both, $e^{-(\beta/4)|\mathbf{P}|}\chi$ and $e^{-(\beta/4)|\mathbf{P}|}\chi\omega^{1/2}$ are Hilbert–Schmidt operators on $L^2(\mathbb{R}^3)$. Hence, the right-hand sides of (28) and (29) are finite, i.e., ΔS has a finite trace, if condition (23) is satisfied.

Combining the results of this section we thus arrive at

PROPOSITION 1. *In a model with mass spectrum m_i, $i \in \mathbb{N}$ all equilibrium states ω_β, $\beta > 0$ are locally normal if and only if the nuclearity condition* (6) *is satisfied.*

(This results holds also if there is no mass gap in the model [9].)

It is noteworthy that the simple class of models considered here can be used to give examples of theories in which no equilibrium states ω_β (i.e., states satisfying the KMS condition) exist above a certain temperature. To this end one must only proceed from the field algebras $\mathfrak{P}(\mathcal{O})$ in the representation π_0 induced by the vacuum state ω_0 to the corresponding von Neumann algebras $\mathfrak{A}_0(\mathcal{O}) = \pi_0(\mathfrak{P}(\mathcal{O}))''$. As long as condition (23) is satisfied, i.e., as long as the states ω_β are locally normal, one can extend these states to the net of algebras $\mathcal{O} \to \mathfrak{A}_0(\mathcal{O})$, and these extensions still satisfy the KMS condition. But if the mass spectrum is such that for some temperature (and, consequently, for all higher temperatures) relation (18) holds, it follows from a result of Takesaki and Winnink [10] that there cannot exist any state on the net $\mathcal{O} \to \mathfrak{A}_0(\mathcal{O})$ satisfying the KMS condition for this temperature. For, such a state would necessarily be normal on $\mathfrak{A}_0(\mathcal{O})$, in contradiction to our results. Similarly if $\mathfrak{A}_\beta(\mathcal{O}) = \pi_\beta(\mathfrak{P}(\mathcal{O}))''$, where π_β is the representation induced by some equilibrium state ω_β in a model with a mass spectrum for which (18) holds, one obtains a net $\mathcal{O} \to \mathfrak{A}_\beta(\mathcal{O})$ for which there exists no other equilibrium state besides ω_β. Of course, these physically awkward models can be ruled out by the nuclearity condition.

4. In conclusion, we want to analyze the local energy content of the equilibrium states ω_β and its relation to the mass spectrum. First of all we notice that, irrespective of the

mass spectrum, one can approximate the states ω_β by states in the vacuum representation. Namely, given ω_β and $\mathfrak{P}(\mathcal{O})$ there exists some generalized sequence $\omega_{0\iota}$, $\iota \in \mathbb{I}$ ($\mathbb{I}$ being some index set) of normal states with respect to the vacuum representation of $\mathfrak{P}$ such that

$$\omega_\beta(A) = \lim_\iota \omega_{0\iota}(A), \quad A \in \mathfrak{P}(\mathcal{O}). \tag{30}$$

This fact can be used to evaluate the local energy content of the states ω_β relative to the vacuum. (Note that the algebra $\mathfrak{P}$ does not contain operators having the meaning of an energy density.) We say that a state ω_β has finite local energy if one can choose an approximating sequence $\omega_{0\iota}$ in (30) in such a way that for any $n \in \mathbb{N}$

$$\omega_{0\iota}(H^n) \leqslant c_n < \infty \tag{31}$$

uniformly in $\iota \in \mathbb{I}$. Here, H denotes the Hamiltonian in the vacuum representation of $\mathfrak{P}$.

Let us assume first that the mass spectrum satisfies the nuclearity condition (6) and let ω_β be any equilibrium state. Then we define a quasi-free state $\omega_{0\beta}$ on $\mathfrak{P}$ which coincides with ω_β on the algebra $\mathfrak{P}(\mathcal{O})$ and with the vacuum ω_0 on any algebra $\mathfrak{P}(\mathcal{O}_1)$ whenever $\mathcal{O}_1$ is contained in the complement of some sufficiently large region $\hat{\mathcal{O}} \supset \mathcal{O}$. The two-point function of $\omega_{0\beta}$ is given by (compare (26))

$$S_{0\beta}(F, G) = S_0(F, G) + \\ + \left(F, \begin{pmatrix} \omega^{1/2}\chi\omega^{-1}(e^{\beta\omega} - 1)^{-1}\chi\omega^{1/2} & 0 \\ 0 & \omega^{-1/2}\chi\omega(e^{\beta\omega} - 1)^{-1}\chi\omega^{-1/2} \end{pmatrix} G\right). \tag{32}$$

Using the criterion of Araki and Yamagami [6] and the methods of the previous section it is easy to verify that $\omega_{0\beta}$ is a normal state with respect to the vacuum representation of $\mathfrak{P}$. By a straight-forward calculation, one can also show that $\omega_{0\beta}$ can be extended to any power H^n, $n \in \mathbb{N}$ of the Hamiltonian, i.e., $\omega_{0\beta}(H^n) < \infty$. To given an example, one has

$$\omega_{0\beta}(H) \\ = \tfrac{1}{2}\mathrm{Tr}_{K_\varphi}\omega\chi\omega^{-1}(e^{\beta\omega} - 1)^{-1}\chi\omega + \tfrac{1}{2}\mathrm{Tr}_{K_\pi}\chi\omega(e^{\beta\omega} - 1)^{-1}\chi, \tag{33}$$

which is finite if (6) holds, cf. the discussion in the previous section. So one may choose for the approximation (30) of ω_β the constant sequence $\omega_{0\beta}$, showing that ω_β has finite local energy.

Let us now assume that the mass spectrum of the model is such that condition (6) is violated for some β. In order to see that the corresponding equilibrium state ω_β does not have finite local energy, we must exhibit a sequence of Hermitean operators $H_n^{\mathcal{O}} \in \mathfrak{P}(\mathcal{O})$, $n \in \mathbb{N}$ such that, firstly,

$$\pm \pi_0(H_n^{\mathcal{O}}) \leqslant k \cdot (1 + H^l) \tag{34}$$

for fixed numbers $k, l > 0$ and, secondly, $\lim_{n\to\infty} \omega_\beta(H_n^{\mathcal{O}}) = \infty$. Then the estimate

$$\liminf_\iota \omega_{0\iota}(1 + H^l) \geqslant \frac{1}{k} \lim_\iota |\omega_{0\iota}(H_n^{\mathcal{O}})| = \frac{1}{k} |\omega_\beta(H_n^{\mathcal{O}})|, \tag{35}$$

which holds for fixed n and any sequence $\omega_{0\iota}$ converging to ω_β in the sense of relation (30), shows, that $\omega_{0\iota}(H^I)$ cannot be uniformly bounded in ι. A natural candidate for the sequence $H_n^{\mathcal{O}}$ is the expression

$$H_n^{\mathcal{O}} = \frac{1}{2} \sum_{k=1}^{n} :\pi^*(\boldsymbol{\omega}\mathbf{g}_k)\pi(\boldsymbol{\omega}\mathbf{g}_k) + \phi^*(\mathbf{P}\mathbf{g}_k)\cdot\phi(\mathbf{P}\mathbf{g}_k) + \phi^*(\mathbf{m}\mathbf{g}_k)\phi(\mathbf{m}\mathbf{g}_k): , \tag{36}$$

where the colon indicates that the vacuum expectation value has to be subtracted. $\mathbf{m}$ is the mass operator on K_ϕ which, on a dense set of vectors $\mathbf{f} = \oplus_{i=1}^{\infty} f_i \in K_\phi$ is defined by

$$\mathbf{m}\cdot\mathbf{f} = \bigoplus_{i=1}^{\infty} m_i f_i , \tag{37}$$

and the functions $\mathbf{g}_k \in K_\phi(\mathcal{O})$ are to be properly chosen such that $\pi_0(H_n^{\mathcal{O}})$ approaches (in the sense of bilinear forms on states with finite energy) the energy density integrated over the region $\mathcal{O}$.

By standard arguments ([11], cf. also [12]) one can establish for $\pi_0(H_n^{\mathcal{O}})$ an energy bound of the form (34). Moreover, from (36) one obtains

$$\omega_\beta(H_n^{\mathcal{O}}) = \sum_{k=1}^{n} (\mathbf{g}_k, \boldsymbol{\omega}^2 (e^{\beta\boldsymbol{\omega}} - 1)^{-1} \mathbf{g}_k) , \tag{38}$$

and taking into account the constraints on the functions $\mathbf{g}_k$ as well as the fact that $\Sigma_{i=1}^{\infty} e^{-\beta m_i} = \infty$ one can show that the sequence $\omega_\beta(\mathrm{H}_n^{\mathcal{O}})$ diverges. Since the arguments are very similar to those given in the previous section, we omit them and just state the final result.

PROPOSITION 2. *In a model with mass spectrum m_i, $i \in \mathbb{N}$ all equilibrium states ω_β, $\beta > 0$ have finite local energy if and only if the nuclearity condition* (6) *is satisfied.*

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