NON-TRIVIAL LATTICE RANDOM SURFACES

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We refine a previously introduced Monte Carlo method for simulating random surfaces. This allows us to calculate with high precision critical indices for planar random surfaces without spikes. We assume standard scaling laws. Within errors of only a few percent our results in four dimensions are: $\nu = \frac{1}{4}$, $\gamma = \frac{1}{4}$, $d_H = 4$, $\eta = 1$. In contrast to planar random surfaces with spikes the model is non-trivial: The two-point function has an anomalous dimension $\eta \neq 0$.

Within the last few years there has been a continuous interest in the study of random surfaces. Most prominently they are studied because of their connection to QCD, but also their relation to surface problems in solid state physics and many other aspects are of interest. For a review see ref. [1]. The simplest lattice model is the one of planar random surfaces (PRS) with fixed Euler characteristics [2]. Unfortunately it has been proven by Durhuus, Fröhlich and Jonsson [3] that PRS are trivial. This means the anomalous dimension η of their two point function vanishes and in the critical limit they describe a (generalized) free field theory.

Independently from the investigations of ref. [3] one of the present authors, in collaboration with Billoire and Foerster [4–6], started a numerical investigation of lattice random surfaces. In this context planar random surfaces without spikes [‡1] (PRSWS), called "fermionic random surfaces" in refs. [4–6], were introduced. For reasons sketched at the end of this letter the triviality proof of ref. [3] does not apply to PRSWS. Our numerical results presented here show that PRSWS are non-trivial and consequently belong to a different universality class than PRS.

Our model is defined by the following loop Green functions:

$$G_\beta(\gamma_1, ..., \gamma_n) = \sum_{S \in \mathcal{S}(\gamma_1,...,\gamma_n)} e^{-\beta|S|}, \quad \beta \geqslant \beta_c . \quad (1)$$

The sum goes over the set $\mathcal{S}(\gamma_1, ..., \gamma_n)$ of all link-wise connected PRSWS with boundary $\gamma_1 \cup ... \cup \gamma_n$ given by n loops $\gamma_1, ..., \gamma_n$ in $\mathbf{Z}^d$. $|S|$ is the area (= number of plaquettes) of surface S. The difference to ref. [3] is that there $\mathcal{S}(\gamma_1, ..., \gamma_n)$ is a set of connected PRS with spikes allowed.

In our numerical work we simulate surfaces with a common fixed plaquette. In other words, the boundary consists of one loop ∂p_0 given by the four links of this plaquette. The partition function of our model is

$$Z = G_\beta(\partial p_0), \quad (2a)$$

and for technical reasons [4,5] we generate surfaces with respect to the modified partition function

$$\hat{Z} = \sum_{S \in \mathcal{S}(\partial p_0)} |S| e^{-\beta|S|} . \quad (2b)$$

The results of refs. [2,3] imply that the partition function converges for β large enough. A critical point β_c is supposed to exist such that in the limit $\beta \to \beta_c + 0$ the loop Green functions (1) define a euclidean quantum field theory.

Critical indices are defined as follows

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[‡1] A spike is consisting out of two connected surface plaquettes, occupying the same lattice plaquette. For a figure see ref. [5].

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$$m(\beta) \sim (\beta - \beta_c)^{\nu}\,, \quad \chi(\beta) \sim (\beta - \beta_c)^{-\gamma}\,, \qquad (3a, b)$$

$$G_{\beta}(\partial p_0, \partial p) \sim |x_p - x_{p_0}|^{-(d-2+\eta)}\,. \qquad (3c)$$

Here $m(\beta)$ is the mass gap, $\chi(\beta)$ the susceptibility and η denotes the anomalous dimension of the two-point function at the critical point ($1 \ll |x_p - x_{p_0}| \ll m(\beta)^{-1}, \beta \to \beta_c + 0$).

Let $n(A)$ be the number of surfaces with area $|S| = A$ contributing to the partition function (2). We assume

$$n(A) = cA^{\epsilon}\, e^{\beta_c A}\,[1 + O(1/A)]\,, \quad \text{as } A \to \infty\,. \qquad (4)$$

If the susceptibility $\chi(\beta)$ diverges, as $\beta \to \beta_c$, i.e. if $\epsilon \geqslant -2$, then

$$\gamma = 2 + \epsilon\,. \qquad (5)$$

Our numerical simulation will directly determine ϵ and the Hausdorff dimension d_H defined by

$$\langle x^2 \rangle_A = cA^{2/d_H} \quad \text{as } A \to \infty\,. \qquad (6)$$

Here the expectation value $\langle 2 \rangle_A$ is given by averaging over all surfaces of area $|S| = A$ and $x^2 = (x_p - x_{p_0})^2$ is the squared distance of the plaquette $p \in S$ from the fixed plaquette p_0. The other critical indices follow from standard scaling relations

$$\nu = d_H^{-1}\,, \quad \gamma = \nu(2 - \eta)\,. \qquad (7a, b)$$

We now present our numerical results. In ref. [6] β_c was calculated in $d = 5$ dimensions to high precision. We obtain now similar results also in $d = 4$ and $d = 8$ dimensions:

$$1.507 > \beta_c > 1.502 \quad (d = 8)\,, \qquad (8a)$$

$$1.195 > \beta_c > 1.180 \quad (d = 4)\,. \qquad (8b)$$

[For completeness: $1.305 < \beta_c < 1.310$, $d = 5$.] As in ref. [6] the results rely on expanding ($\beta < \beta_c$) and contracting ($\beta > \beta_c$) surfaces. The largest surfaces involved had an area of approximately 8000 plaquettes.

In ref. [6] also an attempt was made to determine the entropy coefficient ϵ of eq. (4) by calculating for $\beta > \beta_c$ the average area $\langle A \rangle$ with respect to the modified partition function $\hat{Z}$ (2b). Fitting the asymptotic behaviour $\langle A \rangle \sim (\epsilon + 2)/(\beta - \beta_c)$ for $\beta \to \beta_c + 0$ yields, in principle, ϵ. This approach is, however, plagued by large statistical errors and unknown systematic errors. The reason is the for $\beta > \beta_c$ the MC procedure [4,5] spends most of the CPU time on small surfaces and the divergence is only indicated by occasional excursions to (very) large surfaces. For the present investigation we therefore decided to measure $\langle x^2 \rangle_A$ in a way similar to the microcanonical approach as successfully applied in ref. [7]. (Unfortunately the model considered in ref. [7] has severe diseases [8].) This is done by using our MC method at $\beta < \beta_c$ to generate lattice surfaces. We prevent these surfaces from growing ad infinitum by inserting an *upper bound* A_{max} for the area. Any shift of our MC procedure which would increase the area beyond A_{max} is rejected. This rejection does not destroy detailed balance, but ergodicity is only maintained for surfaces of sufficiently smaller area $A = |S|$ than A_{max}. By this method we calculate $\langle x^2 \rangle_A$ and get reliable results even for surfaces of areas as large as $A \approx 5000$, whereas by simulating $\hat{Z}$ for $\beta > \beta_c$ [6] similar results already became unreliable around $A \approx 50$.

Table 1
Statistics of our MC calculation

β	A_{max}	Iterations
1.170	300	3×10^9
1.160	5000	5.4×10^9
1.165	5000	5.4×10^9
1.170	5000	5.4×10^9

Table 1 contains (in "iterations") the high statistics that we have collected with two different upper bounds A_{max} and for various β-values. Each proposed shift for a single plaquette is counted as one iteration. The acceptance rate is ≈15%. Altogether the calculation relies on approximately 900h CPU time on the Fujitsu–Siemens 7.882 computer of Hamburg University.

Table 2 gives an impression about our measurements of $\langle x^2 \rangle_A$. For close-by surfaces the results are strongly correlated and practically identical. This is expected because these surfaces emerge from one another by small deformations. During the simulation $\langle x^2 \rangle_A$ fluctuates heavily, but moves slowly. This is due to the fact that deforming the shape of the surface requires a large amount of subsequent iterations and unfortunately the entropy of surfaces of area A is not sharply peaked around $x^2 = \langle x^2 \rangle_A$. This can be seen from fig. 1. Nevertheless our very high statistics results are quite indicative for the Hausdorff dimen-

Table 2
Measurements of $\langle x^2 \rangle_A$ for selected values of A. $n_{MC}(A)$ gives the total number of surfaces of area A generated by our MC procedure. The error bars are estimated by dividing each sample into five independent bins. For $A_{max} = 5000$ we have discarded the first 5% of our data to reach equilibrium. In case of $A_{max} = 280$ the minimal surface of area $A = 1$ is reached so often that we can conclude to be in equilibrium immediately.

A_{max}	β	A	$n_{MC}(A)$	$\langle x^2 \rangle_A$
280	1.170	131	1.2×10^6	5.00(0.03)
		135	1.3×10^6	5.06(0.03)
		139	1.4×10^6	5.16(0.03)
		263	1.1×10^7	6.91(0.04)
		267	1.1×10^7	6.69(0.05)
		271	1.2×10^7	7.00(0.05)
5000	1.160	4951	1.1×10^7	29.0(1.8)
		4955	1.3×10^7	29.0(1.8)
		4959	1.4×10^7	29.1(1.8)
		4963	1.6×10^7	29.1(1.8)
		4967	1.8×10^7	29.1(1.8)
		4971	2.0×10^7	29.1(1.8)
	1.165	4951	1.2×10^7	24.9(2.0)
		4955	1.3×10^7	24.9(2.0)
		4959	1.5×10^7	24.9(2.0)
		4963	1.6×10^7	24.9(2.0)
		4967	1.8×10^7	24.9(2.0)
		4971	2.0×10^7	24.9(2.0)
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		4867	1.7×10^7	29.3(1.8)
		4971	1.8×10^7	29.3(1.8)

sion. Using

$$d_H = 2 \ln(A_1/A_2)[\ln(\langle x^2 \rangle_{A_1}/\langle x^2 \rangle_{A_2})]^{-1}, \quad A_1 \gg A_2, \quad A_2 \to \infty, \tag{8}$$

we obtain $d_H = 4.3 \pm 0.2$ for $A_1 = 271, A_2 = 131$ and $d_H = 4.2 \pm 0.2$ for $A_1 = 4955, A_2 = 131$ or $A_2 = 271$. In the last case we have combined our three values for $\langle x^2 \rangle_{4955}$ to $\langle x^2 \rangle_{4955} = 28 \pm 2$. In the same way we may use other values of A and get very similar results. Within statistical errors and a presumably small positive systematic error (due to the finite value of A_{max}) our data give

$$d_H = 4, \tag{9}$$

in agreement with an old conjecture [8].

Remark. If one assumes that $\langle x^2 \rangle_A$ grows logarithmically with A, that is $d_H = \infty$, the results obtained from $A_1 = 271, A_2 = 131$ and $A_1 = 4955, A_2 = 131$ or 271 are not consistent with each other in the sense, that they lead to completely different values for the constants appearing in $\langle x^2 \rangle_A = a \ln A + b$. So we conclude that $d_H = \infty$ is ruled out for PRSWS.

Our method allows to measure accurately the exponent ϵ of eq. (4). The MC procedure generates large surfaces with the probability $n^{\beta}_{MC}(A) = c_\beta A^{\epsilon+1} \times \exp[(\beta_c - \beta)A]$, where we have neglected $O(1/A)$ corrections. This yields:

$$n^{\beta_2}_{MC}(A_2)/n^{\beta_1}_{MC}(A_1) = c^{\beta_1}_{\beta_2}(A_2/A_1)^{\epsilon+1} \times \exp(-\beta_2 A_2 + \beta_1 A_1). \tag{10}$$

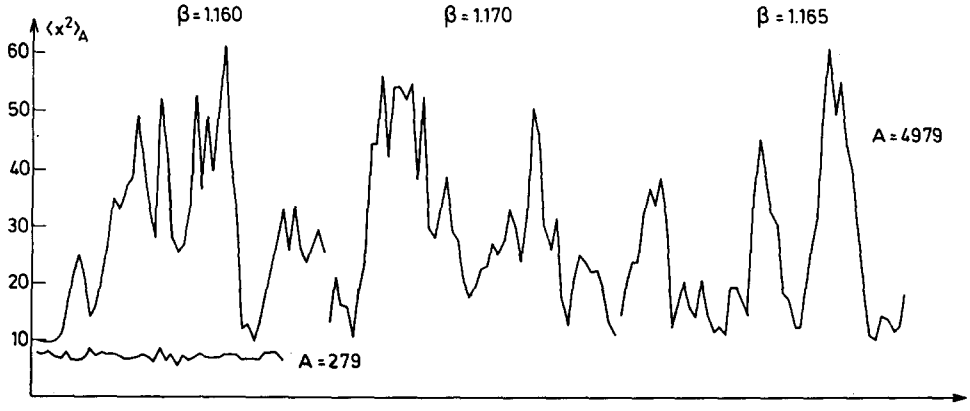

Fig. 1. Fluctuation of $\langle x^2 \rangle_A$ for large ($A = 4979$) and small ($A = 279$) surfaces versus computer time t. (Each set of measurements is represented by 50 bins for large surfaces, 43 for small surfaces.)

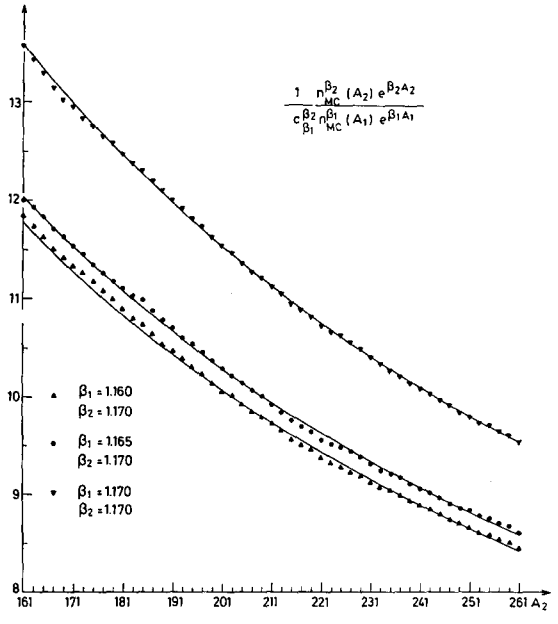

Fig. 2. Least square fits for determining ϵ. For each curve the data points are strongly correlated such that the whole curve fluctuates. $\epsilon = -1.5$ would not fit on the scale of this figure, but would lead to values

$$5.5 \geqslant \frac{1}{c^{\beta_1}_{\beta_2}} \frac{n^{\beta_2}_{MC}(A_2)}{n^{\beta_1}_{MC}(A_1)} \frac{\exp(\beta_2 A_2)}{\exp(\beta_1 A_2)} \geqslant 4.4 .$$

ϵ and the constants $c^{\beta_1}_{\beta_2}$ are unknown. We now take $A_1 \in \{\text{large surfaces}\}$ and $A_2 \in \{\text{"small" surfaces}\}$ with $A_1 - A_2 = \Delta A = 4718$. By varying A_2 in the range $161 \leqslant A_2 \leqslant 261$ least square fits to our data, as depicted in fig. 2, give the result

$$\epsilon = -1.74 \pm 0.03 . \qquad (11)$$

Despite the fact that the constant $c^{\beta_1}_{\beta_2}$ in eq. (10) is $\beta_{1,2}$-dependent, we find consistent ϵ-results for the three different β-values. They are: $\epsilon = -1.723$ ($\beta_2 = 1.160$), $\epsilon = -1.729$ ($\beta_2 = 1.165$) and $\epsilon = -1.765$ ($\beta_2 = 1.170$). Treating these results as independent, we find $\epsilon = -1.739 \pm 0.013$. They are, however, correlated because the small surfaces are identical in all three cases. To analyse these correlations we have divided each dataset into three bins. This gives $3 \times (3 \times 3)$ combinations and for each combination we have separately carried out the least square fit. The thus obtained distribution of ϵ-values is displayed in fig. 3.

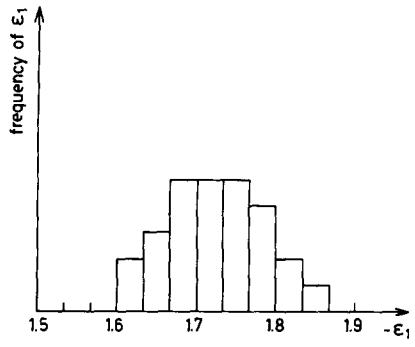

Fig. 3. Distribution of 27 ϵ-values, which are calculated as explained in the text.

It is approximately gaussian and gives confidence in the stability of the final result. Taking the average over these 27 least square fits gives $\epsilon = -1.725$ and the bias, as compared with eq. (11), is well within the given error. The error bar of eq. (11) is obtained by looking for each set of large surfaces at the 3 diagonal bin–bin combinations with the set of "small" surfaces.

Together with the scaling laws (5) and (7) our results (9) and (11) imply:

$$\nu = 1/4 , \quad \gamma = 0.24 \pm 0.03 , \quad \eta = 1.04 \pm 0.12 . \qquad (12)$$

Eq. (12) is the precise formulation of the results stated already in the abstract, and for the Hausdorff dimension we have assumed that it takes on an integer value.

To strengthen the reliability of our method, we have carried out a similar analysis for closed random walks without spikes. In previous equations the area $A = |S|$ of a surface has to be replaced by the length $L = |\omega|$ of a path. We obtain the numerical results $d_H = 2.07 \pm 0.03$ and $\epsilon = -1.97 \pm 0.02$. Within expected small systematic errors this is in good agreement with exact results $d_H = 2$ and $\epsilon = -2$ [9].

In summary our numerical investigation overcomes various shortcomings of previous numerical work [5, 10] concerning lattice surfaces.

The result (11) is to some extent surprising, because naively one may have argued PRSWS to be in the same universality class as PRS. Then ref. [3] would imply either $\epsilon = -1.5$ or $\epsilon < -2$. But universality classes of random surfaces are not at all well-understood, mainly because there is no straightforward

method to determine the corresponding lagrangian field theory (if there is any). It is instructive to pin down the reason, why the proof of ref. [3] collapses for PRSWS. Let now

$$G_\beta(b, b') = G_\beta(\gamma(b), \gamma(b'))$$

be a PRS two-loop function with loops located on the links b, b'. The key step in the proof of ref. [3] is to write

$$G_\beta(b, b') = \sum_{S \in \mathcal{S}(b, b')} e^{-\beta|S|}$$

$$= \sum_{R \in \mathcal{R}(b, b')} e^{-\beta|R|} \prod_{\ell \in R} \left(1 + \sum_{S \in \mathcal{S}(\ell)} e^{-\beta|S|}\right),$$

$$= [1 + G(\beta)]^2 \sum_{R \in \mathcal{R}(b, b')} e^{-\beta_{\mathrm{eff}}|R|}, \qquad (13)$$

where $G(\beta) = G_\beta(b)$ and $\beta_{\mathrm{eff}} = \beta - 2\log(1 + G(\beta))$. Here $\mathcal{R}(b, b') \subset \mathcal{S}(b, b')$ is such that the surfaces $R \in \mathcal{R}(b, b')$ are partially two-link irreducible. The precise definition is: $\mathcal{R}(b, b')$ contains all surfaces, which by cutting any two links either remain connected or decay in surfaces $S_1 \in \mathcal{S}(b, \ell)$ and $S_2 \in \mathcal{S}(b', \ell)$ where ℓ is the link along which is cutted. All surfaces $S \in \mathcal{S}(b, b')$ are obtained by glueing to surfaces $R \in \mathcal{R}(b, b')$ at each link $\ell \in R$ all possible surfaces $S \in \mathcal{S}(\ell)$ or the zero-area surface. For each link $\ell \in R$ we always get the same factor independently of the local geometry. This is of central importance and precisely not true in case of PRSWS. As spikes are forbidden we have to distinguish between links at edges and at flat pieces of R. Consequently the equation for β_{eff} breaks down. This is decisive as for deriving $\eta = 0$ one needs $\beta_{\mathrm{eff}} = \beta - 2\log(1 + c + (\beta - \beta_c)^{1-\gamma})$, $(\beta \to \beta_c)$ and for PRSWS the contribution $(\beta - \beta_c)^{1-\gamma}$ for $\beta \to \beta_c$ cannot be guaranteed anymore.

One may imagine that for large enough d the flat pieces of PRSWS $R \in \mathcal{R}(b, b')$ are no longer important. Then only edges would matter and a similar recursion like eq. (13) would hold. This is satisfactory because the large d expansion [3,11] predicts free field theory behaviour. The upper critical dimension has conjectured [8] to be $d = 8$, and it is tempting to argue $\epsilon = -1.5$ for PRSWS in eight dimensions. To calculate ϵ in eight dimensions is a straightforward extension of our present calculation, which we plan to carry out as a next step.

In conclusion we claim to have identified the simplest non-trivial lattice model of random surfaces. Although our PRSWS do not interact, the corresponding field theory is non-trivial as can be concluded from the anomalous dimension of the two-point function. It is a theoretical challenge to identify the lagrangian version (if there is any) of the corresponding euclidean quantum field theory.

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