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Scaling and String Tension in Pure $SU(2)$ Lattice Gauge Theory

Fritz Gutbrod

Deutsches Elektronensynchrotron DESY, Hamburg 52, Federal Republic of Germany

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Abstract. Wilson loops in $SU(2)$ lattice gauge theory without fermions are determined on lattices of size 12^4, 16^4 and 24^4 at $\beta=2.4$, 2.5 and 2.6. At $\beta=2.6$ the static quark-antiquark potential is extracted for distances up to 8 lattice units. A string tension smaller by a factor 2 than in previous investigations is found. Deviations from asymptotic scaling for multiplicatively improved Creutz ratios are certain, and their magnitude depends on the geometrical size of the ratios. This implies deviations from scaling.

1. Introduction

Although Monte Carlo measurements of planar Wilson loops in pure lattice gauge theory are a straightforward procedure, more effort may still be worthwhile. Especially to find the magnitude of nonperturbative effects, above all of the string tension, requires high precision in the region of small coupling constants g^2. The same is true for the scaling properties of the static $q\bar{q}$-potential (or, more generally, of Creutz ratios [1]). These give, as we believe, presently the most accurate information on how close we are to the continuum limit. We therefore present an investigation of this potential on large lattices with a few thousand Monte Carlo iterations per value of $\beta=4/g^2$. Large spatial extensions are, as it is well known, needed to avoid finite size effects for Wilson loops of large extension, which we need to determine the potential reliably at large distances. Furthermore we have to avoid to cross the finite temperature deconfining phase transition [2], which occurs at a fixed lattice size for increasing β. We shall present evidence that finite size effects are very small at $\beta=2.4$ for lattice sizes $L\geqq 12$, at least for objects (Wilson loops, Creutz ratios etc.) of size 4 or less. Since a shift from $\beta=2.4$ to $\beta=2.6$ corresponds (as we shall see) to a change in scale by a factor 2, it is save to use $L=24$ at $\beta=2.6$ for objects of size 8 or less. On the other hand, there are good indications that at $\beta=2.4$ there is some suppression of Creutz ratios of length $\geqq 3$ on a 8^4-lattice. Therefore our choice of L is necessary for a determination of the potential. We also believe that meaningful scaling tests require large lattices, since it is not excluded that there exist several dimensionful quantities which scale differently.

It has become customary to describe the scaling properties of a physical quantity by quoting $\Delta\beta$, i.e. the shift in β necessary to change the scale by a factor 2. It is by now well established [3–6, 10, 11] that, contrary to early optimism [1, 7–9], $\Delta\beta$ differs substantially from the values predicted by two loop perturbation theory both in $SU(2)$ and $SU(3)$. There is a tendency to approach these values around $\beta=6.6$ in $SU(3)$. The question remains whether this deviation from asymptotic scaling is uniform, i.e. whether scaling of quantities at small and at large distances (and also of different nature) can be described by the same $\Delta\beta$. If not, the relation of finite β lattice studies to continuum physics is obscure.

For such scaling tests, the string tension turns out to be an imperfect candidate, simply because it is rather small in our region of β. The necessary subtraction of nonleading terms of the potential at large R introduces severe systematic errors. On the other hand, scaling of Creutz ratios at finite lattice distances suffers from sizable finite "a" distortions [5] ($a=$lattice unit). It may be possible to remove these distortions by forming linear combinations of Creutz ratios and their generalizations [5, 19]. This, however, makes it difficult, (as a consequence of regularities of these scaling violations), to study scaling of objects of essentially different sizes, given the presently restricted region of available extensions. Therefore we apply a correction for finite "a" effects multiplicatively to individual ratios. We then find a significant dependence of $\Delta\beta$ on the object size in the sense that small ratios change scale closer in accordance with two loop perturbation theory than large ones.

2. Monte Carlo Summary

In Table 1 we summarize the statistics collected for various β and lattice sizes. The action is the standard one plaquette action, boundary conditions are periodic. For $\beta=2.4$ and $\beta=2.5$ the icosahedral subgroup was used with Metropolis updating. One fourth of all Wilson loops was calculated after each sweep. At $\beta=2.6$ 2,000 iterations were performed with the same technique, then the configuration was converted to the continuous group (truncated to 16 bit accuracy) and updated further with the heatbath algorithm. The configuration had to be stored on disc and processed in sequence of timeslices. This might generate certain regularities when passing through the lattice during updating. In order to reduce possible correlations, the lattice was turned after every 12 sweeps. Within the timeslices, a three-dimensional chessboard sequence was followed to update links. After 60 sweeps all Wilson loops in planes orthogonal to the current time direction were measured with the help of the multihit method [12].

The errors of the Wilson loops, which are listed in Table 2, are based on the bin sizes given in Table 1. For $\beta=2.6$ it turned out that when each measurement was considered as being independent, only the errors of the smallest loops decreased. A comparison with 8^4 data from [13] with our 12^4 data indicates moderate finite size effects at $\beta=2.4$, but none between 12^4 and 16^4 for Creutz ratios of size 4×4 at the level of 2%. This can be seen in Table 3, where Creutz ratios are given, with errors derived from averaging Creutz ratios. The ratios are in good agreement with those of [3], and they show little evidence for convergence as function of increasing size, especially at $\beta=2.6$. This was observed at first in [3], and it is now established up to size 6×6

Table 1. Statistics collected at various β and lattice sizes

β	L	Group	No. of sweeps	sweeps discarded	binning
2.4	12	Icos.	27 000	1000	1000
	16	Icos.	6 700	1500	100
2.5	12	Icos.	22 000	1000	1000
	24	Icos.	3 000	1000	150
			(several lattices in parallel)		
2.6	24	Cont.	8 000	2000 (4000)	480
			(2 000 with icosahedral group)		

Table 2a–f. Expectation values of Wilson loops (with statistical errors)

Table 2a: β = 2.4 , L=12, icosahedral group. Binning: 1000 sweeps.

R=	1	2	3	4	5	6	7
T=1	0.629944	0.424972	0.291130	0.200169	0.137751	0.094822	0.065266
	0.000033	0.000058	0.000067	0.000070	0.000067	0.000060	0.000054
2		0.222318	0.123476	0.069802	0.039664	0.022593	0.012865
		0.000087	0.000091	0.000082	0.000066	0.000053	0.000043
3			0.059058	0.029351	0.014750	0.007456	0.003760
			0.000086	0.000071	0.000053	0.000044	0.000036
4				0.013130	0.005977	0.002735	0.001280
				0.000059	0.000041	0.000031	0.000021
5					0.002499	0.001036	0.000409
					0.000032	0.000027	0.000025

Table 2b: β = 2.4 , L=16, icosahedral group. Binning: 100 sweeps.

R=	1	2	3	4	5	6	7
T=1	0.629965	0.425013	0.291195	0.200229	0.137795	0.094854	0.065279
	0.000034	0.000061	0.000074	0.000074	0.000070	0.000062	0.000054
2		0.222417	0.123580	0.069867	0.039712	0.022606	0.012841
		0.000098	0.000102	0.000083	0.000067	0.000051	0.000040
3			0.059100	0.029345	0.014747	0.007482	0.003792
			0.000097	0.000072	0.000050	0.000038	0.000028
4				0.013092	0.005977	0.002712	0.001249
				0.000054	0.000038	0.000031	0.000021
5					0.002513	0.001061	0.000410
					0.000029	0.000023	0.000017

Table 2c: β = 2.5 , L=12, icosahedral group. Binning: 1000 sweeps.

R=	1	2	3	4	5	6	7
T=1	0.651975	0.456438	0.324726	0.231933	0.165820	0.118597	0.084838
	0.000032	0.000057	0.000077	0.000084	0.000082	0.000075	0.000068
2		0.258065	0.154995	0.094759	0.058252	0.035870	0.022140
		0.000106	0.000125	0.000122	0.000109	0.000086	0.000066
3			0.083300	0.046437	0.026203	0.014889	0.008507
			0.000128	0.000121	0.000109	0.000083	0.000057
4				0.024124	0.012773	0.006843	0.003675
				0.000112	0.000086	0.000059	0.000044
5					0.006451	0.003318	0.001700
					0.000064	0.000037	0.000041
6						0.001561	0.000751
						0.000043	0.000027

Table 2d: β = 2.5 , L=24, icosahedral group. Binning: 150 sweeps.

R=	1	2	3	4	5	6	7
T=1	0.651968	0.456450	0.324731	0.231917	0.165798	0.118563	0.084811
	0.000020	0.000040	0.000031	0.000018	0.000020	0.000035	0.000031
2		0.258062	0.154998	0.094738	0.058215	0.035850	0.022087
		0.000075	0.000055	0.000068	0.000063	0.000057	0.000044
3			0.083319	0.046425	0.026171	0.014813	0.008405
			0.000080	0.000067	0.000057	0.000047	0.000050
4				0.024077	0.012714	0.006798	0.003614
				0.000061	0.000045	0.000033	0.000016
5					0.006368	0.003243	0.001636
					0.000039	0.000022	0.000023

Table 2e: β = 2.6 , L=24, full group.
Binning: 480 sweeps, first 2000 sweeps included.

T \ R=	1	2	3	4	5	6	7	8
1	.669947	0.482145	0.352568	0.258798	0.190183	0.139804	0.102777	0.075558
	.000055	0.000052	0.000063	0.000059	0.000059	0.000053	0.000046	0.000040
2		0.287558	0.181825	0.116915	0.075590	0.048978	0.031762	0.020603
		0.000062	0.000070	0.000065	0.000054	0.000042	0.000034	0.000026
3			0.105010	0.062782	0.037981	0.023084	0.014053	0.008561
			0.000072	0.000062	0.000046	0.000036	0.000027	0.000019
4				0.035578	0.020552	0.011958	0.006982	0.004082
				0.000050	0.000037	0.000027	0.000019	0.000013
5					0.011426	0.006418	0.003623	0.002048
					0.000027	0.000021	0.000015	0.000010
6						0.003493	0.001915	0.001053
						0.000016	0.000011	0.000008
7							0.001022	0.000548
							0.000007	0.000005
8								0.000289
								0.000003

Table 2f: β = 2.6 , L = 24, full group.
Binning: 480 sweeps, first 2000 sweeps omitted.

T \ R=	1	2	3	4	5	6	7	8
1	.669941	0.482100	0.352520	0.258742	0.190138	0.139779	0.102760	0.075544
	.000063	0.000067	0.000077	0.000087	0.000083	0.000068	0.000063	0.000050
2		0.287497	0.181787	0.116874	0.075561	0.048958	0.031748	0.020594
		0.000074	0.000093	0.000088	0.000072	0.000058	0.000046	0.000035
3			0.104981	0.062750	0.037956	0.023067	0.014044	0.008556
			0.000102	0.000087	0.000065	0.000050	0.000037	0.000026
4				0.035556	0.020535	0.011950	0.006980	0.004082
				0.000070	0.000050	0.000036	0.000026	0.000017
5					0.011416	0.006416	0.003624	0.002051
					0.000034	0.000025	0.000018	0.000012
6						0.003495	0.001918	0.001057
						0.000017	0.000013	0.000009
7							0.001024	0.000553
							0.000008	0.000006
8								.0002924
								.0000034

Table 3a–e. Creutz ratios $\chi(R, T)$ with statistical errors

Table 3a: β = 2.4, L = 12

T \ R=	2	3	4	5
T=2	0.25431			
	0.00018			
3	0.20981	0.14946		
	0.00026	0.00046		
4	0.19576	0.12880	0.1053	
	0.00037	0.00063	0.0017	
5	0.19151	0.1229	0.0988	0.0851
	0.00043	0.0010	0.0026	0.0093

Table 3b: β = 2.4, L = 16

T \ R=	2	3	4	5
T=2	0.25402			
	0.00023			
3	0.20954	0.15000		
	0.00030	0.00055		
4	0.1958	0.1298	0.1070	
	0.0003	0.0007	0.0020	
5	0.19124	0.1232	0.0960	0.082
	0.00056	0.0012	0.0034	0.012

Table 3c: β = 2.5, L = 12

T \ R=	2	3	4	5
T=2	0.21369			
	0.00022			
3	0.16935	0.11115		
	0.00030	0.00037		
4	0.15553	0.0923	0.0707	
	0.00037	0.0007	0.0012	
5	0.15103	0.0856	0.0638	0.0476
	0.00048	0.0012	0.0013	0.0037

Table 3d: β = 2.5, L = 24

T \ R=	2	3	4	5
T=2	0.21376			
	0.00016			
3	0.16954	0.11085		
	0.00037	0.00054		
4	0.15555	0.0932	0.0711	
	0.00032	0.0008	0.0011	
5	0.15138	0.0860	0.0652	0.0513
	0.00041	0.0011	0.0017	0.0029

Table 3e: β = 2.6, L = 24 , first 2000 sweeps omitted

T \ R=	2	3	4	5	6	7	8
T=2	0.18790						
	0.00012						
3	0.14534	0.09068					
	0.00016	0.00019					
4	0.13246	0.07290	0.05347				
	0.00016	0.00024	0.00034				
5	0.12808	0.06659	0.04628	0.03818			
	0.00015	0.00024	0.00034	0.00050			
6	0.12629	0.06409	0.04336	0.03491	0.03153		
	0.00018	0.00024	0.00038	0.00050	0.00086		
7	0.12547	0.06306	0.04163	0.03349	0.0291	0.0273	
	0.00021	0.00029	0.00048	0.00057	0.0011	0.0020	
8	0.12516	0.06284	0.04078	0.03297	0.0267	0.0215	0.0204
	0.00024	0.00040	0.00057	0.00090	0.0014	0.0021	0.0046

Table 4a–f. Ratios $V_T(R)$ defined in (1) with statistical errors

Table 4a: β = 2.4, L = 12

T=	2	3	4	5	6	7
R=1	0.39361	0.37825	0.37461	0.37371	0.37344	0.37353
	0.00009	0.00010	0.00013	0.00015	0.00018	0.00024
2	0.64791	0.58806	0.57037	0.56522	0.56280	0.5631
	0.00026	0.00035	0.00047	0.00053	0.00092	0.0013
3	0.85773	0.73753	0.6992	0.6881	0.6822	0.6846
	0.00051	0.00075	0.0010	0.0013	0.0029	0.0046
4	1.05349	0.8663	0.8045	0.7869	0.7817	0.760
	0.00085	0.0013	0.0023	0.0032	0.0063	0.012
5	1.2450	0.9892	0.9033	0.872	0.881	0.929
	0.0012	0.0021	0.0042	0.010	0.019	0.045

Table 4b: β = 2.4, L = 16

T=	2	3	4	5	6	7
R=1	0.39355	0.37813	0.37453	0.37370	0.37343	0.37366
	0.00009	0.00012	0.00013	0.00017	0.00019	0.00025
2	0.64756	0.58766	0.57030	0.56494	0.56343	0.5656
	0.00031	0.00040	0.00041	0.00062	0.00091	0.0014
3	0.85710	0.73767	0.70011	0.6881	0.6785	0.6795
	0.00059	0.00087	0.00095	0.0015	0.0025	0.0051
4	1.05287	0.8675	0.8071	0.7841	0.7904	0.775
	0.00086	0.0014	0.0023	0.0037	0.0077	0.014
5	1.2441	0.9906	0.9031	0.866	0.863	0.950
	0.0013	0.0020	0.0044	0.011	0.022	0.045

Table 4c: β = 2.5, L = 12

T=	2	3	4	5	6	7
R=1	0.35655	0.34047	0.33654	0.33554	0.33518	0.33499
	0.00008	0.00012	0.00013	0.00015	0.00017	0.00022
2	0.57024	0.50982	0.49207	0.48657	0.48491	0.48257
	0.00029	0.00041	0.00049	0.00060	0.00065	0.00093
3	0.7396	0.6210	0.5844	0.5723	0.5654	0.5598
	0.0006	0.0007	0.0011	0.0017	0.0017	0.0024
4	0.8951	0.7133	0.6551	0.6361	0.6244	0.6223
	0.0010	0.0014	0.0022	0.0024	0.0034	0.0059
5	1.0466	0.7990	0.7189	0.6837	0.6651	0.674
	0.0014	0.0024	0.0028	0.0046	0.0050	0.016
6	1.1959	0.8796	0.7778	0.7244	0.761	0.736
	0.0018	0.0033	0.0040	0.0056	0.020	0.027

Table 4d: β = 2.5, L = 24

T=	2	3	4	5	6	7
R=1	0.35652	0.34059	0.33649	0.33562	0.33533	0.33502
	0.00009	0.00015	0.00015	0.00011	0.00020	0.00015
2	0.57028	0.51013	0.49205	0.48700	0.48468	0.48436
	0.00021	0.00050	0.00035	0.00048	0.00061	0.00052
3	0.73982	0.62098	0.58521	0.57303	0.5697	0.5684
	0.00048	0.00078	0.00078	0.00077	0.0015	0.0041
4	0.89537	0.71414	0.65626	0.6383	0.6260	0.6302
	0.00069	0.00145	0.00092	0.0022	0.0023	0.0055
5	1.0468	0.8002	0.7215	0.6896	0.6751	0.6844
	0.0011	0.0016	0.0011	0.0043	0.0014	0.0096

Table 4e: β = 2.6, L = 24, first 2000 iterations included

T=	2	3	4	5	6	7	8
R=1	0.32895	0.31300	0.30920	0.30806	0.30774	0.30768	0.30766
	0.00005	0.00006	0.00008	0.00008	0.00009	0.00010	0.00011
2	0.51682	0.45838	0.44160	0.43613	0.43394	0.43310	0.43284
	0.00013	0.00016	0.00017	0.00016	0.00017	0.00021	0.00021
3	0.66220	0.54899	0.51438	0.50257	0.49794	0.49631	0.49562
	0.00022	0.00029	0.00032	0.00032	0.00034	0.00038	0.00044
4	0.79460	0.62178	0.56795	0.54877	0.54159	0.53798	0.53678
	0.00032	0.00044	0.00053	0.00054	0.00053	0.00062	0.00067
5	0.92267	0.68822	0.61415	0.5871	0.5768	0.5718	0.5704
	0.00042	0.00059	0.00074	0.0009	0.0009	0.0009	0.0011
6	1.04887	0.75222	0.65780	0.6223	0.6082	0.6010	0.5979
	0.00054	0.00075	0.00092	0.0012	0.0014	0.0012	0.0017
7	1.17429	0.81543	0.6995	0.6562	0.6373	0.6285	0.6225
	0.00069	0.00088	0.0012	0.0015	0.0016	0.0021	0.0027
8	1.29947	0.8782	0.7406	0.6898	0.6648	0.6531	0.6411
	0.00083	0.0011	0.0014	0.0018	0.0022	0.0030	0.0049

Table 4f: β = 2.6, L = 24, first 2000 iterations excluded

T=	2	3	4	5	6	7	8
R=1	0.32904	0.31305	0.30927	0.30808	0.30769	0.30767	0.30768
	0.00005	0.00008	0.00009	0.00009	0.00010	0.00011	0.00012
2	0.51694	0.45838	0.44173	0.43616	0.43398	0.43313	0.43283
	0.00015	0.00021	0.00021	0.00019	0.00021	0.00025	0.00025
3	0.66227	0.54905	0.51462	0.50274	0.49804	0.49616	0.49563
	0.00029	0.00037	0.00041	0.00039	0.00040	0.00044	0.00055
4	0.79473	0.62194	0.56806	0.54898	0.54137	0.53774	0.53636
	0.00042	0.00057	0.00068	0.00064	0.00061	0.00071	0.00081
5	0.92281	0.68851	0.61430	0.58710	0.57621	0.5711	0.5692
	0.00054	0.00075	0.00090	0.00089	0.00092	0.0011	0.0013
6	1.04910	0.75258	0.6576	0.6219	0.6076	0.6001	0.5957
	0.00070	0.00093	0.0011	0.0012	0.0014	0.0015	0.0020
7	1.1746	0.8156	0.6992	0.6553	0.6366	0.6272	0.6169
	0.0011	0.0012	0.0012	0.0014	0.0019	0.0022	0.0026
8	1.2997	0.8784	0.7399	0.6882	0.6631	0.6484	0.6366
	0.0013	0.0014	0.0014	0.0018	0.0031	0.0024	0.0049

beyond any doubt, and up to 7×7 with 90% confidence.

At $\beta=2.6$ we find, that large Wilson loops from the first 2,000 sweeps are systematically lower than those from the rest. This leads to a 2 standard deviation effect in Creutz ratios with size $\geqq 5$ and to even larger differences for the ratios $V_T(R)$ considered below. Although this trend is not fully significant by itself, we consider it as real as it shows up also in other runs. Therefore the results both for the full sample and for the sweeps after the first 2,000 iterations are quoted separately in Tables 2 and 4.

3. The $q\bar{q}$-Potential

We now proceed to analyze the Wilson loop expectation values $W(R,T)$ with respect to the static $q\bar{q}$-

Table 5. Eigenvalues $\lambda_1(R)$, lattice potentials $V(R)$ and fits to $V(R)$. $V_C(R)$= lattice potential corrected for finite a effects. $V_{CL}(R)$ = fit to $V_C(R)$ of the form $-\alpha/R+C+K*R$. $V_B(R)$= two loop continuum potential with $\Lambda_R=20.78\,\Lambda_L$. $V_{BT}(R)$= one parameter potential of [18] with $K=0.008$. $\beta=2.6$, first 2,000 iterations included with half weight

R	$\lambda_1(R)$	$\lambda_2(R)$	$V_C(R)$	$V_{CL}(R)$	$V_B(R)$	$V_{BT}(R)$
1	0.3077 (1)	1.43(4)	0.3237	0.3217	0.3237	0.324
2	0.4327 (2)	1.38(2)	0.4407	0.4432	0.4353	0.455
3	0.4953 (5)	1.44(2)	0.4980	0.4979	0.4809	0.514
4	0.5359 (7)	1.40(2)	0.5369	0.5359	0.5073	0.551
5	0.5684(10)	1.42(3)	0.5688	0.5672	0.5252	0.579
6	0.5950(16)	1.37(5)	0.5952	0.5951	0.5384	0.601
7	0.6174(28)	1.31(7)	0.6175	0.6211	0.5488	0.620
8	0.6308(55)	1.25(9)	0.6309	0.6459	0.5574	0.637

$\lambda_3(R)$ = 2.64(15)

potential, concentrating on $\beta=2.6$. In Tables 4d, e the logarithms of ratios are given which converge to $V(R)$ for $T\to\infty$:

$$V_T(R)=-\ln W(R,T)/W(R,T-1). \tag{1}$$

Clearly a limit is not reached within errors for $T\leqq 8$ and $R\geqq 4$. We extrapolate to $T\to\infty$ by the ansatz [14]

$$W(R,T)=\sum_{n=1}^{N} c_n(R)\exp(-\lambda_n(R)T). \tag{2}$$

The exponents $\lambda_n(R)$ are the eigenvalues of the transfer matrix [15] with $\lambda_1(R)=V(R)$. The smallness of our errors just allows for a fit with $N=3$, including all $T\leqq 8$. The fit favours the largest eigenvalue $\lambda_3(R)$ to be independent of R. This can be understood by assuming that for very asymmetric $W(R,T)$ the variation with respect to the length of the smaller side is given only by short range correlations among links on the larger side. In Table 5 the values $\lambda_n(R)$ are shown. The analysis includes the first 2,000 iterations with half statistical weight. The errors quoted in Table 5 were obtained by subdividing the sample into 6 subsets. They agree within 20% with those obtained if we leave $\lambda_3(R)$ free and vary $\lambda_1(R)$ such that the fit misses $V_8(R)$ by 1 s.d.

It is impossible to assign systematical downward errors to $V(R)$, since we never can exclude contributions from an eigenvalue $\lambda_n(R)$ close to $\lambda_1(R)$ which would allow to lower $\lambda_1(R)$ and $c_1(R)$. The upward systematical errors are not independent of the statistical ones. The positivity of the $c_n(R)$ requires that the differences between successive $V_T(R)$ decrease at most exponentially. This does not allow to extrapolate to values $V(R)$ higher than those given in Table 5 while reproducing the $V_T(R)$.

Before turning to an interpretation of $V(R)$, a correction for finite lattice spacing should be applied. The propagator, for infinite volume, departs from the Coulomb form $1/R$ by a few percent at small R:

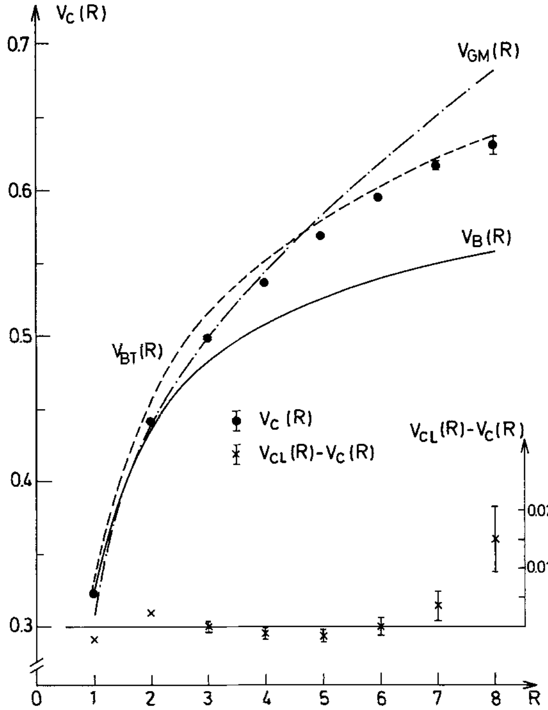

Fig. 1. Lattice potential $V_C(R)$, compared to a) $V_{CL}(R)$, see (5), b) $V_{GM}(r)$, see (5) with the parameters of [3], c) $V_B(R)$, see (9) and d) $V_{BT}(R)$, see text and [18]

$$\Delta_L(R)=\pi\int_{-\pi}^{\pi}\frac{d^3k}{(2\pi)^3}\frac{\cos k_1 R}{\sum\limits_{i=1,3}\sin^2\frac{k_i}{2}} = \begin{cases}1.081 & R=1\\ \frac{1}{R}\left(1+\frac{1}{4R^2}+\dots\right), & R\geqq 2.\end{cases} \tag{3}$$

Since all fits to $V(R)$ will lead to a short range Coulomb contribution of the form $-(0.20\pm0.02)/R$, we add

$$\Delta V(R)=0.2(\Delta_L(R)-1/R) \tag{4}$$

to obtain a potential $V_C(R)$, which is presumably closer to the continuum potential than $V(R)$. This potential is listed in the third column of Table 5.

The potential $V_C(R)$ is sufficiently accurate to exclude the standard Coulomb plus linear approximation

$$V_{CL}(R)=-\alpha/R+C+KR \tag{5}$$

as a good interpolation. In Fig. 1 we show the difference between this ansatz and $V_C(R)$, where $\alpha=0.201$

± 0.003 and $K=0.0213\pm 0.0003$. Better fits are obtained after inclusion of a logarithmic correction to the Coulomb term, which corresponds to a coupling constant increasing with R:

$$V_L(R)=-\alpha(1+\gamma\ln R)/R+C+KR. \tag{6}$$

This ansatz leads to a noticably smaller string tension, namely $K=0.0164\pm 0.0003$ with $\alpha=0.254 \pm 0.003$ and $\gamma=0.3$. The increased value of α is due to the fact, that the inclusion of the logarithmic term decreases the force for small R.

The interpolation $V_{GM}(R)$ of the form (5), obtained in [3] with $K=0.0287$, is also included in Fig. 1 and is seen to be in strong disagreement with $V_c(R)$ for $R>5$. Obviously the string tension K is considerably smaller than previous estimates.

The interpretation of $V_c(R)$ by a fluctuating string [16] suggests to use the ansatz

$$V_S(R)=-\pi/12R+C+KR \tag{7}$$

only for $R\geqq R_0$ with R_0 not too small. We obtain

$$K=\begin{cases}0.0156\pm 0.0015 & R_0=4\\ 0.0145\pm 0.0019 & R_0=5.\end{cases} \tag{8}$$

The fit, however, is not very good, and for α free one obtains $\alpha=0.45\pm 0.07$ instead of $\pi/12$. We observe that the fluctuating string picture has little phenomenological support at these distances. Accepting $\sqrt{K}=0.12/a$ and $\sqrt{K}=0.42$ GeV we have $a=0.06$ Fermi at $\beta=2.6$. Now $\langle P_\perp\rangle$, the transverse momentum of hadrons generated by a breaking of the string is of the order of $1/\delta$, where δ is the thickness of the string. From $\langle P_\perp\rangle=0.4$ GeV we obtain $\delta\sim 0.5$ Fermi, i.e. for $R=8$ the string would be as long as thick. Therefore the nonleading expansion terms of $V(R)$ for $R\to\infty$ are unknown, leading to considerable uncertainty in K. It may be, however, that the inclusion of fermions changes the scale in such a way that our arguments no longer apply.

The potential difference $V_c(2)-V_c(1)$ is already quite close to the two-loop, renormalization group improved result for the force,

$$V'(R)=-c_{2R}/4\pi R^2\{\beta_0\ln(R\Lambda_R)^{-2} + \beta_1/\beta_0\ln\ln(R\Lambda_R)^{-2}\} \tag{9}$$

with

$$c_{2R}=3/4,\quad \beta_0=22/3(2\pi)^2,\quad \beta_1=136/3(2\pi)^4. \tag{10}$$

The scale parameter Λ_R is related to the standard lattice scale parameter Λ_L by [17]

$$\Lambda_R=20.78\,\Lambda_L, \tag{11}$$

where

$$\Lambda_L=a^{-1}(\beta_0 g^2)^{-\beta_1/2\beta_0^2}\exp(-1/2\beta_0 g^2). \tag{12}$$

In order to suppress the singularity of (9) for $R\Lambda_L=1$, we have substituted $R\Lambda_R\to R\Lambda_R/(1+R\Lambda_R)$. With this modification we show the integrated form of (9) in Fig. 1 as the curve $V_B(R)$, normalized to $V_C(R)$ at $R=1$. When the difference between this curve and $V_C(R)$ is interpreted as being due to an almost linear piece in $V_C(R)$ for $R\geqq 4$, we obtain

$$K=0.011\pm 0.0015. \tag{13}$$

Finally in the last column of Table 5 the one parameter potential $V_{BT}(R)$ of Buchmüller and Tye [18] is listed, with K optimized to $K=0.008$. This small value should not be taken seriously, since $V_{BT}(R)$ fails to describe $V_C(R)$ quantitatively, as it is apparent in Fig. 1. The deviations may be understood qualitatively by the large Λ parameter characteristic for the BT-potential.

Although it is remarkable, how close the lattice potential is to reasonable phenomenological potentials, the importance of this should not be overestimated. In the next section we present evidence that the potential does not scale, and therefore the potential for very large β may have a form different from the present one.

4. Scaling

The improved accuracy of Creutz ratios, as compared to [3], will allow more precise statements about deviations from "asymptotic scaling", which means e.g. a β-dependence of K/Λ_L^2. We furthermore are in the position to test scaling itself, although the systematic uncertainties due to finite "a" effects are a series problem. Previously [3] we attempted to test scaling of the potential in the following way: If $V_I(\beta,R)$ are interpolating functions to $V(\beta,R)$ at two different β, we searched for a ξ_{12} such that

$$\xi_{12}^2 V_I'(\beta_1,R\xi_{12})=V_I'(\beta_2,R). \tag{14}$$

With the new data this test works qualitatively for $\beta_1=2.6$ and $\beta_2=2.5$, if we use $V_{CL}(R)$ as interpolating function. There are, however, discrepancies in the order of $5\div 10\%$ between both sides of (14), and we obtain a larger ξ_{12} when restricting the fit to $R\geqq 3$ as when we take $R\geqq 1$. Although this is, we believe, significant evidence for scale breaking, we shall not elaborate on it, since the interpolation of $V_C(R)$ by $V_{CL}(R)$ is not good enough, and furthermore the corrections due to lattice artifacts are quite complicated here. We therefore turn to the ratio test for scale changes by a factor 2, using Wilson loops at $\beta=2.6$ and, for comparison, loops interpolated between those at $\beta=2.3$, 2.35 and 2.4.

The general Creutz ratios $\chi(l)$ with $l=\{l_1 \dots l_8\}$,

$$\chi(l)=-\ln[W(l_1,l_2)W(l_3,l_4)/W(l_5,l_6)W(l_7,l_8)] \quad (15)$$

and

$$l_1+l_2+l_3+l_4=l_5+l_6+l_7+l_8,$$

do not scale in perturbation theory [5] for finite l. The pattern of the scaling violations can be, at least for elongated loops ($l_1>l_2$ etc), read off from the tree level potential (3). For the force at $R \geqq 3$ one obtains a nonscaling factor $1+3/4R^2$ with respect to the scaling $1/R$ potential, which leads to a "positive" violation: An elongated ratio corresponding to a potential difference between R and $R-1$ will, for $R>2$, be larger than the ratio of size $2R$. On the contrary, between $R=1$ and $R=2$ the violation is negative due to the irregular behaviour of the propagator at $R=1$. This pattern of scaling violation holds also for square ratios and survives in one loop approximation, as an analysis of the results of [19] shows.

In the "improved ratio method" one linearly combines, on the one loop level, ratios with positive coefficients [5],

$$\bar{\chi}(l,\beta)=\sum_{i=1,3} c_{l,l_i}\chi(l_i,\beta). \quad (16)$$

One demands that scaling holds for perturbatively calculated ratios $\bar{\chi}_p(l,\beta)$ up to $O(g^4)$:

$$\bar{\chi}_p(l,\beta)=\bar{\chi}_p(2l,\beta+\Delta\beta_p) \quad (17)$$

and

$$\Lambda_L(\beta)=2\Lambda_L(\beta+\Delta\beta_p). \quad (18)$$

The hope is that the same linear combinations of $\chi(l,\beta)$, determined by Monte Carlo simulation, are free of lattice artifacts. Now the general argument given above tells us that in this method we have to combine such ratios, where at least one side has length one, with larger ratios. Thus possible scaling violations will be washed out. Presently only sizes up to 3×4 (6×8 on the large lattice) are available with good accuracy, which leaves, after superposition of various sizes, little room to look for a size dependence of $\Delta\beta$.

In order to overcome this, we shall make a somewhat stronger assumption. Let us define $c_p(l,\beta)$ by

$$c_p(l,\beta)\chi_p(l,\beta)=\chi_p(2l,\beta+\Delta\beta_p) \quad (19)$$

Table 6. Ratio test for small loops at $\beta=2.6$. Unimproved ratios ($C_P=1$), tree level improved ($C_{P,\text{tree}}$) and one loop level improved ratios ($C_{P,\text{loop}}$)

		$\Delta\beta$		
l_1-l_4	l_5-l_8	$C_P=1$	$C_P=C_{P,tree}$	$C_P=C_{P,loop}$
2211	2121	0.2627	0.2339	0.2413(14)
22	31	0.2910	0.2545	0.2409(15)
2221	4111	0.3230	0.2821	0.2480(16)
2222	4121	0.2918	0.2572	0.2428(15)
3122	4121	0.2927	0.2598	0.2448(14)
3211	2221	0.1931	0.1975	0.2481(16)

at the one loop level (in tree approximation one has to take $\Delta\beta_p=0$). We now assume that the Monte Carlo ratios obey the same scaling relations, i.e. that the nonperturbative contributions to ratios violate scaling (due to lattice artifacts) by the same factor as the perturbative χ_p do. This may be called multiplicative improvement. The assumption is as ad hoc as the linear superposition procedure is. Of course, it may happen that higher order perturbative contributions show no scaling violations at all. In this case (17) still holds, whereas our $c_p(l,\beta)$ deviate from 1 stronger than the correct ones. We shall discuss the consequence of this possibility below.

In the following we shall test scaling under the hypothesis that multiplicative improvement is allowed. Then we can test scaling for individual l without the need to combine small l and large l ratios. It is, however, still somewhat arbitrary how to define the $\chi_p(l,\beta)$, given [19] the $W(i,j)$ up to $O(g^4)$. Here we first convert the expansion for $W(i,j)$ into one for $\ln W(i,j)$ and then "Padéize" the $\chi_p(l,\beta)$ according to

$$\begin{aligned}\chi_p(l,\beta)&=g^2\chi_p^{(2)}(l)+g^4\chi_p^{(4)}(l)+O(g^6)\\&\Rightarrow g^2\chi_p^{(2)}(l)/(1-g^2\chi_p^{(4)}(l)/\chi_p^{(2)}(l)).\end{aligned} \quad (20)$$

In this form the $c_p(l,\beta)$ differ from the tree level $c_p(l,\beta)$ in most cases by only few percent.

Our procedure can be shown a posteriori to work reasonably well by comparing scaling for ratios of similar small size. In Table 6 we list $\Delta\beta$ for ratios involving the smallest loops, where $\Delta\beta$ is now defined by

$$c_p(l,\beta_A)\chi(l,2.6-\Delta\beta)=\chi(2l,\beta=2.6). \quad (21)$$

Here $\beta_A=2.323$ is the β-value related to $\beta=2.6$ by asymptotic scaling. The ratios at $\beta=2.6-\Delta\beta$ are derived from linear interpolations between our data at $\beta=2.4$ and those from [13] at $\beta=2.35$. We note substantial improvement in the consistency of the resulting $\Delta\beta$ by using $c_p(l,\beta)$ at the one loop level as compared to the tree level or the uncorrected case.

Table 7. Scaling test for improved ratios at $\beta=2.6$, $L=24$. The numbers l_1-l_8 are defined in (15), $\Delta\beta$ in (21), ΔA = difference of areas of numerator and denominator of ratio. Numbers in second line are errors. First 2,000 iterations included with half weight

l_1-l_4	l_5-l_8	ΔA	$\Delta\beta$	$\chi(2l)$	$C_P*\chi(l)$	$C_{P,tree}$	$C_{P,loop}$
2 2 1 1	2 1 2 1	1	0.2413	0.2897	0.3149	1.0597	1.0431
			0.0015	0.0009			
3 3 2 2	3 2 3 2	1	0.2182	0.1400	0.1742	0.8859	0.8755
			0.0034	0.0017			
4 4 3 3	4 3 4 3	1	0.1921	0.0937	0.1369	0.9128	0.9169
			0.0090	0.0046			
3 2 1 1	2 2 2 1	1	0.2481	0.1903	0.2092	0.9864	0.8521
			0.0016	0.0008			
4 3 2 2	3 3 3 2	1	0.2096	0.1071	0.1494	0.9536	0.9125
			0.0052	0.0028			
3 2 2 1	3 1 2 2	1	0.2270	0.2203	0.2535	1.0149	0.9766
			0.0015	0.0008			
4 3 3 2	4 2 3 3	1	0.2143	0.1231	0.1612	0.8958	0.8946
			0.0050	0.0027			
3 3 1 1	3 2 2 1	2	0.2356	0.3303	0.3809	0.9471	0.8569
			0.0022	0.0022			
4 4 2 2	4 3 3 2	2	0.2012	0.2008	0.2863	0.9332	0.9144
			0.0054	0.0056			
4 2 2 1	3 2 3 1	1	0.2192	0.2043	0.2440	1.0506	1.0036
			0.0019	0.0011			
4 2 3 1	4 1 3 2	1	0.2224	0.2084	0.2454	1.0357	0.9974
			0.0018	0.0010			

In column 3 of Table 7 we finally collect values of $\Delta\beta$ (with errors) for square ratios and for such elongated ratios, which are not too different from potential differences*. Those ratios, which differ by increasing all lengths by one unit, are grouped together. The ratios themselves are listed in column 4, and in column 5 the quantities $c_p(l,\beta_A)\chi(l,\beta_A)$. They are related to the $\chi(2l,\beta=2.6)$ by "improved" asymptotic scaling, and they were gained by interpolation among the data of [13]. We notice a clear tendency of $\Delta\beta$ to decrease with increasing size of the ratios. This is significant, for the comparison of length 2 and length 3 square ratios, on the level of 7 s.d. and, between 3 and 4 on the level of 3 s.d. For elongated ratios the deviations are also on the level of 3 s.d.

Since the differences between $\Delta\beta$'s are fully significant for the square ratios, where furthermore the differences between c_p(tree) and c_p(one loop) are very small (see last 2 columns of Table 7), we have to interpret these results. We notice, that the differences between $\chi(2l,2.6)$ and $c_p(l)\chi(l,\beta_A)$ are increasing with l within the various groups. For ratios with large loops this difference is about $\Delta\chi=0.042\pm0.003$. Since $\Delta\chi=4\Delta V$ for elongated ratios, we see that ΔV is not far from the lower results for K obtained in the last section. A possible interpretation of the observed nonscaling behaviour is therefore, that the ratios and the potential are a superposition of an asymptotically scaling and asymptotically free Coulomb term plus an approximately linear term, which does not scale asymptotically but decreases between $\beta=2.323$ and $\beta=2.6$ faster than Λ_L^2 by a factor close to 2. If scaling would hold both for the Coulomb term and for the linear term with a single β-function differing from the asymptotic one, the differences $\Delta\chi$ ought to be larger for small l than for large l. This is so because via the decreasing coupling constant also the Coulomb term will contribute to $\Delta\chi$, but dominantly at small l. We observe the opposite. If insufficient corrections for finite "a" effects are to be blamed for this effect, the small ratios are the most suspicious ones. In order to make the differences $\Delta\chi$ consistent with scaling, the c_p would have to be increased by about 10% for the smallest loops, which is against the trend of the changes from c_p(tree) to c_p(one loop). If higher order contributions have smaller scaling violations than those up to $O(g^4)$, the corresponding c_p would be closer to 1, which would increase the scaling violations for square ratios.

Finally we note that a return to asymptotic scaling for objects of size 8 or less is expected for $\beta>2.8$, since if the linear term continues to decrease rapidly, it will then be smaller then the Coulomb term.

5. Discussion

If we can rely on the error estimates based on grouping the data at $\beta=2.6$ into bins containing up to 500 sweeps, the statistical accuracy of the Monte Carlo data presented here is quite good. If on the other hand the deviations observed for the first 2,000 iterations are real, our total of 8,000 iterations (including the 2,000 iterations with the icosahedral group) may be insufficient. Accepting the quoted errors we can conclude the following:

For $R\geqq 4$ and $T\leqq 8$ planar Wilson loops have not yet converged to the exponentially decreasing form $\exp(-V(R)T)$, and our extrapolated values for $V(R)$ are, strictly speaking, upper limits only for the correct potential. Since the fit with three exponentials works successfully, we can be optimistic and

* The ratios at $\beta=2.6$ include the contributions from the first 2,000 iterations with half weight

use $V(R)$ as the real $q\bar{q}$-potential (in $SU(2)$ without quarks of course). From these potential values again an upper limit for the string tension can be obtained by forming potential differences. Doing so between $R=5$ and $R=7$ gives

$$K<0.0245\pm0.0016 \tag{22}$$

at $\beta=2.6$. Since, however, the potential is not linear between $R=4$ and $R=8$, a subtraction of nonleading terms is necessary, yielding values

$$K=\begin{cases}0.015\pm0.0019\\0.011\pm0.0015.\end{cases} \tag{23}$$

The first value is due to the fluctuating string picture, the second one to the continuum two loop potential. Taking the average value of $K=0.013\pm0.003$, we obtain

$$\Lambda_L=(0.027\pm0.003)\sqrt{K}. \tag{24}$$

This is twice as large as old estimates [1, 7] and 50% higher than the value given in [3].

Tests for scaling require to correct ratios for finite "a" effects, which is possible only in low order perturbation theory. If we perform the corrections multiplicatively, we find significantly different variation with β for ratios with small areas and with large areas. The pattern of scaling violations is such that ratios at large β are too small (as compared to the value predicted by asymptotic scaling) by an amount which slightly increases with the area of the ratios and which is close to the lower value of K given in (23). A possible description of this is that the string tension vanishes relatively to the short distance Λ-parameter of the potential.

Whether this observation is a consequence of not completely removed finite "a" effects, cannot be answered convincingly until high statistical accuracy of loops with sizes 8×8 up to 10×10 becomes available. Then the dangerous loops with small lengths can be omitted from the scaling test. In the present situation we have to rely on the small loops, and we only can remark that for small loops perturbative improvement seems to work quantitatively well.

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