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Current Commutators for the Non-Linear σ-Model with Wess-Zumino Term

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Abstract. We consider equal-time commutation relations of chiral $SU(N)_L \times SU(N)_R$ charge densities in the non-linear σ-model. These commutators are derived using the cocycle formalism and from the usual canonical theory. Both methods give the same result. The charge density commutator of the symmetry currents contains operator valued Schwinger terms arising from the Wess-Zumino term.

1. Introduction

Many years ago, Wess and Zumino [1] derived a low energy effective action for pseudoscalar mesons in the presence of external gauge fields. The result was given in terms of a certain five-dimensional integral containing the Bardeen anomaly [2]. It was observed that the integral is non-vanishing, even when the gauge fields go to zero. This interaction term (in the following referred to as WZT) induced abnormal parity interactions such as $K^+K^- \to \pi^+\pi^-\pi^0$ and, if coupled to an external photon field, processes as $\pi^0 \to 2\gamma$ and $\gamma \to \pi^+\pi^-\pi^0$ etc. Further phenomenological discussion can be found in [3].

Witten [4] has given a simple intuitive reason for the presence of the Wess-Zumino action, namely, that it removes a certain symmetry from the non-linear σ-model. This symmetry, $\pi_i \to -\pi_i$, where π_i is the octet of pseudoscalar fields in the case of $SU(3)_L \times SU(3)_R$ chiral symmetry, for example, is not a symmetry of QCD and, therefore, should not be present in the effective action. The term, which breaks this symmetry with the fewest number of derivatives, in order to act as a low-energy effective action, is the Wess-Zumino action with gauge fields set equal to zero. In the equations of motion the WZT makes a local contribution, which, however, cannot be derived from a local Lagrangian. Witten [4] showed that it can be written more symmetrically as an integral over a five-disk. In this way it is seen that the action depends on the orientation of the five-disk. This leads to an ambiguity which depends on the winding number of the pseudoscalar meson configuration, and has the consequence that the overall coefficient is equal to an integer up to a known normalization constant.

In the σ-model the WZT induces modifications of the conventional $SU(3)_L \times SU(3)_R$ currents. The complete current $J_\mu^{R,L}(x)$ is a sum of two parts, the old current $j_\mu^{R,L}(x)$ and the anomalous current $\tilde{j}_\mu^{R,L}(x)$ originating from the WZT

$$J_\mu^{R,L}(x) = j_\mu^{R,L}(x) + \tilde{j}_\mu^{R,L}(x). \tag{1.1}$$

These complete currents have been written down by Witten [4] as a function of the non-linear field U. Their expansion in terms of physical particle states can be found in [3]. One might expect that the WZT anomaly not only modifies the currents but the equal-time current commutation relations as well. Several authors considered these commutators for the non-linear σ-model with WZT. Bars [5] has calculated the equal-time commutators in two dimensions and obtained the modified current algebra. In particular he obtained c-number Schwinger terms also in the local charge commutators. They are proportional to the quantized interaction strength of the Wess-Zumino term. Rajeev [6] has given the modification of the equal-time commutators of the naive currents $j_\mu^{R,L}(x)$ in the presence of a WZT in four

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dimensions. Under the assumption that the Hamiltonian for this theory has the Sugawara [7] form in terms of the $j_\mu^{R,L}$ be could show that his commutators lead to the correct equations of motion. These commutators for the $j_\mu^{R,L}$-currents contain anomalous pieces proportional to the WZT strength. His algebra, however, is not the algebra of the physical currents $J_\mu^{R,L}(x)$, the currents that couple to the electroweak interactions and which enter into the usual current algebra formulation of low energy processes. One of us and Palmer [8] calculated $[J_0^{R,L}(x), J_0^{R,L}(y)]$ at equal-time starting from the commutator of $J_0^{R,L}(x)$ with the non-linear field U as given by the properties of U under $SU(N)_L \times SU(N)_R$ transformations. These modified commutators have operator values Schwinger terms which are functionals of the non-linear field U.*

From a completely different viewpoint Fadeev and Shatashvili [9] have shown that a unified mathematical description of anomalies can be given in terms of cocycles. Considering as underlying theory massless chiral fermions interacting with a Yang-Mills field the second cocycle, for example, is in infinitesimal form equal to the anomalous equal-time commutator of the gauge generators in odd-dimensional space, if a Hamiltonian description in even-dimensional space is given. The odd-dimensional second cocycle is derived from the Chern-Pontryagin density in three dimensions higher. This dimensional descend is produced by the coboundary operation. It would be interesting to see how these different derivations of anomalous equal-time commutators are related and whether they produce the same result for $[J_0^{R,L}(x), J_0^{R,L}(y)]$. This is the purpose of this paper.

First we study how the Schwinger terms for the non-linear σ-model with WZ anomaly emerge in the approach of Fadeev and Shatashvili [9]. Then we derive the equal-time commutators of the naive currents $j_0^{R,L}$ starting from canonical commutation relations. This way we obtain Rajeev's commutators without any additional assumptions. They can be used to derive the commutators of the full currents.

The outline of the paper is as follows. In Sect. 2 we derive the anomalous equal-time commutators on the basis of the topological approach following closely the work of Fadeev and Shatashvili [9]. The canonical theory is developed in Sect. 3. Here we rely on the differential geometric approach as introduced by one of us many years ago [10]. The connection of the commutators for the naive currents $j_\mu^{R,L}(x)$ with the full current is also established in this section. We close with a summary and some concluding remarks in Sect. 4.

* The anomalous contributions to the time-time component commutator of the current, under discussion here, are usually also called "Schwinger terms". This may be confusing, but is common practice now. Originally Schwinger studied non-canonical contributions to the commutator between time and space components of the current

2. Topology

An anomalous term in the time-time commutator of the current algebra is related to a projective representation of the gauge group on the space of functionals depending on gauge potentials in three-dimensional space [9]. With $U(g)$ implementing a representation of the gauge group element g the composition law reads

$$U(g_1)U(g_2)=\exp(i\alpha_2(A;g_1,g_2))U(g_1g_2). \tag{2.1}$$

$A=A^aX^a$ is the gauge field with values in the Lie algebra $su(N)$ of $SU(N)$. We shall use an anti-Hermitean basis X^a of $su(N)$ satisfying

$$[X^a,X^b]=f^{abc}X^c; \quad \mathrm{Tr}(X^aX^b)=-\tfrac{1}{2}\delta^{ab}. \tag{2.2}$$

(In terms of the Gell-Mann matrices λ^a we have $X^a=\lambda^a/2i$.) The expansion

$$g(\mathbf{x})=1+\vartheta^a(\mathbf{x})X^a+\dots, \tag{2.3a}$$

$$U(g)=1-i\int d^3x\,\vartheta^a(\mathbf{x})J_0^a(\mathbf{x})+\dots \tag{2.3b}$$

leads to the equal-time commutation relations [9]

$$[J_0^a(\mathbf{x}),J_0^b(\mathbf{y})] =if^{abc}J_0^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})+S^{ab}(A;\mathbf{x},\mathbf{y}). \tag{2.4}$$

The anomalous Schwinger term S^{ab} can be calculated by expansion of the phase α_2 in (2.1).

It has been shown that the phase $\alpha_2(A;g_1,g_2)$ is a 2-cocycle with regard to the cohomology of the gauge group [9]. α_2 can be traced back to a 0-cocycle $\alpha_0(A)$ that does not depend on gauge group elements. The latter is a functional of gauge fields A in five-dimensional space. It can be expressed by the integral of a 5-form $\Omega_5(A)$ over five-dimensional space,

$$\alpha_0(A)=2\pi\int\Omega_5(A). \tag{2.5}$$

Ω_5 is the so-called Chern-Simons density, related locally to the Chern density Ω_6 by exterior derivation

$$\Omega_6=-\frac{i}{48\pi^3}\mathrm{Tr}\,F^3=d\Omega_5, \tag{2.6a}$$

$$\Omega_5=-\frac{i}{48\pi^3}\mathrm{Tr}(AF^2-\tfrac{1}{2}FA^3+\tfrac{1}{10}A^5). \tag{2.6b}$$

In (2.6a, b) F is the curvature (field strength) of the connection (gauge field) A,

$$F = dA + A \wedge A \tag{2.7}$$

where $A \wedge A$ is a matrix product in $su(N)$ and a wedge product with respect to differential forms. The symbol $\wedge$ is omitted in (2.6a, b) to simplify the notation. To make contact with WZT, which will be studied in the next section, we consider a flat connection with $F=0$. Then Ω_5 is closed and is an element of the cohomology class $H^5(M)$, where M is the five-dimensional base space of the $SU(N)$ principal bundle. For a flat connection we have

$$F = dA + A \wedge A = 0 \tag{2.8}$$

so that the connection A is*

$$A = U^{-1} dU = \omega(R) \quad \text{with} \quad U \in SU(N) \tag{2.9}$$

i.e. we can identify the connection A with the left-invariant Maurer-Cartan form $\omega(R)$ on the group-manifold of $SU(N)$ restricted to a five-dimensional submanifold. Hence

$$\Omega_5(\omega) = -\frac{i}{480\pi^3} \operatorname{Tr} \omega^5. \tag{2.10}$$

The 0-cocycle $\alpha_0(\omega)$ is an integral number

$$\alpha_0(\omega)/2\pi = \int \Omega_5(\omega)$$
$$= -\frac{i}{480\pi^3} \int \operatorname{Tr} \omega^5 = n(U) \in \mathbb{Z} \tag{2.11}$$

$n(U)$ is the degree of mapping U: $M \to SU(N)$. If we take $M \cong S^5$ we know from Bott's periodicity theorem that the homotopy group is isomorphic to the group of integral numbers: $\Pi_5(SU(N)) \cong \mathbb{Z} (N > 5/2)$. The normalization factor of Ω_5 in (2.10) corresponds to the axiomatic normalization of the Chern classes (see e.g. [11]).

Starting from the 5-form $\Omega_5(A)$ in (2.6b) we arrive at the 3-form

$$\Omega_3(A; g_1, g_2) = -\frac{i}{12(2\pi)^3} \operatorname{Tr}\{A(dg_1 dg_2 g_2^{-1} g_1^{-1}$$
$$- g_1 dg_2 g_2^{-1} g_1^{-1} dg_1 g_1^{-1}\} + \text{terms independent of } A \tag{2.12}$$

by a twofold application of the group coboundary operator followed each time by an inversion of the spatial exterior derivative [9]. Integrating $2\pi\Omega_3$ over three-dimensional space we obtain the 2-cocycle α_2,

$$\alpha_2(A; g_1, g_2) = 2\pi \int \Omega_3(A; g_1, g_2). \tag{2.13}$$

* The notation $\omega(R)$ indicates that $\omega(R)$ is changed under right chiral transformations

Inserting the expansion (2.3a, b) yields the anomalous Schwinger term

$$S^{ab}(A; \mathbf{x}, \mathbf{y}) = -\frac{1}{24\pi^2} \operatorname{Tr}\{[X^a, X^b]_+ X^c\}$$
$$\cdot \varepsilon^{ijk} \partial_i A_j^c(\mathbf{x}) \partial_k^x \delta(\mathbf{x}-\mathbf{y}); \quad i,j,k = 1,2,3 \tag{2.14}$$

or for the flat connection (2.9)

$$S^{ab}(\omega; \mathbf{x}, \mathbf{y})$$
$$= \frac{1}{24\pi^2} \operatorname{Tr}\{[X^a, X^b]_+ \omega_i \omega_j\} \varepsilon^{ijk} \partial_k^x \delta(\mathbf{x}-\mathbf{y}) \tag{2.15}$$

with $\omega_i = U^{-1} \partial_i U$. The terms independent of A in (2.12) do not contribute. To obtain (2.15) from (2.14) we used $\partial_i \omega_j(\mathbf{x}) = -\omega_i \omega_j + U^{-1} \partial_i \partial_j U$ (see also (3.3a) below). It is remarkable that the explicit form of the Schwinger term in the time-time component current commutator can be obtained from topological structures without going back to the explicit form of the Lagrangian. In the next section we shall see how this commutator can be derived from the Lagrangian on the basis of the canonical commutation relations for the pseudoscalar fields.

3. Canonical Theory

In the canonical theory we have to introduce coordinates on the group manifold $SU(N)$ that parameterize the matrices U, e.g. we may use the perturbative expansion near the unit matrix

$$U(\pi) = \exp\left(-\frac{4}{F_\pi} \pi^a X^a\right). \tag{3.1}$$

For the case of $SU(3)$ symmetry $\{\pi^a\}$ $(a = 1, 2, \ldots, 8)$ is the octet of pseudoscalar mesons and $F_\pi = 186$ MeV. With respect to the coordinates π^i the Maurer-Cartan form (2.9) can be decomposed as

$$\omega = U^{-1} dU = X^a \omega^a = X^a \omega_i^a d\pi^i. \tag{3.2}$$

The coefficients ω^a are 1-forms on the group manifold and obey the structure relations

$$d\omega = -\omega \wedge \omega, \tag{3.3a}$$

$$\Leftrightarrow d\omega^a = -\tfrac{1}{2} f^{abc} \omega^b \wedge \omega^c, \tag{3.3b}$$

$$\Leftrightarrow \omega_{i|j}^a - \omega_{j|i}^a = f^{abc} \omega_i^b \omega_j^c. \tag{3.3c}$$

From (3.1) we obtain the power series

$$\omega_i^a(\pi) = -\frac{4}{F_\pi} \delta_i^a + \frac{8}{F_\pi^2} f^{iba} \pi^b + \ldots. \tag{3.4}$$

The duals of the left-invariant 1-forms ω^a are the left-invariant vector fields V^a,

$$V^a=\xi^{ai}\frac{\partial}{\partial\pi^i},\qquad \xi^{ai}\omega_i^b=\delta^{ab}. \tag{3.5}$$

They satisfy Lie-algebra commutation relations dual to (3.3a–c)

$$[V^a,V^b]=f^{abc}V^c \tag{3.6a}$$

$$\Leftrightarrow \xi^{aj}\frac{\partial}{\partial\pi^j}\xi^{bi}-\xi^{bj}\frac{\partial}{\partial\pi^j}\xi^{ai}=f^{abc}\xi^{ci}. \tag{3.6b}$$

Let us now turn to the Lagrangian. We write

$$L=L_0+L_a \tag{3.7}$$

where L_0 is the normal and L_a the anomalous part. The normal part is based on the canonical metric of $SU(N)$. A convenient normalization of the metric g_{ij} is

$$g_{ij}(\pi)=-\frac{F_\pi^2}{8}\,\mathrm{Tr}(\omega_i\omega_j)=\delta_{ij}+\dots. \tag{3.8}$$

Hence we take

$$\begin{aligned}L_0&=-\frac{F_\pi^2}{16}\int d^4x\,\mathrm{Tr}\{U^{-1}\partial_\mu U U^{-1}\partial^\mu U\}\\ &=-\frac{F_\pi^2}{16}\int d^4x\,\mathrm{Tr}\{\omega_i\omega_j\}\partial_\mu\pi^i\partial^\mu\pi^j\\ &=\tfrac{1}{2}\int d^4x\,g_{ij}(\pi)\partial_\mu\pi^i\partial^\mu\pi^j.\end{aligned} \tag{3.9}$$

The action L_0 leads to the field equations

$$\partial^\mu(\omega_i\partial_\mu\pi^i)=\partial^\mu\omega_\mu=0 \tag{3.10}$$

with $\omega_\mu=\omega_i\partial_\mu\pi^i$. We obtain, of course, the same Lagrangian, if we use the right-invariant form

$$\omega(L)=dUU^{-1},\qquad d\omega(L)=\omega(L)\wedge\omega(L) \tag{3.11}$$

instead of (3.2). The right-invariant fields $V^a(L)$ satisfy

$$[V^a(L),V^b(L)]=-f^{abc}V^c(L). \tag{3.12}$$

The Lagrangian L_0 is clearly invariant under the chiral group $SU(N)_L\times SU(N)_R\equiv G$ acting on the matrices U as follows

$$G\ni(g_1,g_2):U\to g_1Ug_2^{-1}. \tag{3.13}$$

The corresponding conserved right- and left-handed currents are

$$j_\mu^a(R)=g_{ij}\partial_\mu\pi^i t^{aj}(R), \tag{3.14a}$$

$$j_\mu^a(L)=g_{ij}\partial_\mu\pi^i t^{aj}(L) \tag{3.14b}$$

where t^{aj} are the components of the tangent vectors to the transformations (3.13)

$$t^{aj}=\frac{\partial\pi^j}{\partial\vartheta^a},\qquad g=1+\vartheta^aX^a+\dots. \tag{3.15}$$

One easily shows

$$t^{aj}(L)=\xi^{aj}(L),\qquad t^{aj}(R)=-\xi^{aj}(R). \tag{3.16}$$

We then have from (3.5) and (3.8)

$$j_\mu^a(R)=-\frac{F_\pi^2}{16}\omega_\mu^a(R),\qquad j_\mu^a(L)=\frac{F_\pi^2}{16}\omega_\mu^a(L). \tag{3.17}$$

The definitions (3.14a, b) correspond to those in [3], the axial current A_μ is

$$A_\mu^a=\frac{F_\pi}{2}\partial_\mu\pi^a+\dots. \tag{3.18}$$

The generators of the symmetry are $Q_{R,L}^a=\int d^3x\,j_0^a(R,L)$, (see also (2.3b)).

In the quantum theory the equal-time commutators for the currents can be reduced to the canonical commutation relations

$$[p_j(\mathbf{x}),\pi^k(\mathbf{y})]=\frac{1}{i}\delta_j^k\delta(\mathbf{x}-\mathbf{y}) \tag{3.19}$$

by means of the structure relations (3.6) resp. (3.12). The conjugate field following from L_0 is

$$p_j(\mathbf{x})=g_{jk}\partial_0\pi^k=\partial_0\pi_j. \tag{3.20}$$

We obtain for right- and left-handed currents [10]

$$\begin{aligned}[j_0^a(\mathbf{x}),j_0^b(\mathbf{y})]&=i(\xi^{bj}\partial_j\xi^{ai}-\xi^{aj}\partial_j\xi^{bi})\\ &\cdot\partial_0\pi_i(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})=if^{abc}j_0^c(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}),\end{aligned} \tag{3.21a}$$

$$\begin{aligned}[j_0^a(\mathbf{x}),j_p^b(\mathbf{y})]&=if^{abc}j_p^c(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y})\\ &+i\delta^{ab}\frac{F_\pi^2}{16}\delta_p^x\delta(\mathbf{x}-\mathbf{y}),\qquad p=1,2,3.\end{aligned} \tag{3.21b}$$

The anomalous part in Witten's effective Lagrangian is related to the Chern-Simons density Ω_5 ((2.10)). According to Witten [4] we may write

$$L_a=2\pi N\int_{Q^5}\Omega_5(\omega)=-\frac{iN}{240\pi^2}\int_{Q^5}\mathrm{Tr}\,\omega^5 \tag{3.22}$$

where N is an integral number equal to the number of colours and Q^5 is a five-dimensional disk in the group manifold whose boundary is four-dimensional space-time: $\partial Q^5=M^4$. We know already that Ω_5 is closed but not exact. Nevertheless we can assume that the submanifold $Q^5\subset SU(N)$ is completely covered by an allowable chart of the group manifold. We then introduce a 4-form D_4 that satisfies on Q^5

$$dD_4=\mathrm{Tr}\,\omega^5 \tag{3.23}$$

(D_4 is, of course, not unique). We obtain from (3.22) by Stoke's theorem

$$L_a=\lambda \int_{Q^5} dD_4=\lambda \int_{M^4} D_4; \quad \lambda=\frac{-iN}{240\pi^2}. \tag{3.24}$$

In terms of local coordinates π^i (not necessarily those defined by (3.1)) D_4 can be expressed as

$$D_4=\sum_{i<j<k<l} D_{ijkl} d\pi^i \wedge d\pi^j \wedge d\pi^k \wedge d\pi^l$$
$$=\frac{1}{4!}\sum_{i,j,k,l} D_{ijkl} d\pi^i \wedge d\pi^j \wedge d\pi^k \wedge d\pi^l \tag{3.25}$$

where the coefficients of the second representation are totally antisymmetric. We parametrize the submanifold M^4 by $\{x_\mu\}$ ($\mu=0,1,2,3$) and write the Lagrangian (3.24) in terms of the fields $\pi^i(x)$,

$$L_a=\lambda \int_{M^4} D_4=\frac{\lambda}{4!}\int d^4x D_{ijkl}(\pi)$$
$$\cdot \partial_\mu \pi^i \partial_\nu \pi^j \partial_\rho \pi^k \partial_\sigma \pi^l \varepsilon^{\mu\nu\rho\sigma}. \tag{3.26}$$

It is not difficult to show that the Lagrangian $L=L_0+L_a$ leads to the field equations

$$\partial^\mu(g_{ij}(\pi)\partial_\mu \pi^j)+5\lambda \operatorname{Tr}\{\omega_i\omega_j\omega_k\omega_l\omega_m\}$$
$$\cdot \partial_\mu \pi^j \partial_\nu \pi^k \partial_\rho \pi^l \partial_\sigma \pi^m \varepsilon^{\mu\nu\rho\sigma}=0. \tag{3.27}$$

Because of (3.8) they are equivalent to the following coordinate-free form

$$\partial^\mu\left(-\frac{F_\pi^2}{8}\omega_\mu\right)+5\lambda\omega_\mu\omega_\nu\omega_\rho\omega_\sigma\varepsilon^{\mu\nu\rho\sigma}=0. \tag{3.28}$$

Equation (3.28) means conservation of the total right-handed current $J_\mu^a(R)$ ([3]),

$$J_\mu^a(R)=\frac{F_\pi^2}{8}\operatorname{Tr}(X^a\omega_\mu(R))$$
$$+5\lambda \operatorname{Tr}\{X^a\omega^\nu(R)\omega^\rho(R)\omega^\sigma(R)\}\varepsilon_{\mu\nu\rho\sigma}$$
$$=j_\mu^a(R)+5\lambda\left(-\frac{16}{F_\pi^2}\right)^3 \operatorname{Tr}\{X^aX^bX^cX^d\}$$
$$\cdot j_\nu^b(R) j_\rho^c(R) j_\sigma^d(R)\,\varepsilon_\mu{}^{\nu\rho\sigma}. \tag{3.29}$$

The corresponding expression for the left-handed current is [3]

$$J_\mu^a(L)=-\frac{F_\pi^2}{8}\operatorname{Tr}(X^a\omega_\mu(L))$$
$$+5\lambda \operatorname{Tr}\{X^a\omega^\nu(L)\omega^\rho(L)\omega^\sigma(L)\}\varepsilon_{\mu\nu\rho\sigma}$$
$$=j_\mu^a(L)+5\lambda\left(\frac{16}{F_\pi^2}\right)^3 \operatorname{Tr}\{X^aX^bX^cX^d\}$$
$$\cdot j_\nu^b(L) j_\rho^c(L) j_\sigma^d(L)\,\varepsilon_\mu{}^{\nu\rho\sigma}. \tag{3.30}$$

The first terms in (3.29) and (3.30), respectively, are the normal currents and the second terms are the anomalous parts.

Let us now turn to the equal-time commutator $[J_0^a(\mathbf{x}), J_0^b(\mathbf{y})]$. We see from (3.29) and (3.30) that the latter can be reduced to the commutators

$$[j_0^a(\mathbf{x}), j_0^b(\mathbf{y})] \quad \text{and} \quad [j_0^a(\mathbf{x}), j_p^b(\mathbf{y})] \quad (p=1,2,3),$$

which in turn can be calculated via the caonical commutation relations. To do so we need the momenta p_i for the full Lagrangian $L=L_0+L_a$. We obtain from (3.9) and (3.26)

$$p_i=\partial_0\pi_i+\frac{\lambda}{3!}D_{ijkl}\partial_\nu\pi^j\partial_\rho\pi^k\partial_\sigma\pi^l\varepsilon^{0\nu\rho\sigma}. \tag{3.31}$$

The important point to note is that the momenta are no longer equal to the velocities. The additional term in (3.31) does not depend on the velocities and corresponds to the vector potential for a charged particle moving in an electromagnetic field.

Right- and left-handed components j_0^a are defined in terms of the velocities as before (3.14a, b). Consequently we get instead of (3.21a)

$$[j_0^a(\mathbf{x}), j_0^b(\mathbf{y})]=if^{abc}j_0^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})$$
$$+\xi^{ai}(\pi(\mathbf{x}))[\partial_0\pi_i(\mathbf{x}),\partial_0\pi_j(\mathbf{y})]\xi^{bj}(\pi(\mathbf{y})) \tag{3.32}$$

while the commutators (3.21b) remain unchanged. Because of (3.31) the commutator of the velocities does not vanish as in the case with no anomalous term. Using (3.31), (3.23) and the canonical commutation relations we obtain

$$[\partial_0\pi_i(\mathbf{x}),\partial_0\pi_j(\mathbf{y})]=20i\lambda \operatorname{Tr}\{\omega\lfloor_i\omega_j\omega_k\omega_l\omega_m\rfloor\}$$
$$\cdot\partial_\nu\pi^k\partial_\rho\pi^l\partial_\sigma\pi^m\varepsilon^{0\nu\rho\sigma}\delta(\mathbf{x}-\mathbf{y}). \tag{3.33}$$

(The symbol $\lfloor\ldots\rfloor$ means total antisymmetrization of all indices between the brackets, i.e. application of $1/5!\sum_P \delta_P$ where P is the permutation of 5 elements and δ_P is the signature of P).

Inserting (3.33) into (3.32) and observing (3.5) we finally get

$$[j_0^a(\mathbf{x}), j_0^b(\mathbf{y})]=if^{abc}j_0^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})$$
$$+5i\lambda \operatorname{Tr}\{X^aX^b\omega_\nu\omega_\rho\omega_\sigma-X^a\omega_\nu X^b\omega_\rho\omega_\sigma$$
$$+X^a\omega_\nu\omega_\rho X^b\omega_\sigma-X^a\omega_\nu\omega_\rho\omega_\sigma X^b\}\varepsilon^{0\nu\rho\sigma}\delta(\mathbf{x}-\mathbf{y}). \tag{3.34}$$

These are exactly the commutation relations proposed by Rajeev [6]. He assumed that the Hamiltonian H corresponding to $L=L_0+L_a$ has the Sugawara form [7] in terms of the normal currents j_μ^a. With this assumption he showed that (3.34) and (3.21b) produce the correct equations of motion, i.e. (3.28).

Rajeev's assumption about the current times current form can easily be derived in the formalism presented here. Indeed, using (3.31) it can be seen that the anomalous term in L cancels out and the Hamiltonian has the form

$$H=\frac{8}{F_\pi^2}\int d^3x\{j_0^a(\mathbf{x})j_0^a(\mathbf{x})+j_p^a(\mathbf{x})j_p^a(\mathbf{x})\}. \tag{3.35}$$

Thus, although the structure of $L=L_0+L_a$ is quite complicated, i.e. L contains the anomalous Wess-Zumino term, the Hamiltonian expressed in terms of the normal currents looks extremely simple. The anomaly appears only in the commutation relations (3.34) which again can be expressed solely by the normal currents.

Now we turn to the calculation of the equal-time commutator of the complete current $J_0^a(\mathbf{x})$. This is obtained from

$$\begin{aligned}[J_0^a(\mathbf{x}), J_0^b(\mathbf{y})] &= [j_0^a(\mathbf{x}), j_0^b(\mathbf{y})] + 5\lambda\left(-\frac{16}{F_\pi^2}\right)^3 \varepsilon_0{}^{\nu\rho\sigma} \\ &\cdot \{[j_0^a(\mathbf{x}), j_\nu^{b'}(\mathbf{y}) j_\rho^{c'}(\mathbf{y}) j_\sigma^{d'}(\mathbf{y})]\,\mathrm{Tr}(X^b X^{b'} X^{c'} X^{d'}) \\ &+ \mathrm{Tr}(X^a X^{b'} X^{c'} X^{d'})[j_\nu^{b'}(\mathbf{x}) j_\rho^{c'}(\mathbf{x}) j_\sigma^{d'}(\mathbf{x}), j_0^b(\mathbf{y})]\}.\end{aligned} \tag{3.36}$$

The first term on the right-hand side of (3.36) is given by (3.34). The remaining term can be calculated with the help of (3.21b) since the anomalous part of J_0^a contains only space components of j_μ^a. After some lengthy computations using (3.21b) and (3.3a) we arrive at

$$\begin{aligned}&[J_0^a(\mathbf{x}), J_0^b(\mathbf{y})] \\ &= [j_0^a(\mathbf{x}), j_0^b(\mathbf{y})] + 5i\lambda f^{abc}\left(-\frac{16}{F_\pi^2}\right)^3 \varepsilon_0{}^{\mu\rho\sigma} \\ &\quad\cdot \mathrm{Tr}(X^c X^{b'} X^{c'} X^{d'}) j_\nu^{b'}(\mathbf{x}) j_\rho^{c'}(\mathbf{x}) j_\sigma^{d'}(x)\,\delta(\mathbf{x}-\mathbf{y}) \\ &\quad -5i\lambda\left(-\frac{16}{F_\pi^2}\right)^3 \varepsilon_0{}^{\nu\rho\sigma}\mathrm{Tr}\{X^a X^b X^{b'} X^{c'} X^{d'} \\ &\quad - X^a X^{b'} X^b X^{c'} X^{d'} + X^a X^{b'} X^c X^b X^{b'} \\ &\quad - X^a X^{b'} X^{c'} X^{d'} X^b\} j_\nu^{b'}(\mathbf{x}) j_\rho^{c'}(\mathbf{x}) j_\sigma^{d'}(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}) \\ &\quad +10i\lambda\left(\frac{16}{F_\pi^2}\right)^2 \varepsilon_0{}^{\nu\rho\sigma}\mathrm{Tr}\{[X^a, X^b]_+ X^c X^d\} \\ &\quad \cdot j_\nu^c(\mathbf{x}) j_\rho^d(\mathbf{x})\,\partial_\sigma^x \delta(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{3.37}$$

Now we insert (3.34) into (3.37). Then most of the terms cancel and we obtain

$$\begin{aligned}[J_0^a(\mathbf{x}), J_0^b(\mathbf{y})] &= i f^{abc} J_0^c(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}) \\ &+ 10i\lambda\varepsilon_0^{\nu\rho\sigma}\mathrm{Tr}\{[X^a, X^b]_+ \omega_\nu \omega_\rho\}\,\partial_\sigma \delta(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{3.38}$$

The second term in (3.38) is the operator valued Schwinger term arising from the anomaly. It agrees exactly with the result (2.15) obtained on the basis of topological considerations and also with the result in [8] derived from the transformation properties of U. The derivations in this section also show that (3.38) is consistent with Rajeev's anomalous term in the commutator $[j_0^a(\mathbf{x}), j_0^b(\mathbf{y})]$.

In this and the previous section we considered only the commutators of the right-handed current. The derivations for the left-handed or mixed commutators are analogous. With them the complete current algebra for vector and axial currents can be derived.

4. Summary

We have calculated the equal-time commutation relations of the chiral $SU(N)_L \times SU(N)_R$ charge densities in the non-linear σ-model with Wess-Zumino term. We employed two completely different formalisms. First, using the cohomology of the gauge group the anomalous term in the commutator is obtained from the 2-cocycle α_2. Secondly, the canonical theory for the non-linear σ-model with Wess-Zumino term is formulated. From this the commutator of the normal part of the charge densities is deduced. It has a very simple structure. The modification due to the anomaly is trilinear in the normal currents. In these currents the Hamiltonian has the current times current form as in the case with no anomaly. This remarkable simple structure for the equal-time commutators and the Hamiltonian suggests that further algebraic reduction of the model could be possible.

Finally the commutator of the complete charge densities, normal plus anomalous part, is calculated with the known commutators of the normal currents as input. The resulting operator valued Schwinger term agrees with that obtained from topology.

After completion of this work we learned that the calculation of the anomalous commutator has been attempted in perturbation theory by Jo [12] and by Kobayashi and Sugamoto [13] with differing results. Sonoda obtained the correct result of Fadeev and Shatashvili by computing Berry's phase in chiral gauge theories.

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