A SCAN FOR MODELS WITH REALISTIC FERMION MASS PATTERNS

J. BIJNENS [1] and C. WETTERICH
Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg, Fed. Rep. Germany

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Models are considered which have no small Yukawa couplings unrelated to symmetry. This situation is generic in higher dimensional unification where Yukawa couplings are predicted to have a strength similar to the gauge couplings. Generations have then to be differentiated by symmetry properties and the structure of fermion mass matrices is given in terms of quantum numbers alone. Possible symmetries leading to realistic mass matrices are scanned.

Most of the free parameters of the standard SU(3) × SU(2) × U(1) model are Yukawa couplings between quarks, leptons and the Higgs scalar. Ideas of further unification of all forces aim for an explanation of those Yukawa couplings and thereby a resolution of the old puzzle about the origin of the difference between muon and electron. In particular, unification in more than four dimensions relates the number of generations to topological properties of internal space [1]. As a consequence, the differentiations between generations should also be explained by symmetries and topology of internal space (including ground state configurations of other bosonic fields) [2–4].

In this letter we describe a computerized search for realistic fermion mass matrices whose structure is entirely explained by quantum numbers of quarks and leptons. Although motivated by higher dimensional theories the framework of our discussion is in four dimensions. Our central assumption is that all Yukawa couplings are of the same order as the gauge coupling g unless they are zero because of symmetry or topology (this is the generic situation resulting from higher dimensional unification). If generations are not distinguished by the order of magnitude of their Yukawa couplings, they must be differentiated by some symmetry G larger than SU(3) × SU(2) × U(1). Such a symmetry G may consist of local or global continuous symmetries or be discrete.

In the limit of unbroken G the top quark and the electron must couple to different scalar doublets d_i, which are distinguished by their G transformation properties – otherwise our assumption implies that the electron couples with a Yukawa coupling of order g to the (VEV) responsible for the top quark mass m_t and therefore $m_e \approx m_t$ in contradiction to observation. The Higgs doublet ϕ responsible for weak symmetry breaking will in general not be in a definite representation of G, but rather consist of some linear combination of fields d_i, $\phi = \sum_i \gamma_i^* d_i$.

The basis of mass eigenstates (ϕ, heavy doublets) differs from the basis of fields d_i with given G transformation properties. Re-expressed in the d_i basis the VEV's are

$$\langle d_i \rangle = \gamma_i \langle \phi \rangle \approx \gamma_i \times 174 \text{ GeV}. \quad (1)$$

Masses of quarks and leptons are given by the product of a Yukawa coupling (of order g) and the VEV $\langle d_i \rangle$ which couples to them. The apparent small Yukawa coupling for the electron is caused by the small mixing coefficient γ_i for the corresponding doublet.

In the limit of unbroken G the doublets d_i cannot mix because of their different G quantum numbers and all γ_i vanish except one ($\gamma_0 = 1$). (This "leading" doublet should only couple to the heaviest fermion, i.e. the top quark in the three-generation case). Nonvanishing mixings γ_i are induced by symmetry breaking of G. Let us denote by M the typical scale of mass terms for the

[1] Ere aspirant, NFWO, Belgium.

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doublets d_i in the limit of unbroken G and by M_G the scale of symmetry breaking of G. The dimensionless γ_i are then of order

$$\gamma_i \approx (M_G/M)^{P_i}, \qquad (2)$$

with P_i some integer calculable from symmetry considerations [4]. A small ratio M_G/M should be responsible for all small quantities in the fermion mass matrices ‡1.

There is a first necessary criterion for these ideas to work: the symmetry G must differentiate enough between the various quarks and leptons to allow for realistic structures of the mass matrices. There should be at least one choice of $\langle d_i \rangle$ (the calculations of these VEV's (or γ_i) are not attempted at this stage) which produces correctly all orders of magnitude for the various elements of the quark and lepton mass matrices. This requirement leads to many restrictions on possible quantum numbers of fermions with respect to G. For a given symmetry G and given representations under G for the quarks and leptons a computerized scan for acceptable mass matrices becomes possible and useful in this context. For this we require that all elements in the mass matrices that differ by an order of magnitude are produced by a different scale of VEV $\langle d_i \rangle$, which VEV produces a given element in a mass matrix being determined by the symmetry G. This concerns masses as well as mixing angles differing by orders of magnitude.

Our scanning program is based on the observation that for the three-generation case the structure of fermion mass matrices is well described by four (or five) scales. These scales are separated by about an order of magnitude. The highest scale is the top quark mass, which we assume to be several tens GeV. The second scale is a few GeV, where we find the bottom quark, the charm quark and the tau lepton. The strange quark and the muon constitute the third level of a few hundred MeV. The fourth level consists of the up and down quarks and the electron below a few tens MeV (one may argue that a fifth scale below a few MeV is needed for the electron mass) ‡2. The only other information about the mass matrices comes from the measured mixing angles. We have (a fairly large) Cabibbo angle and a few percent of mixing between the second and third generation. The limit on mixing between the first and third generation is somewhat less than one percent. No information about the lepton mass matrix besides its eigenvalues is available. We will denote these scales by n_s, every scale being a few 10^{n_s} MeV. So the n_s are

$$m_t : 4; \quad m_b,\ m_c,\ m_\tau : 3; \quad m_\mu,\ m_s : 2;$$
$$m_u,\ m_d,\ m_e : 1. \qquad (3)$$

Upper bounds on the size of the elements of the mass matrices are given by

$$M_U \lesssim \begin{pmatrix} 4 & 3 & 2 \\ 4 & 3 & 2 \\ 4 & 3 & 1 \end{pmatrix}, \quad M_D \lesssim \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} \qquad (4a,b)$$

$$M_L \lesssim \begin{pmatrix} 3 & 2 & 2 \\ 3 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \qquad (4c)$$

The mass matrices (4a)–(4c) have been ordered here in the standard way. The bounds on M_{11} come from the maximal size of the mass eigenvalues. An entry of this size is also required to produce the heaviest mass. The bounds on M_{12} and M_{13} come from the observed small mixing with the third generation. The bounds on M_{21} and M_{31} are the same as for M_{11} since they can always

‡1 For the purpose of this letter the choice of M is arbitrary. It could be a very high unification scale – the compactification scale in higher dimensional theories, the string tension for superstrings or the GUT scale for some extended version of grand unification/family unification. In this case the structure of fermion mass matrices is related to a fine structure of scales around the unification scale [4] and only the "weak doublet" survives at low energies. The other extreme case is a low energy (≈ TeV) scale M only somewhat above the weak scale and $M_G \approx M_W$. This scenario requires several doublets in the range below a few TeV. It may be realized in supersymmetric theories with M the gravitino mass. Between these two scenarios one can of course consider a scale M and M_G in some intermediate range.

‡2 We use a normalization for the quark and lepton mass matrices at the weak scale M_W. If we assume that the size of couplings is generated at a very high scale (M_{GUT}, M_P) the different renormalization of the quarks and leptons is a factor of 2.5–3. This does not change the relative orders of magnitude.

be removed by left multiplication with a unitary matrix which is unobservable. The bounds on M_{22} and M_{33} come from the second and third generation mass and the one on M_{23} from the smallness of the Cabibbo angle. In the lepton mass matrix bounds only come from the eigenvalues. Hence, whenever M_L is acceptable, M_L^T (transposed) is too. The second and third generation masses can also be generated by paired off-diagonal elements [5] and we have to impose additional "quadratic" constraints. In terms of n_s they read

$$(M_U)_{13} + (M_U)_{31} \leqslant 5, \quad (M_U)_{23} + (M_U)_{32} \leqslant 4, \tag{5a,b}$$

$$(M_D)_{23} + (M_D)_{32} \leqslant 3, \tag{5c}$$

$$(M_L)_{13} + (M_L)_{31} \leqslant 4, \quad (M_L)_{23} + (M_L)_{32} \leqslant 3. \tag{5d,e}$$

We can now describe our scanning procedure: The fermion masses are generated level by level, starting with $n_s = 4$. At each step we try all possible assignments of the required scale n_s to various VEV's $\langle d_i \rangle$. From G symmetry we then calculate the scale of elements of the fermion mass matrices. They are of order n_s whenever the d_i chosen is allowed to couple to the corresponding fermion bilinear by G quantum numbers and zero otherwise. Consistency is then checked by comparing the pattern of scales thus generated with the bounds (4a)–(5e). A model is rejected if no consistent assignment of scales is found. We note that rows in the mass matrices and columns in M_L can be permuted arbitrarily to reach the standard pattern in (4a)–(5e). For the quark mass matrices, columns in M_U and M_D have to be permuted simultaneously in order to keep track of mixing angles.

First we look for a VEV only coupling to one column in M_U and not to M_L or M_D. This defines the top mass with $n_s = 4$. At the second level we first assign an $n_s = 3$ VEV to produce m_b. We veto if the label $n_s = 3$ then appears in more than one column in M_D or M_L and if it appears in more than one column differing from the top column in M_U. If m_τ is not produced by the same VEV as m_b we try an additional $n_s = 3$ VEV in M_L. The same procedure is then applied to m_c which can be generated either by diagonal or paired off-diagonal $n_s = 3$ entries. The combined set of all $n_s = 3$ entries is subject to the consistency veto described for m_b. At the third level we first generate m_μ by an $n_s = 2$ entry. We again allow for diagonal or paired off-diagonal entries. We veto if an $n_s = 2$ element appears in the last column of M_D or in $(M_U)_{33}$ or $(M_L)_{33}$ or if one of the quadratic bounds (5a)–(5e) is violated. If m_s is not yet generated we assign additional $n_s = 2$ entries in M_D with the same veto. At the end of this level all generation labels, t, c, u, etc. are assigned to the various rows and columns. It is now easy to check by inspection of the various $n_s = 4$, 3, 2 entries in the mass matrices if sufficient mixings θ_{23} and θ_{12} are already generated. If not, we have to assign an appropriate $n_s = 2$ or 3 entry in M_U or $n_s = 2$ or 1 in M_D to generate θ_{23}. The same holds for θ_{12} with an $n_s = 2$, 1 entry in M_U, M_D, respectively. Possible $n_s = 2$, 3 entries are, of course, subject to the same consistency requirements as earlier mentioned. Finally we check if all first generation masses can be generated by $n_s = 1$ entries. This will always be the case unless "topological" or other reasons enforce the absence of scalar doublets coupling to the first generation bilinears.

As an example we discuss a simple higher dimensional model, namely monopole solutions of the six-dimensional SO(12) theory [2]. (This can be considered as a subgroup analysis for the $E_8 \times E_8$ superstring for appropriate deformation classes of the ground state [4].) Monopole solutions with $SU(3) \times SU(2) \times U(1)$ symmetry are characterized by three integers n, m and p with $n + p$ even. We list [2] the numbers of chiral fermions with given charge $q = \pm\frac{1}{2}$ (corresponding to the abelian subgroup of SO(12) commuting with SO(10)):

$$\begin{aligned}
&q_L: [1/2(n+p)]_{1/2} + [1/2(n-p)]_{-1/2},\\
&u_L^c: [1/2(n-p+2m)]_{1/2}\\
&\qquad + [1/2(n+p-2m)]_{-1/2},\\
&d_L^c: [1/2(n-p-2m)]_{1/2}\\
&\qquad + [1/2(n+p+2m)]_{-1/2},\\
&L_L: [1/2(n-3p)]_{1/2} + [1/2(n+3p)]_{-1/2},\\
&e_L^c: [1/2(n+3p-2m)]_{1/2}\\
&\qquad + [1/2(n-3p+2m)]_{-1/2}.
\end{aligned} \tag{6}$$

Negative integers in the brackets correspond to the corresponding number of mirror particles q_L, u^c_L, ..., with charge q opposite to the indicated index. Possible mirrors q_L have therefore the same $U(1)_q$ charge as q_L and mass terms between standard fermions and mirrors require breaking of $U(1)_q$.

Besides $U(1)_q$ and $U(1)_{B-L}$ within SO(12) there is another possible abelian symmetry group $U(1)_G$ commuting with $SU(3)_c \times SU(2)_L \times U(1)_Y$. This comes from an isometry of rotations on two-dimensional internal space. The charge I with respect to $U(1)_G$ for q_L is given by ($n+p>0$, $n-p>0$)

$$I = 1/4\,(n+p) - 1/2,\, 1/4\,(n+p) - 3/2, \ldots, -1/4\,(n+p) + 1/2 \quad (q = 1/2),$$
$$I = 1/4\,(n-p) - 1/2,\, 1/4\,(n-p) - 3/2, \ldots, -1/4\,(n-p) + 1/2 \quad (q = -1/2), \qquad (7)$$

and correspondingly for mirrors and the other fermions. (I is the third component of $SU(2)_G$ spin for $SU(2)_G$ representations with dimension given in the brackets in the list (6).) We want to know if the abelian charges q and I can differentiate sufficiently between various quarks and leptons to allow for realistic mass matrices. (This is the abelian part of a more complete non-abelian analysis as sketched in ref. [4].) We restrict ourselves to the three-generation case $n = 3$.

In the first column of table 1 we give the number of acceptable solutions for various values of m and p for $n = 3$. We observe that no assignments of scales produce acceptable mass matrices for low values of m and p where we have no mirror particles. For high m and p the number of solutions increases rapidly. This is due to a large number of mirror particles. In fact, for this first investigation, we have treated the mechanism giving masses to mirrors as independent from the mechanism mixing the various doublets coupling to the "surviving" chiral fermions. This means for n_q quarks and $n_q - 3$ mirror quarks we have added the number of possible scale assignments from all patterns picking arbitrarily three "surviving" quarks out of n_q quarks and assuming that the remaining $n_q - 3$ quarks from heavy masses with the $n_q - 3$ mirror quarks. (Of course, in a more detailed analysis the SU(3) × SU(2) × U(1) singlet operators responsible for the heavy masses of mirror quarks also lead to the mixing between different doublets by G symmetry breaking – in our case $G = U(1)_q \times U(1)_G$ – and additional restrictions for acceptable models have to be imposed [4].) As an example we have found the following solution for $m = 1$, $p = 3$.

Table 1
Number of solutions for various compactifications of a six-dimensional model. n_{sol} is the number of solutions and n'_{sol} is the number of solutions with m_b and m_τ generated by the same VEV

m	p	n_{sol}	n'_{sol}
−3	1	4750	1386
−2	1	0	0
−1	1	0	0
0	1	0	0
1	1	0	0
2	1	0	0
3	1	72	72
−3	3	⩾ 100000	0
−2	3	⩾ 100000	58724
−1	3	26856	0
0	3	0	0
1	3	3132	1796
2	3	22636	13324
3	3	12840	8780

$$M_U:\ \begin{pmatrix} (1,-1) & (1,1) & (1,0) \\ (0,-1/2) & (0,3/2) & (0,1/2) \\ (0,-3/2) & (0,1/2) & (0,-1/2) \end{pmatrix},$$
$$\begin{pmatrix} 4 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad (8a)$$

$$M_D:\ \begin{pmatrix} (0,-3/2) & (0,1/2) & (0,-1/2) \\ (0,1/2) & (0,5/2) & (0,3/2) \\ (0,-1/2) & (0,3/2) & (0,1/2) \end{pmatrix},$$
$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad (8b)$$

$$M_L: \begin{pmatrix} (0, -3/2) & (0, 1/2) & (0, -1/2) \\ (0, 1/2) & (0, 5/2) & (0, 3/2) \\ (0, -1/2) & (0, 3/2) & (0, 1/2) \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \tag{8c}$$

Here the first matrix exhibits the quantum numbers (q, I) for the various fermion bilinears in the mass matrices, whereas the second matrix gives the chosen assignment of n_s. (Note that if d_i is allowed to couple to M_U, only d_i^* can couple to M_D or M_L.)

In addition we have illustrated imposing additional constraints by requiring that m_b and m_τ are generated by the same VEV $\langle d_i \rangle$. The numbers of solutions with this extra constraint are given in the second column of table 1.

We conclude that a computerized scan for mass matrices whose structure is only determined by G quantum numbers is possible – at least for abelian G. This analysis should certainly be extended to a more complete connection between the mirror masses and the mass matrices M_U, M_D and M_L as well as to an inclusion of the calculable powers P_i in eq. (2) [4]. Nevertheless, we find that already at this stage of the scanning many models are excluded since the symmetry G does not differentiate enough the generations to account for realistic mass matrices.

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