GAUSS' LAW AND THE INFRAPARTICLE PROBLEM

Detlev BUCHHOLZ

II. Institut für Theoretische Physik der Universität Hamburg, D-2000 Hamburg 50, Fed. Rep. Germany

Received 8 March 1986

It is shown that a state carrying an "electric" charge which can be determined with the help of Gauss' law cannot be an eigenstate of the mass operator.

1. There is ample evidence, both from the study of exactly soluble models and from more abstract arguments, that particles carrying an electric charge are inevitably accompanied by clouds of soft photons, and therefore cannot be described by eigenstates of the mass operator. (For a review of this "infraparticle problem" cf. ref. [1].) It is the aim of the present letter to provide a general argument which traces back this fact to its very origin: Gauss' law.

2. If one wants to determine the electric charge of a physical state with the help of Gauss' law one must be able to measure the spacelike asymptotic electromagnetic field of this state with sufficient precision. Let us first discuss how this requirement can be expressed in terms of a simple condition which must be satisfied by the vectors describing such states. To this end we consider the smoothed-out electromagnetic field operators

$$F_{\mu\nu}(\varphi_R) = \int d^4x \, \frac{1}{R^2} \, \varphi(x/R) F_{\mu\nu}(x), \tag{1}$$

where φ is an arbitrary real test function which has compact support in the spacelike complement of the origin of Minkowski space, and $R > 0$ is a scaling parameter. As R increases, the electromagnetic field in (1) is averaged over regions whose diameter and spacelike distance from the origin grows like R, and this average is rescaled by the factor R^{-2} according to the engineering dimension of the field.

The class of vectors Φ describing *physical* states in which the asymptotic electromagnetic field can reliably be determined, can now be characterized by the property that, firstly, the expectation values of the operators $F_{\mu\nu}(\varphi_R)$ converge for all abovementioned test functions φ,

$$\lim_{R\to\infty} (\Phi, F_{\mu\nu}(\varphi_R)\Phi) = f_{\mu\nu}(\varphi), \tag{2a}$$

and, secondly, the mean square deviations of these quantities stay bounded,

$$\limsup_{R\to\infty} \| [F_{\mu\nu}(\varphi_R) - f_{\mu\nu}(\varphi)\cdot \mathbf{1}]\Phi\|^2 < \infty. \tag{2b}$$

Although these conditions seem to be a minimal requirement if Gauss' law is to be verifiable in experiments, it is worthwhile to examine whether they are satisfied in models of physical interest, such as quantum electrodynamics. There exist two theoretical obstructions to these conditions [2]. Firstly, the state Φ may be packed with a multitude of particles, giving rise to large absolute values of the electromagnetic field. But since we are only interested in the elementary systems of the theory, we do not have to worry about this possibility here.

The second theoretical obstruction to be discussed is the possibility that the quantum effect of the measuring process described by $F_{\mu\nu}(\varphi_R)$ gives rise to fluctuations of the electromagnetic field which are in conflict with condition (2b). In order to get an idea of the magnitude of this effect let us consider the fluctuations of $F_{\mu\nu}(\varphi_R)$ in the vacuum state Ω. It follows from the first Maxwell equation $\partial^\nu \tilde{F}_{\mu\nu} = 0$ that the Källen–Lehmann representation of the two-point

0370-2693/86/$ 03.50 © Elsevier Science Publishers B.V. (North-Holland Physics Publishing Division)

Wightman function of $F_{\mu\nu}$ has the form ($\mu \neq \nu$ and no summations involved)

$$(F_{\mu\nu}(x)\Omega, F_{\mu\nu}(y)\Omega)$$
$$= \int d\mu(m) \int \frac{d^3p}{2p_0} (-p_\mu^2 g_{\nu\nu} - p_\nu^2 g_{\mu\mu}) \exp[ip(x-y)], \tag{3}$$

where $p_0 = (\boldsymbol{p}^2 + m^2)^{1/2}$, and $d\mu(m)$ is some positive measure which has a discrete part $\delta(m)dm$ due to the intermediate one-photon states contributing to (3). Proceeding as in ref. [3] it is then straightforward to show that

$$\lim_{R\to\infty} \|F_{\mu\nu}(\varphi_R)\Omega\|^2$$
$$= \int \frac{d^3p}{2|\boldsymbol{p}|} (-\bar{p}_\mu^2 g_{\nu\nu} - \bar{p}_\nu^2 g_{\mu\mu}) |\widetilde{\varphi}(\bar{p})|^2, \tag{4}$$

where we have put $\bar{p} = (|\boldsymbol{p}|, \boldsymbol{p})$. Thus the fluctuations of $F_{\mu\nu}(\varphi_R)$ in the vacuum state stay bounded in the limit $R \to \infty$. Using the spacelike commutativity of observables, the same result can be established for a dense set of vectors in the superselection sector of states carrying the charge quantum numbers of the vacuum (cf. the discussion below). But one finds in this sector locally already all possible configurations of charged particles (the compensating charges sitting "behind the moon"), so this result provides evidence to the effect that the quantum fluctuations of $F_{\mu\nu}(\varphi_R)$ stay bounded also in charged sectors. We therefore hold that condition (2) characterizes all states of interest here.

We note that the functional $f_{\mu\nu}(\cdot)$ in (2a), being the scaling limit of a distribution, is again a distribution which is defined on the region $\{x : x^2 < 0\}$ and which is homogeneous of degree-2, corresponding to the scaling transformation in (1). Knowing this spacelike asymptotic electromagnetic field of a state Φ one can determine its electric charge by means of Gauss' law $j_\mu = \partial^\nu F_{\nu\mu}$, giving

$$(\Phi, Q\Phi) = \lim_{R\to\infty} \int d^4x \frac{1}{R} \chi(x/R) (\Phi, j_0(x)\Phi)$$
$$= f_{i0}(\partial^i \chi). \tag{5}$$

Here χ is any test function whose spatial derivatives $\partial^i \chi$ have support in the region $\{x : x^2 < 0\}$, and which is normalized in such a way that $\int d^4x\, \chi(x) \times \delta^{(3)}(\boldsymbol{x}) = 1$. A simple example of such a function is $\chi(x) = \alpha(x_0)\beta(\boldsymbol{x})$, where α, β are test functions with $\beta(\boldsymbol{x}) = 1$ for $|\boldsymbol{x}| \leqslant 1$, $\alpha(x_0) = 0$ for $|x_0| \geqslant 1/2$, and $\int dx_0\, \alpha(x_0) = 1$. Actually, the expression (5) should not depend on the specific choice of χ within the above limitations. This is the case if $f_{\mu\nu}(\partial^\nu \varphi) = 0$ for all test functions φ with support in $\{x : x^2 < 0\}$, i.e. if there are no sources of the electromagnetic field in the state Φ at spacelike infinity. But we will make no use of this assumption here. What will be used, however, is the fact that $f_{\mu\nu}$ must be different from 0 if Φ carries a non-zero electric charge.

3. Let us now turn to the discusssion of the implications of condition (2). Of central importance for our argument is the notion of superselection sector [4], whose various aspects are briefly recalled for later reference: according to its basic definition, a superselection sector is a closed subspace $\mathcal{H}_s$ of the physical Hilbert space, which is stable under the action of the algebra $\mathfrak{A}$ generated by all *local* observables of the theory [‡1], and in which the superposition principle holds unrestrictedly, i.e. every unit vector in $\mathcal{H}_s$ induces a pure state on $\mathfrak{A}$. Equivalently, one can characterize the superselection sectors $\mathcal{H}_s$ by the fact that for every non-zero vector $\Psi \in \mathcal{H}_s$ the set of vectors $\mathfrak{A}\Psi$ is dense in $\mathcal{H}_s$. Still another characterization of superselection sectors is based on Schur's lemma, saying that every hermitian (but not necessarily bounded) operator on $\mathcal{H}_s$ commuting with all elements of $\mathfrak{A}$ is necessarily a multiple of the unit operator $\mathbf{1}_s$ on $\mathcal{H}_s$. From the latter fact it is obvious that all vectors in $\mathcal{H}_s$ carry the same charge quantum numbers. It should also be noticed that the energy-momentum operators P_μ, being the limit of local observables [5], leave each superselection sector invariant.

Now let $\mathcal{H}_s$ be any superselection sector and let $\Phi \in \mathcal{H}_s$ be any unit vector satisfying the conditions (2). It then follows that the sequence of vectors $F_{\mu\nu}(\varphi_R)\Phi \in \mathcal{H}_s$, φ as in (1), converge weakly as R tends to infinity, and

‡1 In order to avoid discussions of domain questions we assume that $\mathfrak{A}$ is a $*$-algebra of bounded operators, and that the unbounded local observables, such as $F_{\mu\nu}(\varphi_R)$, are affiliated with this algebra in the sense that their bounded functions are elements of $\mathfrak{A}$.

$$\text{w} - \lim_{R\to\infty} F_{\mu\nu}(\varphi_R)\Phi = f_{\mu\nu}(\varphi)\cdot\Phi. \qquad (6)$$

The proof of this statement is based on standard arguments and is given only for completeness: according to condition (2b) the norms $\|F_{\mu\nu}(\varphi_R)\Phi\|$ are uniformly bounded in R. Since the unit ball in a Hilbert space is weakly compact, we must therefore only show that all weakly convergent subsequences of the sequence $F_{\mu\nu}(\varphi_R)\Phi$ have the same limit (6). So let $F_{\mu\nu}(\varphi_{R_i})\Phi$, $i \in \mathbf{I}$ ($\mathbf{I}$ being some index set) be any such subsequence and let

$$\text{w} - \lim_{i} F_{\mu\nu}(\varphi_{R_i})\Phi = \Phi_{\mathbf{I}}. \qquad (7)$$

Because of the localization properties of the operators $F_{\mu\nu}(\varphi_R)$ and the spacelike commutativity of observables we have that for any given $A \in \mathfrak{A}$ the commutator $[F_{\mu\nu}(\varphi_R), A]$ vanishes for sufficiently large R. Thus it follows from (7) that for all $A \in \mathfrak{A}$

$$\text{w} - \lim_{i} F_{\mu\nu}(\varphi_{R_i})A\Phi = A\Phi_{\mathbf{I}}. \qquad (8)$$

But, as was discussed, the set of vectors $\mathfrak{A}\Phi$ is dense in $\mathcal{H}_s$, hence relation (8) shows that the sequence of hermitian operators $F_{\mu\nu}(\varphi_{R_i})$ converges weakly on the domain $\mathfrak{A}\Phi$, the limit being again a hermitian operator. It is also clear from the previous remarks that this limit operator commutes with all elements of $\mathfrak{A}$. According to Schur's lemma it must therefore be a multiple of the identity $\mathbf{1}_s$, and consequently $\Phi_{\mathbf{I}} = c_{\mathbf{I}}\Phi$, where $c_{\mathbf{I}}$ is some constant. Taking scalar products of the vectors in (7) with Φ and making use of condition (2a) it follows that $c_{\mathbf{I}} = f_{\mu\nu}(\varphi)$. This shows that the limit in (7) does not depend on the specific choice of the subsequence $F_{\mu\nu}(\varphi_{R_i})\Phi$, and thereby proves the assertion (6).

In the course of this argument we have seen that all states in a superselection sector have the same asymptotic electromagnetic field $f_{\mu\nu}$. This result is physically quite plausible since these states are obtained from a fixed one by the effect of local operations. But such operations cannot change the field at spacelike infinity according to Einstein's principle of causality.

In the final step of our argument we will show that a vector Φ satisfying relation (6) can only be an eigenstate of the mass-operator $P_\sigma P^\sigma$ if $f_{\mu\nu} = 0$. So let us assume that $P_\sigma P^\sigma\Phi = m^2\Phi$, and therefore also $P_\sigma P^\sigma \times \Phi(y) = m^2\Phi(y)$ where $\Phi(y) = \exp(\mathrm{i}y\cdot P)\Phi$, $y \in \mathbf{R}^4$. Starting from the trivial equation (in the sense of distributions)

$$(\Phi(y), [P_\sigma P^\sigma, F_{\mu\nu}(x)]\Phi) = 0 \qquad (9)$$

and taking into account that $\mathrm{i}[P_\sigma, F_{\mu\nu}(x)] = \partial_\sigma F_{\mu\nu}(x)$, it is obvious that

$$2\mathrm{i}(P^\sigma\Phi(y), \partial_\sigma F_{\mu\nu}(x)\Phi) + (\Phi(y), \Box F_{\mu\nu}(x)\Phi) = 0. \qquad (10)$$

We integrate this expression with test functions of the form $(1/R)\varphi(x/R)$, where φ has the properties assumed in (1), giving

$$2\mathrm{i}(P^\sigma\Phi(y), F_{\mu\nu}((\partial_\sigma\varphi)_R)\Phi) = (1/R)(\Phi(y), F_{\mu\nu}((\Box\varphi)_R)\Phi). \qquad (11)$$

Proceeding in this equation to the limit of large R and making use of relation (6) we thus find that

$$(P^\sigma\Phi(y), \Phi)\cdot f_{\mu\nu}(\partial_\sigma\varphi) = 0. \qquad (12)$$

Now if $f_{\mu\nu}(\partial_\sigma\varphi)$ would be different from 0 it would follow from (12) that the Fourier transform of $y \to (\Phi(y), \Phi)$ has support on the (at most) two-dimensional manifold

$$\{p : p^\sigma p_\sigma = m^2, p^\sigma f_{\mu\nu}(\partial_\sigma\varphi) = 0\}. \qquad (13)$$

But this is impossible since the joint spectrum of the spatial momentum operators P_i, $i = 1, 2, 3$ is Lebesque-absolutely continuous on the orthogonal complement of the vacuum (as a consequence of the locality of observables) [5]. Hence $f_{\mu\nu}(\partial_\sigma\varphi) = 0$, and bearing in mind that $f_{\mu\nu}$ is a homogeneous distribution of degree-2, we conclude that $f_{\mu\nu} = 0$.

We emphasize that one arrives at the same conclusion even if one relaxes the assumption that Φ is an element of a particular superselection sector, i.e. if one allows for the possibility of mixed states. The proof is, however, slightly more involved: one must first decompose Φ into a direct sum (integral) of vectors Φ_p inducing primary states on $\mathfrak{A}$. Since the energy-momentum operator P_μ is affiliated with $\mathfrak{A}^-$ [5], the components Φ_p appearing in this decomposition are again eigenstates of the mass operator, and they still satisfy condition (2b). The latter fact is sufficient to show (using the uniform boundedness principle for distributions) that all weak limit-points of the "central" sequence $F_{\mu\nu}(\varphi_R)$ on $\mathfrak{A}\Phi_p$ are c-number distributions $f^{\rm p}_{\mu\nu}(\varphi)$. Arguing now as before one finds that $f^{\rm p}_{\mu\nu}(\partial_\sigma\varphi) = 0$, showing that $F_{\mu\nu}((\partial_\sigma\varphi)_{\rm R})\Phi_{\rm p}$ converges weakly to 0. Since this holds true for all components $\Phi_{\rm p}$ of Φ it then follows from condition (2a) that $f_{\mu\nu}(\partial_\sigma\varphi) = 0$, and hence $f_{\mu\nu} = 0$. This completes our proof of the statement that particles carrying an electric charge cannot be described by eigenstates of the mass operator.

4. It is noteworthy that by a similar reasoning one can also establish the spontaneous breakdown of the Lorentz symmetry in superselection sectors of states carrying an electric charge [6]. (For a different argument, which is based on the timelike asymptotics of the radiation field, cf. ref. [7].) Namely, if there exist unitary operators $U(\Lambda)$ on $\mathcal{H}_{\rm s}$ implementing the Lorentz-transformations Λ, i.e.

$$U(\Lambda)\cdot F_{\mu\nu}(x) = \Lambda^{-1}{}_{\mu}{}^{\mu'}\Lambda^{-1}{}_{\nu}{}^{\nu'}F_{\mu'\nu'}(\Lambda x)\cdot U(\Lambda), \qquad (14)$$

one finds, by taking matrix elements of this equation with respect to the dense set of vectors $\mathfrak{A}\Phi \subset \mathcal{H}_{\rm s}$ and using eq. (6), that the spacelike asymptotic field $f_{\mu\nu}$ of the states in $\mathcal{H}_{\rm s}$ must satisfy

$$f_{\mu\nu}(x) = \Lambda^{-1}{}_{\mu}{}^{\mu'}\Lambda^{-1}{}_{\nu}{}^{\nu'}f_{\mu'\nu'}(\Lambda x). \qquad (15)$$

But in view of the antisymmetry of $f_{\mu\nu}$ in μ and ν this is only possible if $f_{\mu\nu} = 0$.

By the above discussion the well-known infrared problems in quantum electrodynamics have been traced back to the fact that the spacelike asymptotic electromagnetic field is a superselection rule of the theory. On the other hand it should be noticed that, due to the presence of this superselection rule, the physical state space of the theory splits into an abundance of superselection sectors [6], which brings its structure close to that of a classical theory. Some aspects of this simplifying feature of quantum electrodynamics, which is frequently ignored, well be discussed elsewhere.

References

[1] G. Morchio and F. Strocchi, Infrared problem, Higgs phenomenon and long range interactions, in: Fundamental problems of gauge field theory. Erice lecture notes (1985), to be published.

[2] D. Buchholz and K. Fredenhagen, unpublished notes.

[3] M. Requardt, Commun. Math. Phys. 50 (1976) 259.

[4] G.C. Wick, A.S. Wightman and E.P. Wigner, Phys. Rev. 88 (1952) 101;
R. Haag and D. Kastler, J. Math. Phys. 5 (1964) 848.

[5] H.J. Borchers, Commun. Math. Phys. 2 (1966) 49.

[6] J.L. Gervais and D. Zwanziger, Phys. Lett. B 94 (1980) 389;
D. Buchholz, Commun. Math. Phys. 85 (1982) 49.

[7] J. Fröhlich, G. Morchio and F. Strocchi, Phys. Lett. B 89 (1979) 61.