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Open Bosonic Strings in General Background Fields

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Abstract. We discuss the renormalization properties of the 2 dim. field theory describing an open bosonic string in the background fields corresponding to its massless excitations. The relevant β-functions are calculated for gravitational, antisymmetric tensor and Yang-Mills background on 1-loop level, for pure Yang-Mills background an 2-loop level and in the Abelian case up to 3 loops. We find a renormalization scheme dependence starting at 2 loop order. Putting β to zero yields the equation of motion for the non-linear electrodynamics of Fradkin and Tseytlin.

1. Introduction

At present supersymmetric string theories are good candidates for a unified theory of all interactions. Living in 10 space-time dimensions they have to be compactified to make contact with the real world. The compactification has been studied for the low energy effective field theory describing the massless excitations of the string. Along this line phenomenological attractive compactification patterns have been explored [1]. Another approach starts directly with strings living in curved space-time and tries to get restrictions on the space-time manifold by requiring 2-dimensional world sheet conformal invariance [2–4]. The corresponding calculations have been done on 1-loop and partially on 2-loop level of the 2 dim. field theory. They yield as conditions forced by conformal invariance of the string just the equations of motion of 10-dim. supergravity coupled to a Yang-Mills supermultiplet (in the heterotic case). The 2-loop results have been shown to contain the crucial Lorentz and Yang-Mills Chern-Simons completions of the field strengths of the antisymmetric tensor field necessary for anomaly cancellation in the effective theory.

Due to better phenomenological prospects of the heterotic string compared to the type I superstring the investigations have been restricted to closed strings. However, this restriction has also pure technical reasons [3]. Mainly to fill this gap we consider in this paper as a first step the open bosonic string in general gravitational, antisymmetric tensor and Yang-Mills background.

Our starting point is the (Euclidean) action [3]

$$\begin{aligned} S &= S_M + S_{\partial M} \\ S_M &= \frac{1}{4\pi\alpha'} \int_M d^2 z \{\sqrt{g}\, g^{mn}(z) G_{\mu\nu}(x(z)) \partial_m x^\mu \partial_n x^\nu(z) \\ &\quad + \varepsilon^{mn} B_{\mu\nu}(x(z)) \partial_m x^\mu \partial_n x^\nu + \alpha' \sqrt{g}\, R^{(2)} \phi(x)\} \\ S_{\partial M} &= -\log \operatorname{tr} P \exp i \int_{\partial M} A_\mu dx^\mu. \end{aligned} \tag{1.1}$$

$G_{\mu\nu}$ is the metric in the $D=26$ dimensional space-time manifold, $B_{\mu\nu}$ an antisymmetric tensor field, A_μ the Yang-Mills vector potential and ϕ the scalar dilaton field. This set of fields corresponds to the massless excitations of open and closed bosonic strings. The 2 dimensional metric $g^{mn}(z)$ and the string position $x^\mu(z)$ are the quantum fields. M denotes the 2 dim. parameter domain with boundary ∂M. The case of interacting strings is included if M is allowed to be multiconnected. For fixed g^{mn} we have a generalized σ-model [5, 6]. Due to general covariance the trace of the 2 dim. energy-momentum tensor has the structure ($\partial M = 0$) [3]

$$\begin{aligned} T_m^m &= \beta^\phi \sqrt{g}\, R^{(2)} + \beta^G_{\mu\nu} \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu \\ &\quad + \beta^B_{\mu\nu} \varepsilon^{mn} \partial_m x^\mu \partial_n x^\nu. \end{aligned} \tag{1.2}$$

Conformal invariance requires to put all the 3β-functions defined in (1.2) equal to zero. This yields the set of equations for the background fields ϕ, G, B mentioned above.

If the Yang-Mills field couples only via the boundary it is difficult to pick up the correct response to a change of g^{mn} defining T_{mn}. We have no direct solution to this problem. Instead we prefer to use the alternative definition of the β-functions via the renormalization properties of generalized σ-models [5]. In the case of β^{ϕ}, β^G and β^B this is equivalent to (1.2) and for the wanted β^A it yields sensible results.

Since our main object is the treatment of the Yang-Mills coupling via the Wilson loop along ∂M we restrict ourselves for simplicity to flat 2 dim. metric g^{mn} and consequently neglect the dilaton field ϕ. In Sect. 2 we sketch the background field method (with respect to the string configuration) as used in our context and study the system in G, B and A background in 1-loop approximation. The requirement $\beta^A=0$ yields the ordinary equation of motion for a Yang-Mills field in gravitational background. Section 3 is devoted to a 2-loop study of the problem with pure A_μ background. Here we find a renormalization scheme dependence. The result is compared to the corresponding equation derived on the basis of $\alpha'\to 0$ considerations of dual S-matrix elements [7]. Section 4 extends the investigation for Abelian A_μ to 3 loops. The equation for A_μ agrees in this order with the equation of motion for a generalization of the Born-Infeld action derived in [8] by a quite different method for constant electromagnetic fields. The concluding Sect. 5 discusses the open problems and makes some comments on the renormalization scheme dependence.

2. The General Method

We renormalize the generalized σ-model under discussion in the sense of [5]. Counter terms are classified according to their 2 and D-dimensional index structure and e.g. all counter terms of the type $H_{\mu\nu}\partial^m x^\mu \partial^m x^\nu$ are interpreted as a renormalization of the D-dim. metric. Till now practical calculations have been done in the framework of the background field method only.

The effective action $\Gamma(x)$ generating the 1-particle irreducible vertex functions of the string position field is equal to the sum of the 1-particle irreducible vacuum bubbles in string configuration background $x^\mu_{\text{class}}=x^\mu$: $\tilde{\Gamma}(0,x^\mu)$ [10]. Two remarks concerning the relevant diagrams are in order.

First, the fundamental relation between Γ and $\tilde{\Gamma}$ is a relation for Green functions of local fields. Hence it applies to the action (1.1) only after the introduction of an auxiliary field which allows to express the Wilson loop as a two point function [9]. We make no further use of this formalism but only notice the general rule that 1-point vertices of the string fluctuation y^μ lying on the boundary and coupling to the Yang-Mills background have to be taken into account. (The boundary acts effectively as a further leg of the corresponding vertex.) On the other side 1-point vertices in the interior of M or on ∂M not coupling to A_μ are irrelevant.

Second, the method in a first step yields counter terms as functions of the background configuration x^μ. But the same functional structure then gives the counter terms as operators, i.e. as functions of the full quantum field. Hence going to the next order of perturbation theory the counter terms of the lower order have to be included into the background-quantum split $x^\mu \to x^\mu + y^\mu$.

The expansion of $S_M[x+y]$ (g^{mn} flat, $\phi=0$) can be taken from the standard literature. One only has to take care of partial integrations to keep possible boundary effects. We follow the appendix A of [6].

$$\begin{aligned} S_M[x+y] = {} & S_M[x] + \text{term linear in } \xi \\ & + \frac{1}{4\pi\alpha'}\int_M d^2 z \{G_{\alpha\beta}(x)\hat{D}_m\xi^\alpha \hat{D}_m \xi^\beta \\ & + \hat{R}_{\alpha\beta\gamma\delta}\xi^\beta\xi^\gamma(\delta_{mn}-\varepsilon_{mn})\partial_m x^\alpha \partial_n x^\delta \\ & + \partial_m[\xi^\gamma (D_\gamma B_{\alpha\beta})\xi^\alpha \varepsilon_{mn}\partial_n x^\beta \\ & + B_{\alpha\beta}\xi^\alpha \varepsilon_{mn} D_n \xi^\beta]\} + O(\xi^3). \end{aligned} \tag{2.1}$$

Here ξ^α is the Riemann normal coordinate corresponding to y^α. The covariant derivative D_γ refers to the usual symmetric Levi-Civita connection Γ. D_n is the projection on 2-space. The quantities with a hat refer to a connection $\hat{\Gamma}$ which contains a torsion part according to

$$\hat{\Gamma}_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma} - S_{\alpha\beta\gamma}; \qquad S_{\alpha\beta\gamma} = \tfrac{3}{2}\partial_{[\alpha}B_{\beta\gamma]}. \tag{2.2}$$

The quantum field ξ^α is still not convenient since its kinetic part in the action is multiplied by $G_{\alpha\beta}(x)$. Following [6] we switch to the tangent space quantity ζ^A via

$$\zeta^A = V^A_\alpha \xi^\alpha. \tag{2.3}$$

V^A_α is the vielbein field: $G_{\alpha\beta} = V^A_\alpha V^B_\beta \delta_{AB}$. Then the kinetic term for the ζ^A field is

$$\frac{1}{4\pi\alpha'}\int_M d^2 z\, \partial_m \zeta^A \partial_m \zeta^B \delta_{AB}.$$

The boundary term produced by partial integration vanishes if one chooses Neumann boundary condition for the quantum field ζ^A together with the normalization condition $\int_{\partial M} \zeta^A ds = 0$ which means the inclusion of constant modes into the background configuration x [8, 11]. We want to emphasize that this choice is motivated by pure technical reasons. For the field before doing the background-quantum split it would correspond to a conditions involving the vielbein and connection on a certain reference configuration, just that which later on is used as background. We have to be careful what concerns both dependence on and possible renormalization of the boundary condition.

Our propagator is now

$$\overline{\zeta^A(z)\zeta^B}(z') = 2\pi\alpha'\,\delta^{AB} N(z, z'), \tag{2.4}$$

with $N(z, z')$ denoting the Neumann function for M. To read off from (2.1) the vertices one has to eliminate ξ^α in favour of ζ^A: $\xi^\alpha = V^{\alpha A}\zeta^A$ ($V^{\alpha A}$ the inverse of V_α^A). We prefer to treat the total derivative term in (2.1) as an integral over the boundary. The vertices living in M are that of [6], i.e.

$$\begin{aligned}
\text{Fig. 1a} &= +\frac{1}{4\pi\alpha'}\, C_m^{AB} \overleftrightarrow{\partial}_m \\
\text{Fig. 1b} &= -\frac{1}{4\pi\alpha'}\, C_m^{DA} C_m^{DB} \\
\text{Fig. 1c} &= -\frac{1}{4\pi\alpha'}\, V^{\beta B} V^{\gamma C} \\
&\quad \cdot \hat{R}_{\alpha\beta\gamma\delta}(\delta_{mn} - \varepsilon_{mn})\partial_m x^\alpha \partial_n x^\delta
\end{aligned} \tag{2.5}$$

with $C_m^{AB} = (\partial_m x^\mu)\omega_\mu^{AB} - \varepsilon_{mn}(\partial_n x^\gamma) V_\alpha^A S_{\beta\gamma}^\alpha V^{\beta B}$ and $\omega_\mu^{AB} = V_\alpha^A(\partial_\mu V^{\alpha B} + \Gamma_{\mu\nu}^\alpha V^{\nu B})$ the spin connection.

In addition we have the vertices living on the boundary ∂M

$$\begin{aligned}
\text{Fig. 2a} &= -\frac{1}{4\pi\alpha'}\, V^{\alpha A} V^{\beta B} D_\alpha B_{\beta\lambda} \dot{x}^\lambda \\
\text{Fig. 2b} &= -\frac{1}{4\pi\alpha'}\, V^{\alpha A} V^{\beta B} B_{\alpha\beta} \dot{z}_n \partial_n \\
\text{Fig. 2c} &= -\frac{1}{4\pi\alpha'}\, V^{\alpha A} V^{\gamma C} B_{\alpha\gamma} \omega_\lambda^{CB} \dot{x}^\lambda.
\end{aligned} \tag{2.6}$$

The dot indicates differentiation with respect to the contour parameter s along the boundary ∂M.

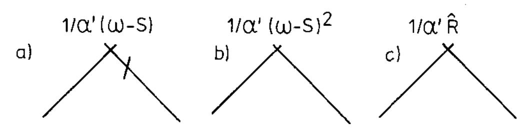

Fig. 1a–c. Vertices living in M. A slash indicates differentiation. Each vertex is characterized by a short hand notation of the background field involved

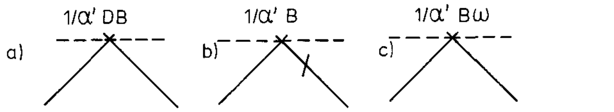

Fig. 2a–c. Vertices arising from S_M and living on the boundary ∂M. The dashed line indicates ∂M

The expansion of $S_{\partial M}[x+y]$ up to second order is easily performed using the well known expressions for the first and second functional derivatives of the Wilson loop $U = P \exp i \int A^\mu dx_\mu$, see e.g. [9].

$$\begin{aligned}
\operatorname{tr} U[x+y] = \operatorname{tr} P(U[x] \exp i \textstyle\int ds[&F_{\mu\lambda}\dot{x}^\lambda y^\mu \\
&+ \tfrac{1}{2} D_{\mu_2}^{(A)} F_{\mu_1\lambda}\dot{x}^\lambda y^{\mu_1} y^{\mu_2} \\
&+ \tfrac{1}{2} F_{\mu_1\mu_2} y^{\mu_1}\dot{y}^{\mu_2} + O(y^3)]).
\end{aligned} \tag{2.7}$$

Solving the geodesic equation defining ζ^A perturbatively we find

$$y^\alpha = V^{\alpha A}\zeta^A - \tfrac{1}{2}\Gamma_{\mu\lambda}^\alpha V^{\mu M} V^{\lambda L}\zeta^M\zeta^L + O(\zeta^3). \tag{2.8}$$

Putting (2.8) into (2.7) our aim is of course to produce the total covariant derivative referring to both gauge and gravitational background $D^{(\Gamma+A)}$. Since it applies to a tensor we need a further Christoffel symbol. We add and subtract the corresponding term and get

$$\begin{aligned}
\operatorname{tr} U[x+y] = \operatorname{tr} P(U[x] \exp i \textstyle\int ds[&F_{\mu\lambda}\dot{x}^\lambda V^{\mu M}\zeta^M \\
&+ \tfrac{1}{2}(D_\beta^{(A+\Gamma)} F_{\alpha\lambda})\dot{x}^\lambda V^{\alpha A} V^{\beta B}\zeta^A\zeta^B \\
&+ \tfrac{1}{2} F_{\alpha\gamma}\omega_\lambda^{CB}\dot{x}^\lambda V^{\gamma C} V^{\alpha A}\zeta^A\zeta^B \\
&+ \tfrac{1}{2} F_{\alpha\beta} V^{\alpha A} V^{\beta B}\zeta^A\dot{\zeta}^B + O(\zeta^3)]).
\end{aligned} \tag{2.9}$$

This yields the following additional vertices on ∂M

$$\begin{aligned}
\text{Fig. 3a} &= i/2 (D_\alpha^{(A+\Gamma)} F_{\beta\lambda})\dot{x}^\lambda V^{\alpha A} V^{\beta B} \\
\text{Fig. 3b} &= i/2 F_{\alpha\beta} V^{\alpha A} V^{\beta B} \dot{z}_n \partial_n \\
\text{Fig. 3c} &= i/2 F_{\alpha\gamma}\omega_\lambda^{CB}\dot{x}^\lambda V^{\gamma C} V^{\alpha A} \\
\text{Fig. 3d} &= i \quad F_{\alpha\lambda}\dot{x}^\lambda V^{\alpha A}.
\end{aligned} \tag{2.10}$$

These vertices have to be applied under the operation $\operatorname{tr} P(U[x]\ldots)$.

Having set the general framework let us start with the 1-loop diagrams which by power counting might be divergent. The 2 dim. propagator behaves logarithmically for short distances. Those diagrams containing only vertices inside M are shown in Fig. 4. They heve been studied in [6].

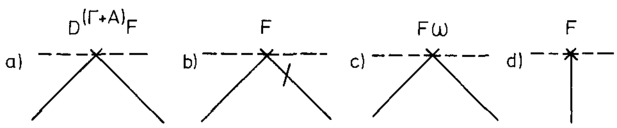

Fig. 3a–d. Vertices arising from $S_{\partial M}$

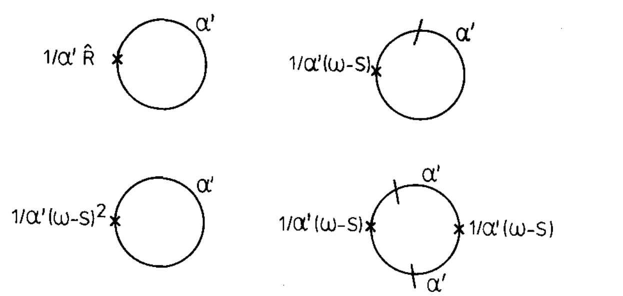

Fig. 4. Potentially divergent 1-loop diagrams with vertices in M. Powers of α' originating from vertices and propagators are indicated

$$\text{Fig. 4} = \tfrac{1}{2}\int d^2z N(z,z)[R_{\alpha\beta} - S_\alpha^{\mu\nu} S_{\mu\nu\beta})\partial_m x^\alpha \partial_m x^\beta - D^\mu S_{\mu\alpha\beta}\varepsilon_{mn}\partial_m x^\alpha \partial_n x^\beta] + uv \text{ finite.} \quad (2.11)$$

Here $\hat{R}$ has been separated in the usual Riemannian R and the torsion contribution. $R_{\alpha\beta}$ is the Ricci tensor.

Potentially divergent diagrams containing vertices on ∂M are shown in Fig. 5. One immediately gets

$$\text{Fig. 5a} = -\tfrac{1}{2}\int ds N(z(s), z(s)) D^\alpha B_{\alpha\beta}\dot{x}^\beta(s) + \text{fin.}$$

$$\text{Fig. 5b} = 0 \quad (2.12)$$

$$\text{Fig. 5c} = -\tfrac{1}{2}\int ds N(z(s), z(s)) V^{\alpha A} V^{\beta B} B_{\alpha\beta}\omega_\lambda^{BA}\dot{x}^\lambda(s) + \text{fin.}$$

A little bit more effort is necessary to evaluate Fig. 5d. We sketch the calculation in our Appendix A.

$$\text{Fig. 5d} = \tfrac{1}{2}\int ds N(z(s), z(s))(V^{\alpha A} V^{\beta B} B_{\alpha\beta}\omega_\lambda^{BA}\dot{x}^\lambda - \dot{z}_m \varepsilon_{mk}(\partial_k x^\beta) S_\beta^{\alpha\delta} B_{\delta\alpha}) + \text{fin.} \quad (2.13)$$

In the sum the term $\sim B\omega$ cancels. For this effect the interference diagram 5d is crucial.

The diagrams 5e–h have exactly the same structure as 5a–d, one only has to replace $D_\alpha \to D_\alpha^{(A+\Gamma)}$, $B_{\alpha\beta} \to -2\pi i\alpha' F_{\alpha\beta}$ and to put the whole expression under $\operatorname{tr} P(U\ldots)$. Summarizing (2.11)–(2.13) we get for

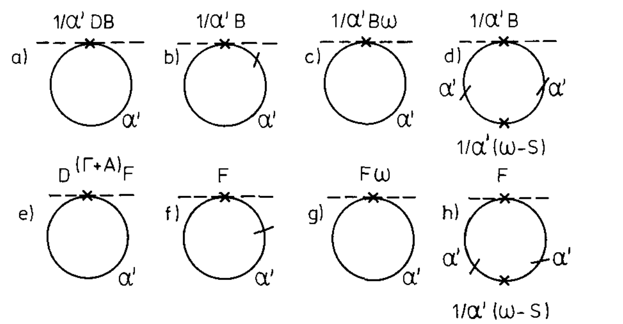

Fig. 5a–h. Possible divergent 1-loop diagrams involving vertices on ∂M

the effective action $\Gamma(x)$

$$\Gamma(x) = \tfrac{1}{2}\int d^2z N(z,z)[(R_{\alpha\beta} - S_\alpha^{\mu\nu} S_{\mu\nu\beta})\partial_m x^\alpha \partial_m x^\beta - D^\mu S_{\mu\alpha\beta}\varepsilon_{mn}\partial_m x^\alpha \partial_n x^\beta]$$
$$-\tfrac{1}{2}\int_{\partial M} ds N(z(s), z(s))(D^\alpha B_{\alpha\beta}\dot{x}^\beta + \dot{z}_m\varepsilon_{mk}(\partial_k x^\beta) S_\beta^{\alpha\delta} B_{\delta\alpha})$$
$$+\pi i\alpha' \int_{\partial M} ds N(z(s), z(s)) \operatorname{tr} P(U[D^{(A+\Gamma)\alpha} F_{\alpha\beta}\dot{x}^\beta + \dot{z}_m\varepsilon_{mk}(\partial_k x^\beta) S_\beta^{\alpha\delta} F_{\delta\alpha}) + \text{fin.} \quad (2.14)$$

In the closed string case where only the integral over M is present one can cancel the uv divergence due to $N(z,z)$ by counter terms $G_{\alpha\beta} \to G_{\alpha\beta} + \delta G_{\alpha\beta}$, $B_{\alpha\beta} \to B_{\alpha\beta} + \delta B_{\alpha\beta}$ leading to lowest order β-functions (for a discussion of the relation between coefficients of counter terms and β-functions compare Appendix B)

$$\beta_{\mu\nu}^G = -2\alpha'(R_{\mu\nu} - S_\mu^{\alpha\beta} S_{\alpha\beta\nu}) + O(\alpha'^2)$$

$$\beta_{\mu\nu}^B = 2\alpha' D^\alpha S_{\alpha\mu\nu} + O(\alpha'^2).$$

In our open string case we are seeking for a further independent β_μ^A function. However, there are some subtleties. In the Abelian case by Stokes theorem the boundary integrals can be transformed in integrals over M. These integrals have at least for on shell background configuration x index structure fitting into β^G and β^B. Hence, in some sense, the electromagnetic field disappears. This is the reflection in our language of an effect discussed at various occasions already [12]. In the non-Abelian case there is no suitable Stokes theorem [13]. This we take as a justification for the absenxe of any interference between the non-Abelian ∂M-integral and the remaining terms of (2.14). Even if we go to infinitesimal M where we can use an expansion of the ∂M-integral, the coefficient multiplying the area of M is projected to zero by the trace operation.

The first term in the non-Abelian boundary integral can be cancelled by a counter term ($N(z,z) = N_{\text{div}} + N_{\text{fin}}$)

$$\delta A_\mu = -\pi\alpha' N_{\text{div}} D^{(A+\Gamma)\nu} F_{\nu\mu} + \text{pure gauge}$$

leading to

$$\beta_\mu^A = -\alpha' D^{(A+\Gamma)\nu} F_{\nu\mu} + O(\alpha'^2). \quad (2.15)$$

Till now we have no convenient interpretation for the second term. It may signal the need of a further independent coupling or may be connected with the renormalization of the boundary condition. Leaving this problem for further study we must restrict ourselves (to be selfconsistent) to an open string coupling only to $G_{\mu\nu}$ and A_μ. Then the absence of uv-

singularities requires the background fields to fulfil

$$R_{\mu\nu}=O(\alpha')$$
$$D^{(A+\Gamma)\nu}F_{\nu\mu}=O(\alpha'). \tag{2.16}$$

The exclusion of the interaction with the antisymmetric tensor field could be justified by a restriction to nonorientable strings [14]. In this connection one should add that type I superstrings are nonorientable.

3. Pure Yang-Mills Background in 2-Loop Order

We start with the straightforward generalization of (2.7) to all orders

$$\operatorname{tr} U[x+y]$$
$$=\operatorname{tr} P\Big(U[x]\exp i\int ds\Big[F_{\mu\lambda}\dot{x}^{\lambda}y^{\mu}$$
$$+\tfrac{1}{2}D_{\mu_2}F_{\mu_1\lambda}\dot{x}^{\lambda}y^{\mu_1}y^{\mu_2}+\tfrac{1}{2}F_{\mu_1\mu_2}y^{\mu_1}\dot{y}^{\mu_2}$$
$$+\sum_{n=3}^{\infty}\Big(\frac{1}{n!}D_{\mu_1}\dots D_{\mu_{n-1}}F_{\mu_n\lambda}\dot{x}^{\lambda}y^{\mu_1}\dots y^{\mu_n}$$
$$+\frac{n-1}{n!}D_{\mu_1}\dots D_{\mu_{n-2}}F_{\mu_{n-1}\mu_n}y^{\mu_1}\dots y^{\mu_{n-1}}\dot{y}^{\mu_n}\Big)\Big]\Big). \tag{3.1}$$

The A_λ counterterm we write as

$$\delta A_\lambda=\alpha'\delta_1 A_\lambda+\alpha'^2\delta_2 A_\lambda+\dots. \tag{3.2}$$

As already known

$$\delta_1 A_\lambda=-\pi N_{\rm div}D^{\mu}F_{\mu\lambda}, \tag{3.3}$$

where $N(z,z)=N_{\rm div}+N_{\rm fin}(z)$. The concrete splitting in $N_{\rm div}$ and $N_{\rm fin}$ is fixed by the renormalization scheme used. The vertices on ∂M can be directly read off from (3.1). To a given number of legs $\geqq 2$ there is always one vertex without any derivative and one vertex where one leg is differentiated. The one loop counter term at the two loop level has to be included into the background-quantum split. This yields the following counter vertices contributing to the total order α'^2 (Fig. 6)

$$\begin{aligned}
\text{Fig. 6a} &= i\alpha'\delta_1 A_\lambda\dot{x}^{\lambda}=-i\alpha'\pi N_{\rm div}D^{\mu}F_{\mu\lambda}\dot{x}^{\lambda}\\
\text{Fig. 6b} &= i\alpha'^2\delta_2 A_\lambda\dot{x}^{\lambda}\\
\text{Fig. 6c} &= i\alpha'\dot{x}^{\lambda}(D_\mu\delta_1 A_\lambda-D_\lambda\delta_1 A_\mu)\\
\text{Fig. 6d} &= i/2\alpha'\dot{x}^{\lambda}(D_{\mu_1}D_{\mu_2}\delta_1 A_\lambda-D_{\mu_1}D_\lambda\delta_1 A_{\mu_2}\\
&\quad -i[\delta_1 A_{\mu_1},F_{\mu_2\lambda}])\\
\text{Fig. 6e} &= i/2\alpha'(D_{\mu_1}\delta_1 A_{\mu_2}-D_{\mu_2}\delta_1 A_{\mu_1}).
\end{aligned} \tag{3.4}$$

In Fig. 7 we list all α'^2 diagrams which by power counting might be divergent.

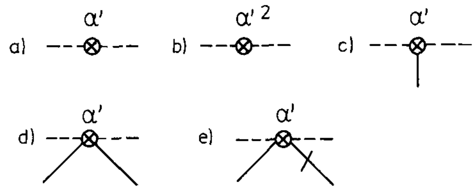

Fig. 6a–e. Counter vertices relevant for a 2-loop calculation in the pure Yang-Mills case

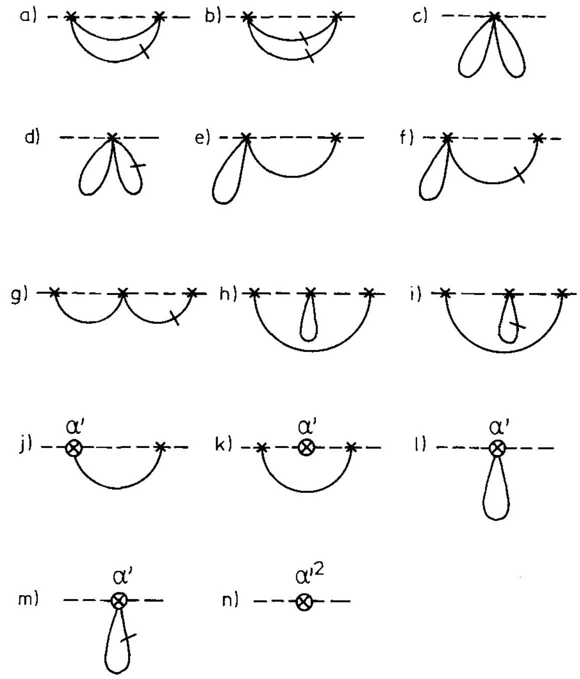

Fig. 7a–n. By power counting possible divergent 1-particle irreducible 2-loop diagrams

There are some trivial cases. Figures 7a, d, i and m are zero due to the symmetry properties of indices involved. Further the sum of h and k is ultraviolet finite, of course. We list now the results for the remaining diagrams, omitting Fig. 7b which requires a special discussion.

$$\text{Fig. 7c} =\frac{i\pi^2\alpha'^2}{6}\int ds N(z(s),z(s))^2\operatorname{tr}P(U(D^2D^{\mu}F_{\mu\lambda}$$
$$+D^{\nu}D^{\mu}D_{\nu}F_{\mu\lambda}+D^{\mu}D^2F_{\mu\lambda}))\dot{x}^{\lambda}+\text{fin}. \tag{3.5}$$

$$\begin{aligned}
\text{Fig. 7e} &= -\tfrac{2}{3}\pi^2\alpha'^2\int ds_1 ds_2 N(z(s_1),z(s_1))\\
&\quad\cdot\operatorname{tr}P(U(D^{\mu}D^{\nu}F_{\nu\lambda}\dot{x}^{\lambda}(s_1)+D^{\nu}D^{\mu}F_{\nu\lambda}\dot{x}^{\lambda}(s_1)\\
&\quad+D^2F^{\mu\lambda}\dot{x}_\lambda(s_1))F_{\mu\kappa}\dot{x}^{\kappa}(s_2))\\
&\quad\cdot N(z(s_1),z(s_2))+\text{fin}.
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\text{Fig. 7j} &= 2\pi^2\alpha'^2N_{\rm div}\int ds_1 ds_2\\
&\quad\cdot\operatorname{tr}P(U(D^{\mu}D^{\nu}F_{\nu\lambda}\dot{x}^{\lambda}(s_1)\\
&\quad-D^{\lambda}D_{\nu}F^{\nu\mu}\dot{x}_\lambda(s_1))F_{\mu\kappa}\dot{x}^{\kappa}(s_2))\\
&\quad N(z(s_1),z(s_2))+\text{fin}
\end{aligned} \tag{3.7}$$

$$\text{Fig. 7l} \quad = i\pi^2\alpha'^2 N_{\text{div}} \int ds N(z(s), z(s)) \cdot \operatorname{tr} P(U(-D^2 D^\nu F_{\nu\lambda} + D^\mu D_\lambda D^\nu F_{\nu\mu} + i[D^\nu F_{\nu\mu}, F^\mu_\lambda]))\dot{x}^\lambda(s) + \text{fin.} \tag{3.8}$$

The above four equations are valid for any regularization. If we make a slight restriction on the allowed regularizations and require

$$\frac{d}{dz_m} N(z, z') = -\frac{d}{dz'_m} N(z, z') + \text{terms not producing } uv \text{ singularities}$$

then

$$\text{Fig. 7f} \quad = \tfrac{4}{3}\pi^2\alpha'^2 \int ds N(z(s), z(s))^2 \cdot \operatorname{tr} P(U[D^\mu F_{\mu\nu}, F^{\nu\lambda}])\dot{x}_\lambda(s) + \tfrac{4}{3}\pi^2\alpha'^2 \int ds_1 ds_2 N(z(s_1), z(s_1)) \cdot \operatorname{tr} P(U D_\kappa D^\mu F_{\mu\nu}\dot{x}^\kappa(s_1) \cdot F^{\nu\lambda}\dot{x}_\lambda(s_2)) \cdot N(z(s_1), z(s_2)) + \text{fin.} \tag{3.9}$$

$$\text{Fig. 7g} \quad = 2i\pi^2\alpha'^2 \int ds_1 ds_2 N(z(s_1), z(s_1)) \cdot \operatorname{tr} P(U[F_{\mu\lambda}\dot{x}^\lambda(s_1), F^{\mu\nu}(s_1)] \cdot F_{\nu\kappa}\dot{x}^\kappa(s_2)) N(z(s_1), z(s_2)) + \text{fin.} \tag{3.10}$$

This gives altogether

Fig. 7 without b and n

$$= \frac{\pi^2\alpha'^2}{2} N_{\text{div}} \int ds N_{\text{fin}} \operatorname{tr} P(U[F_{\mu\nu}, D_\lambda F^{\mu\nu}])\dot{x}^\lambda(s) + \pi^2\alpha'^2 N_{\text{div}}^2 \int ds \operatorname{tr} P\left(U\left(-\frac{i}{2} D^2 D^\mu F_{\mu\lambda} - [D^\mu F_{\mu\nu}, F^\nu_\lambda] + \tfrac{1}{4}[F_{\mu\nu}, D_\lambda F^{\mu\nu}]\right)\right)\dot{x}^\lambda(s) + \text{fin.} \tag{3.11}$$

To get this sum use has been made of the formula

$$[D_\mu, D_\nu] F_{\alpha\beta} = -i[F_{\mu\nu}, F_{\alpha\beta}]$$

as well as the Bianchi identity. Note that the double integrals already have been cancelled.

The diagram 7b disregarded so far is special since it is the only diagram which requires a definite choice of the regularized Neumann function. We specialize the domain M to the unit circle in complex z-plane. Then [11, 8]

$$N(z, z') = -\frac{1}{2\pi} \log(|z - z'|\,|z - \bar{z}'^{-1}|).$$

Since no vertices inside M are involved we only need

$$N(z, z') = -\frac{1}{2\pi} \log|z - z'|^2, \qquad z, z' \in \partial M. \tag{3.12}$$

In the a-regularization we choose

$$N^{(a)}(z, z') = -\frac{1}{2\pi} \log(|z - z'|^2 + a^2) \tag{3.13}$$

and in the dimensional regularization

$$N^{(n)}(z, z') = 2 \cdot \frac{m^{n/2-1}}{(2\pi)^{n/2}} |z - z'|^{1-n/2} K_{n/2-1}(m|z - z'|), \tag{3.14}$$

The mass m serves in the case of unrestricted parameter space z as an intermediate infrared cutoff [5, 6]. Although in our case no infrared regularization is necessary, we need $m \neq 0$ since only then $N^{(n)}(z, z) \neq 0$. At the end of the calculation m can be put to zero. The crucial factor 2 is necessary to ensure for $n \to 2$ the correct factor in front of the short distance singularity. It is in some sense a consequence of the restriction of y-field propagation to the interior of M.

As a consequence of the symmetry properties of the regularized propagator near $z = z'$, diagram 7b is finite in the Abelian case*. However, for non-Abelian fields due to the path ordering the left and right limits of the factor multiplying the propagators are different. After some calculation we find

$$\text{Fig. 7b} \quad = \alpha'^2 \left(\frac{\pi^2}{4}(N_{\text{div}} + N_{\text{fin}})^2 + K\right) \int ds \cdot \operatorname{tr} P(U[D_\lambda F_{\mu\nu}, F^{\mu\nu}])\dot{x}^\lambda + \text{fin.} \tag{3.15}$$

with

$$K = \frac{1}{2(n-2)} \tag{3.16}$$

in dimensional regularization and

$$K = \log a \tag{3.17}$$

in a-regularization.

We further need

$$N_{\text{div}}^{(a)} = -1/\pi \log a, \qquad N_{\text{div}}^{(n)} = -1/\pi \frac{1}{n-2}. \tag{3.18}$$

The different factors relating $\frac{1}{n-2}$ and $\log a$ in (3.16), (3.17) on one side and (3.18) on the other lead to a renormalization scheme dependence.

Adding (3.11) and (3.15) we get the two loop counter term $\delta_2 A_\lambda$. This yields the β^A-function (compare Appendix B)

$$\beta_\lambda^{A(n)} = -\alpha' D^\mu F_{\mu\lambda} - \frac{i}{2}\alpha'^2 [D_\lambda F_{\mu\nu}, F^{\mu\nu}] + O(\alpha'^3) \tag{3.19}$$

* We ignore linear divergencies (poles at $n = 1$)

in minimal subtracted dimensional regularization and

$$\beta_\lambda^{A(a)} = -\alpha' D^\mu F_{\mu\lambda} - i\alpha'^2 [D_\lambda F_{\mu\nu}, F^{\mu\nu}] + O(\alpha'^3) \quad (3.20)$$

in minimal subtracted a-regularization. Putting $\beta^A = 0$ e.g. for (3.20) gives the following equation for the background field A

$$D^\mu F_{\mu\lambda} + i\alpha' [D_\lambda F_{\mu\nu}, F^{\mu\nu}] = O(\alpha'^2). \quad (3.21)$$

Let us compare with the equation of motion related to the action derived in the $\alpha' \to 0$ limit of dual S-matrix elements in [7] (after fitting the different normalizations of the Yang-Mills field and use of the Bianchi identity)

$$D^\mu F_{\mu\lambda} + 12 i\alpha' [D_\mu F^{\mu\nu}, F_{\nu\lambda}] + 6 i\alpha' [D_\lambda F_{\mu\nu}, F^{\mu\nu}] = O(\alpha'^2). \quad (3.22)$$

Since $D_\mu F^{\mu\nu} = O(\alpha')$ the second term in (3.22) can be neglected. Then (3.21) and (3.22) differ only by a numerical coefficient. Having already established a renormalization scheme dependence, this difference is no surprise, however. We make some general comments on the scheme dependence in Sect. 5.

4. Abelian Gauge Field Background to 3-Loop Order

The order α' non-linearity in (3.21) vanishes for Abelian A_μ. To get contact with the "non-linear electrodynamics" of [8] we have to analyze the next order. Having in mind the renormalization scheme dependence this makes only sense if both approaches are based on the same scheme. In [8] ζ-regularization is performed. Using the formulas presented there one finds in a straightforward manner that minimal subtracted a-regularization reproduces the ζ-result exactly. The action of [8] derived to all orders in α' but for constant fields is

$$[\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})]^{1/2}. \quad (4.1)$$

Taking this expression seriously for arbitrary $F_{\mu\nu}$ one gets the equation of motion

$$\partial^\mu (F_{\mu\lambda} + \alpha'^2 \pi^2 (4 F_{\mu\alpha} F^{\alpha\beta} F_{\beta\lambda} - F_{\mu\lambda} F_{\alpha\beta} F^{\beta\alpha})) = O(\alpha'^3). \quad (4.2)$$

Let us turn to the study of ultraviolet divergencies. For the total sum of α'^3 contributions from diagrams of pure tadpole type we easily find

$$\frac{i\alpha'^3 \pi^3}{6} N_{\text{div}}^3 \int ds (\partial^2)^2 \partial^\mu F_{\mu\lambda} \dot{x}^\lambda(s) + \text{fin}. \quad (4.3)$$

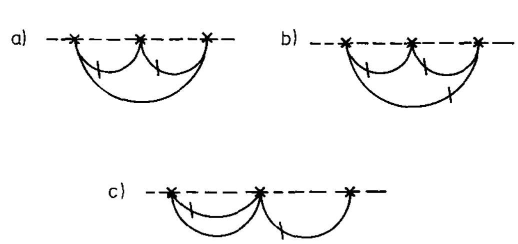

Fig. 8a–c. Possible divergent α'^3-diagrams without tadpoles (Abelian case). Each diagram stands for all contractions of the vertices involved

Since the coefficients of $\log a$ arising in single diagrams have cancelled, (4.3) is without any influence on the β-function we are looking for. As discussed in the previous section due to the symmetry properties of the propagator the coincidence of two vertices cannot produce a (logarithmic) divergence in the Abelian case. Hence we have to look for an effect from a triple coincidence of vertices. The diagrams potentially divergent in this region are shown in Fig. 8. We find

$$\begin{aligned} \text{Fig. 8a} &= -4 i\alpha'^3 \pi^3 \log a \int ds (\partial^\mu F_{\nu\lambda}) F_{\mu\rho} F^{\rho\nu} \dot{x}^\lambda(s) + \text{fin.} \\ \text{Fig. 8b} &= \text{fin.} \qquad (4.4) \\ \text{Fig. 8c} &= \text{fin.} \end{aligned}$$

This divergence is balanced by a counter term

$$\delta_3 A_\lambda = 4\pi^2 \log a (\partial^\mu F_{\nu\lambda}) F_{\mu\rho} F^{\rho\nu}. \quad (4.5)$$

The total β^A function to order α'^3 is

$$\beta_\lambda^A = -\alpha' \partial^\mu F_{\mu\lambda} - \alpha'^3 4\pi^2 (\partial^\mu F_{\nu\lambda}) F_{\mu\rho} F^{\rho\nu} + O(\alpha'^4). \quad (4.6)$$

(In dimensional regularization: $4\pi^2 \to 2\pi^2$), and the equation for F following from $\beta^A = 0$

$$\partial^\mu F_{\mu\lambda} + 4\pi^2 \alpha'^2 (\partial^\mu F_{\nu\lambda}) F_{\mu\rho} F^{\rho\nu} = O(\alpha'^3). \quad (4.7)$$

To compare (4.2) and (4.7) we write the α'^2 term of (4.2) as

$$\begin{aligned} &\alpha'^2 \pi^2 \partial^\mu (4 F_{\mu\alpha} F^{\alpha\beta} F_{\beta\lambda} - F_{\mu\lambda} F_{\alpha\beta} F^{\beta\alpha}) \\ &= 4\pi^2 \alpha'^2 (\partial^\mu F_{\nu\lambda}) F_{\mu\rho} F^{\rho\nu} - \alpha'^2 \pi^2 (\partial^\mu F_{\mu\lambda}) F_{\alpha\beta} F^{\beta\alpha} \\ &\quad + \alpha'^2 \pi^2 F_{\beta\lambda} (4 \partial^\mu (F_{\mu\nu} F^{\nu\beta}) - \partial^\beta (F_{\mu\nu} F^{\mu\nu})). \end{aligned}$$

The second term on the rhs is effectively $\sim \alpha'^3$. The third term has the structure $F_{\beta\lambda} f^\beta$. Such renormalizations of A_λ can be generated by a change of the coordinates x and an accompanying gauge transformation. Hence, in the same spirit as for the metric β-function one discards contributions which are due to a diffeomorphism of the manifold [5], we can neglect contributions to the β_λ^A function of the structure $F_{\beta\lambda} f^\beta$ as unphysical.

In this sense we get the remarkable result, that up to α'^2 the absence of ultraviolet divergencies in the 2-dimensional field theory yields the equation of motion of the "non-linear electrodynamics" of [8].

5. Conclusions

We have demonstrated how by the use of functional derivatives of the Wilson loop and careful treatment of other boundary effects the calculation of β-functions for a string in background fields can be extended from closed to open strings coupled at the end points to the Yang-Mills field. Future work should complete this first step analysis by inclusion of the 2-dimensional metric as well as the dilaton field. Furthermore, the possible impact of changing the boundary conditions has to be clarified. Finally, one should extend the analysis to type I superstrings.

The renormalization scheme dependence found by explicit calculation is a quite familiar property of any kind of renormalization. In the context of interpreting the background field equations gained by setting $\beta=0$ as equation of motion it certainly requires more attention. Here we have in mind studies looking for symmetry fixed numbers. As an example one would expect this in the superstring case for the Chern-Simons completion of dB.

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Appendix A

Evaluation of the Ultraviolet Divergent Part of Fig. 5d

Using the formulated Feyman rules one finds

$$\begin{aligned}\text{Fig. 5d} &= \tfrac{1}{2}\int ds \int d^2z'\, V^{\alpha A} V^{\beta B} B_{\alpha\beta}(z(s))(\partial_m x^\mu(z') \\ &\quad \cdot \omega_\mu^{BA} - \varepsilon_{mn}\partial_n x^\gamma(z') V_\mu^B S_{\nu\gamma}^\mu V^{\nu A}(z')) \\ &\quad \cdot \left(N(z,z')\frac{d}{ds}\frac{d}{dz'_m} N(z(s),z') \right. \\ &\quad \left. -\frac{d}{ds} N(z(s),z')\frac{d}{dz'_m} N(z,z')\right). \qquad \text{(A1)}\end{aligned}$$

The d^2z' integral is near $z'=z(s)$ logarithmically divergent. To study the ultraviolet divergent part we can put the background fields ω, S and V on the boundary. The second term in the remaining z' integral we treat by partial integration. The boundary term is ultraviolet convergent due to the antisymmetry of the derivated propagator. Hence we arrive at

$$\begin{aligned}\text{Fig. 5d} &= \int ds (V^{\alpha A} V^{\beta B} B_{\alpha\beta}\partial_m x^\mu \omega_\mu^{BA} \\ &\quad - \varepsilon_{mn}\partial_n x^\gamma S_\gamma^{\beta\alpha} B_{\alpha\beta})\dot{z}_k(s) \\ &\quad \cdot \int d^2z' N(z(s),z')\frac{d^2}{dz_k dz'_m} N(z,z') + \text{fin.} \qquad \text{(A2)}\end{aligned}$$

Let us call the d^2z' integral I_{km}. Then the ultraviolet divergent part of I_{km} has the structure

$$I_{km}=\delta_{km}A+\text{fin.}$$

A further contribution $\dot{z}_k\dot{z}_m B$ we can exclude since the divergence does not depend on the tangent orientation in the z-plane. From $A=1/2 I_{mm}$, $\frac{d^2}{dz_k dz'_m}N(z,z')=-\frac{d^2}{dz'_k dz'_m}N(z,z')$ near $z=z'$ and $\partial^2 N(z,z')=-\delta(z-z')$ we get

$$I_{km}=1/2\,\delta_{km}N(z(s),z(s))+\text{fin.} \qquad \text{(A3)}$$

Putting this into (A2) we arrive at (2.13).

Appendix B

Relation of β-Function Coefficients to Counter Terms

The relation of generalized β-function coefficients to that of single and multiple poles in dimensional regularization has been discussed in [5] for the metric contribution. We have some problems in correctly handling the corresponding dimensional scaling arguments for the Yang-Mills field coupling on the boundary. The main obstacle is how to handle one dimensional ∂M while M gets dimension $\neq 2$.

Therefore we define

$$\beta_\mu^A(A^{\text{bare}})=-\frac{d}{d\log a}A_\mu^{\text{bare}}|A\,\text{fix} \qquad \text{(B1)}$$

or

$$\beta_\mu^A(A^{\text{bare}})=-\frac{d}{d\left(\frac{1}{n-2}\right)}A_\mu^{\text{bare}}|A\,\text{fix}.$$

Writing

$$A_\mu^{\text{bare}}=A_\mu+\sum_n K_\mu^{(n)}(A)(\log a)^n \qquad \text{(B2)}$$

one gets

$$\beta_\mu^A(A^{\text{bare}})=-\sum_n K_\mu^{(n)}(A)n(\log a)^{n-1}.$$

Inverting ($B2$) and eliminating A in favour of A^{bare} we get $\beta^A(A^{\text{bare}})$ as a function of A^{bare} and $\log a$. The explicit $\log a$ dependence must cancel. This yields (replacing at the end the argument A^{bare} by A)

$$\beta_\mu^A(A) = -K_\mu^{(1)}(A) \tag{B3}$$

and relations among the $K_\mu^{(n)}$ starting with

$$K_\mu^{(2)}(A)(x) = \frac{1}{2}\int \frac{\delta K_\mu^{(1)}(A)(x)}{\delta A_\nu(y)} K_\nu^{(1)}(A)(y)\,dy. \tag{B4}$$

Relation (B4) is a welcome test of our calculations. From (3.3), (3.18) we know

$$K_\mu^{(1)}(A) = \alpha' D^\nu F_{\nu\mu} + O(\alpha'^2).$$

Then (B4) leads to

$$K_\mu^{(2)}(A) = \frac{\alpha'^2}{2}(D^2 D^\nu F_{\nu\mu} - 2i[D_\nu F^{\nu\rho}, F_{\rho\mu}]) + O(\alpha'^3). \tag{B6}$$

Picking up the $(\log a)^2$ terms in (3.11) and (3.15) we find consistency with this relation.

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Note Added in Proof. Meanwhile we realized that $\alpha' \to \alpha' \mu^{2-n}$ is the correct scaling required for dimensional regularization in Appendix B. This leads to $\beta_\lambda^{A(n)} \equiv -\mu \frac{\partial}{\partial \mu} A_\lambda = -\alpha' \frac{\partial K_\lambda^{(1)(n)}}{\partial \alpha'}$ and hence to the same 2-loop β^A-function as in a-regularization. The two β-functions differ then at the 3-loop level, however.