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FERMION MASSES FROM SYMMETRY

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We present a model with continuous or discrete abelian generation group G, which predicts all orders of magnitude for fermion masses and mixings correctly as a function of only one small parameter $\phi_S/M \approx \frac{1}{10}$. Here ϕ_S is the scale of G symmetry breaking and M the fundamental mass scale of the theory. No small Yukawa couplings or special choices of the scalar potential are needed.

We perform a systematic computerized scan for similar models with abelian generation group and we find a few other examples. However, for a wide range of charges we find no anomaly free continuous symmetry and also none without mixed $SU(3) \times SU(2) \times U(1), U(1)_G$ anomalies consistent with realistic fermion masses if G is broken by one field with definite charge. We also scan a class of models with generation symmetry derived from a higher dimensional framework.

1. Introduction

Most of the free parameters of the standard model [1] arise from fermion masses and their mixings appearing in the form of Yukawa couplings of the scalar doublet ϕ responsible for weak symmetry breaking and the scale $\phi_L \approx 174$ GeV. It is hoped since a long time that additional symmetries could restrict these couplings and explain the observed mass and mixing pattern for the fermions. In SU(5) gauge theories it was realized that masses of down type quarks and charged leptons are equal ($m_b = m_\tau$, $m_s = m_\mu$, $m_d = m_e$) if ϕ belongs to a 5-plet of SU(5). For the heaviest generation this mass relation proved successful once scaled down from a scale near the Planck mass to present energies. However, the relations $m_s = m_\mu$ and $m_d = m_e$ are not obeyed and additional multiplets (45) had to be introduced, which in turn also made the relation $m_b = m_\tau$ depend on special assumptions. In parallel, there have been various attempts to use discrete symmetries [3] for an understanding of some particular features of the fermion mass matrices. In general, one considers a

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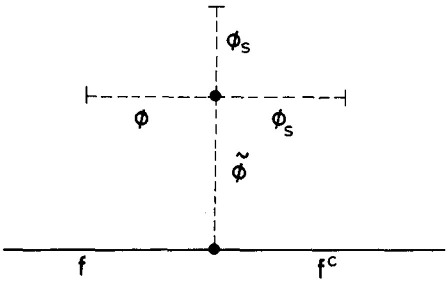

Fig. 1. A graphical representation of the generation of a mass of order $(\phi_S/M)^2\phi_L$.

restricted set of Higgs fields and forbids some Yukawa couplings by suitable discrete symmetries. Such a procedure can reproduce certain mass relations, but it gives no explanation about the whole structure of fermion mass matrices and leaves the question unanswered why certain fermion masses are much smaller than others.

There is a mechanism relating small fermion masses to a small scale of symmetry breaking for some symmetry G beyond $SU(3)\times SU(2)\times U(1)$. If the Yukawa couplings of the low-energy weak doublet ϕ to certain fermions vanish for reasons of G symmetry, these fermions would remain massless in the limit of unbroken G. For G spontaneously broken at a scale ϕ_S, the masses of such fermions must be proportional to some power of ϕ_S. Thus, for ϕ_S smaller than the overall mass scale M of the model, these fermions obtain masses suppressed by powers $(\phi_S/M)^P$ compared to the "natural scale" of fermion masses which is of the order of the W-boson mass.

There are different ways to describe this situation. For example, we can represent an expansion in ϕ_S/M graphically: Assume that the fermions f, f^c and the scalar doublet ϕ transform under G so that the Yukawa coupling $f^c f\phi$ is forbidden. Assume further another doublet $\tilde{f}$ with (positive) mass $\sim M^2$ (which belongs to a different G representation than ϕ), so that the Yukawa coupling $f^c f\tilde{\phi}$ is allowed. Consider a quartic coupling $\tilde{\phi}\phi\phi_S^2$, where ϕ_S is the field whose vacuum expectation value (VEV) is responsible for spontaneous breaking of G. Exchange of $\tilde{\phi}$ gives then a tree contribution to the mass of f in order $(\phi_S/M)^2\ \langle\phi\rangle$ as depicted in fig. 1. Alternatively, we could construct the full scalar potential for ϕ, $\tilde{\phi}$ and ϕ_S and find that the low-energy doublet acquires a small admixture of $\tilde{\phi}$ in order $(\phi_S/M)^2$. This leads in turn to a small mass for f. Still another way of saying this states that $\tilde{\phi}$ acquires a VEV of order $(\phi_S/M)^2\ \langle\phi\rangle$. (We describe this in detail in the next section.)

This mechanism for small fermion masses has first been considered [4] to show that VEV's of all $SU(2)_L$ triplets are very small of order ϕ_L^2/M – thus proving that the relation [5] for left-handed neutrino masses $m_\nu \sim \phi_L^2/M$ is obtained naturally and independent of the field content and specific couplings used in a model. The idea was subsequently applied [6] to understand in a general setting with scalars in various representations why $m_b = m_\tau$ is valid up to small corrections $(\phi_S/M)^P$ whereas a similar relation breaks down for the lower generation*. Early attempts to understand the generation pattern by this mechanism failed at this time, since no suitable symmetry G could be identified. In a similar spirit, small fermion masses due to a broken flavour symmetry were discussed in ref. [8] in models with heavy fermions**.

As a consequence of these developments, the philosophy with respect to scalars has shifted: instead of a special choice of scalar multiplets one rather works with many scalar fields in arbitrary representations and tries to understand specific properties in terms of symmetries and scales of their spontaneous breaking.

In this paper we take the more radical approach that *all* small fermion masses and mixings should be explained by a symmetry G and a small ratio ϕ_S/M involving the scale of its spontaneous breaking. We will not allow for small Yukawa couplings nor consider a particular selection of scalar multiplets and particular conditions on their interactions and masses, except those needed to obtain the required scales of spontaneous symmetry breaking.

This approach has first been advocated [9] in the context of higher dimensional unification. Indeed, dimensional reduction predicts (in the generic case) that all Yukawa couplings are of the order of the gauge coupling g or they vanish due to reasons of symmetry or topology [9,10]. Higher dimensional theories give also motivation for the existence of (infinitely) many scalar fields in various representations. They are obtained as modes of an expansion on internal space. A crucial ingredient in our understanding of fermion masses is the existence of an (approximate) symmetry G beyond SU(3) × SU(2) × U(1). This may be a local generation gauge symmetry, a global Peccei-Quinn-type symmetry [11] or a discrete symmetry. There are examples for all these different types of symmetries originating in higher dimensions [9,10,12] from properties of internal space (including other bosonic field configurations if necessary). It was shown that in the context of higher dimensional theories the existence of several chiral fermion generations is related to a chirality index [13] which depends on topology and symmetry of internal space. It is then natural that the differentiation between the fermion generations should also be related to properties of internal space.

The absence of small Yukawa couplings and the existence of many scalar fields imply that our approach to fermion masses is probably necessary in the context of all higher dimensional theories (including string theories). In four-dimensional theories other options remain open, but our approach seems nevertheless appealing.

* For an account of attempts to obtain naturally the top quark mass relation $(m_t/m_c) = (m_b/m_s)$ see ref. [7] and references therein.

** Our approach differs from [8] in several aspects: We are not committed a priori to a Fritzsch texture and we want to explain all small quantities (including m_b/m_t) by symmetries rather than small Yukawa couplings. We do not select certain representations for the Higgs fields and consider instead the most general Higgs sector consistent with the symmetries.

Since it relies entirely on symmetry concepts, we can always formulate it in a four-dimensional language, which we will use throughout this paper.

Our scenario is roughly as follows: The generation symmetry G is spontaneously broken at a scale ϕ_S somewhat smaller than the characteristic scale M of the model. (For the purpose of this paper the choice of M is arbitrary. It could be a very high unification scale – the compactification scale in higher dimensional theories, the string tension for superstrings or the GUT scale for some extended version of grand unification/family unification. In this case the structure of fermion mass matrices is related to a fine structure of scales around the unification scale [9]. Only one Higgs doublet ϕ survives at low energies. The other extreme case is a "low-energy" ($\approx$ TeV) scale M only somewhat above the weak scale $\phi_S \approx \phi_L$. This scenario requires several doublets in the range below a few TeV. It may be realized in supersymmetric theories with M the gravitino mass. Between these extreme scenarios one can of course consider possible combinations or a scale M in some intermediate range. We note, however, that the existence of several scalars at a low scale M has to be discussed carefully in view of possible problems with baryon number violation, strangeness violating neutral currents etc., as typical in models with low-energy supersymmetry. Also the general discussion of the next section has to be modified in the case of $\phi_S \approx \phi_L$.) The small scale ratio ϕ_S/M will appear with various powers in the fermion mass matrices. We try to find models where in leading order only the top quark acquires a mass, whereas m_b, m_τ and m_c are suppressed by ϕ_S/M. The masses for muon and strange quark should be generated by terms of order $(\phi_S/M)^2$, whereas the first generation masses should be $\sim (\phi_S/M)^{3-4}$. The same ratio ϕ_S/M has also to account for the observed size of all mixings. It is obviously not an easy task to find a symmetry G predicting all small fermion masses and mixings in terms of a single small parameter ϕ_S/M. On the other hand the observed structure in fermion mass matrices [14] suggests that this may indeed be possible.

This paper is organized as follows: In sect. 2 we describe the general pattern of weak symmetry breaking in presence of many scalar doublets in different representations of G and its consequences for fermion masses. In sect. 3 we present a model with abelian G which predicts correctly the observed order of magnitude of all fermion masses and mixings in terms of a single small parameter ϕ_S/M. In sect. 4 we describe a systematic search for such models by a computerized scanning procedure. In sect. 5 we give the results of such a scan for G an abelian U(1) symmetry and in sect. 6 we describe a scan of a six-dimensional SO(12) model.

2. Weak symmetry breaking with many scalar doublets

Let us now discuss systematically spontaneous symmetry breaking of weak interactions in presence of many scalar doublets and its consequences for fermion mass matrices. We assume N_G generations of standard quarks and leptons $(u, d)_i$, u_i^c, d_i^c, $(\nu, e)_i$ and e_i^c. (We consider only left-handed fermions.) These fermions couple to various scalar doublets ϕ_m in the usual way:

$$\mathscr{L}_{\text{Yuk}} = h_{ijm}^{(\text{U})} u_i^c u_j \phi_m + h_{ijm}^{(\text{D})} d_i^c d_j \phi_m + h_{ijm}^{(\text{L})} e_i^c e_j \phi_m^* + \text{h.c.}. \tag{2.1}$$

Here we have only listed the Yukawa couplings to the electrically neutral components of the scalar doublets. We note that because of weak hypercharge conservation down-type quarks and charged leptons couple to ϕ^* whereas up-type quarks couple to ϕ. If one or several doublets acquire VEV's, the fermion mass matrices are:

$$(M_\text{U})_{ij} = h_{ijm}^{(\text{U})} \langle \phi_m \rangle,$$

$$(M_\text{D})_{ij} = h_{ijm}^{(\text{D})} \langle \phi_m \rangle,$$

$$(M_\text{L})_{ij} = h_{ijm}^{(\text{L})} \langle \phi_m \rangle. \tag{2.2}$$

Let us assume that the scalar doublets are all distinguished by quantum numbers of some symmetry G. In the limit of unbroken G, the most general scalar potential up to quartic order is

$$V(\phi) = \mu_m^2 (\phi_m^\dagger \phi_m) + \lambda_{mnpq} (\phi_m^\dagger \phi_p)(\phi_n^\dagger \phi_q), \tag{2.3}$$

with

$$\mu_m^2 = (\mu_m^2)(\mu_m^2)^*,$$

$$\lambda_{mnpq} = \lambda_{nmqp} = (\lambda_{pqmn})^*. \tag{2.4}$$

The symmetry G imposes constraints on the quartic couplings λ. For example, a continuous abelian symmetry $\text{U}(1)_\text{G}$ with m, n etc. labeling integer charges requires

$$\lambda_{mnpq} = \lambda_{mnp} \delta_{m+n,\, p+q}. \tag{2.5}$$

We now add a scalar χ which is a singlet with respect to $\text{SU}(3)_c \times \text{SU}(2)_\text{L} \times \text{U}(1)_\text{Y}$ but transforms nontrivially with respect to G. The additional piece for the scalar potential is

$$V(\chi) + V(\chi, \phi) = m^2 \chi^* \chi + \tilde{V}(\chi) + (\tilde{\alpha}_{mn} (\phi_m^\dagger \phi_n) \chi + \text{h.c.}) + (\beta_{mn} (\phi_m^\dagger \phi_n) \chi^2 + \text{h.c.}) + \rho_m (\phi_m^\dagger \phi_m) \chi^* \chi. \tag{2.6}$$

Again, the couplings $\tilde{\alpha}$, β and ρ are restricted by G symmetry. There is a range of parameters where G symmetry is broken by

$$\langle \chi \rangle = \phi_S \tag{2.7}$$

and $SU(2)_L \times U(1)_Y$ is broken by some linear combination of ϕ_m

$$\langle \gamma_m^* \phi_m \rangle = \phi_L \tag{2.8}$$

which is chosen so that all orthogonal linear combinations of ϕ_m do not acquire a VEV. This defines the coefficients γ_m by

$$\langle \phi_m \rangle = \gamma_m \phi_L \,. \tag{2.9}$$

We are interested in a situation where

$$|\phi_L|^2 \ll |\phi_S|^2 \tag{2.10}$$

and furthermore ϕ_L consists mostly of one ϕ_{m_0} with small contributions from other ϕ_m

$$\begin{aligned} &\gamma_{m_0} \simeq 1 \,, \\ &|\gamma_m| \ll 1 \qquad \text{for} \quad m \neq m_0 \,. \end{aligned} \tag{2.11}$$

The first condition (2.10) usually requires fine tuning of one parameter (which we hope can be understood within a more fundamental theory) whereas the conditions for (2.11) will become clear in a moment. We use ϕ_L and ϕ_S instead of $\mu_{m_0}^2$ and m^2 to parametrize the potential. We further assume that all remaining parameters with dimension of mass are of the order of only one mass scale M (which we may identify with the fundamental mass scale of the theory – but this is not necessary):

$$\begin{aligned} &\mu_m^2 = \nu_m M^2 \qquad \text{for} \quad m \neq m_0 \,, \\ &\tilde{\alpha}_{mn} = \alpha_{mn} M \,. \end{aligned} \tag{2.12}$$

In addition, we take all ν_m positive.

We remain with dimensionless parameters ν, λ, α, β and ρ all assumed to be of order g^2 with g the gauge coupling of $SU(2)_L$. Similarly, all Yukawa couplings are assumed of order g. Is it possible to reproduce realistic fermion masses as a consequence of G symmetry, without any small dimensionless couplings (except the scale ratios $|\phi_L|/M$ and $|\phi_S|/M$)?

In presence of ϕ_S and $\langle \phi_{m_0} \rangle \approx \phi_L$ the static field equations for ϕ_m, $m \neq m_0$, read

$$\begin{aligned} &\nu_m M^2 \phi_m + \left(\alpha_{mn}\chi + \alpha_{nm}^* \chi^*\right) M \phi_n \\ &\quad + \left(\beta_{mn}\chi^2 + \beta_{nm}^* \chi^{*2}\right)\phi_n + \rho_m \chi^* \chi \phi_m \\ &\quad + \lambda_{mnpq}(1 + \delta_{mn})\left(\phi_n^\dagger \phi_q\right)\phi_p \,. \end{aligned} \tag{2.13}$$

For $\phi_L^2 \ll \phi_S^2$, M the last term is completely negligible. Assuming real ϕ_S for simplicity one finds

$$\phi_m = -\frac{\alpha_{mn} + \alpha^*_{nm}}{\nu_m + \rho_m(\phi_S/M)^2}\frac{\phi_S}{M}\phi_n - \frac{\beta_{mn} + \beta^*_{nm}}{\nu_m + \rho_m(\phi_S/M)^2}\left(\frac{\phi_S}{M}\right)^2\phi_n . \tag{2.14}$$

We are interested in a situation where the scale of G symmetry breaking ϕ_S is small compared to M

$$\phi_S^2 \ll M^2, \tag{2.15}$$

so that approximate G symmetry can be exploited. In this case we can solve (2.14) stepwise. In first order ϕ_S/M only fields ϕ_m with non-vanishing α_{m,m_0} or $\alpha_{m_0,m}$ couplings acquire a VEV $\sim \phi_L\phi_S/M$. We denote them by ϕ_{m_1}

$$\langle\phi_{m_1}\rangle \simeq -\frac{\alpha_{m_1m_0} + \alpha^*_{m_0m_1}}{\nu_{n_1}}\frac{\phi_S}{M}\langle\phi_{m_0}\rangle .$$

This is easily generalized to contributions with higher powers of ϕ_S/M. One has the recursive relation at level N

$$\langle\phi_{m_N}\rangle \simeq -\frac{\alpha_{m_Nm_{N-1}} + \alpha^*_{m_{N-1}m_N}}{\nu_{m_N}}\frac{\phi_S}{M}\langle\phi_{m_{N-1}}\rangle - \frac{\beta_{m_Nm_{N-2}} + \beta^*_{m_{N-2}m_N}}{\nu_{m_N}}\left(\frac{\phi_S}{M}\right)^2\langle\phi_{m_{N-2}}\rangle . \tag{2.16}$$

The solution is

$$\langle\phi_{m_p}\rangle = \eta_{m_p}\left(\frac{\phi_S}{M}\right)^P\phi_L ,$$

$$\gamma_{m_p} = \eta_{m_p}\left(\frac{\phi_S}{M}\right)^P \tag{2.17}$$

with η_{m_p} a ratio of dimensionless coupling constants which by assumption is of order one. As a consequence of G symmetry, the contribution of certain doublets ϕ_m to ϕ_L can be small. Weak symmetry breaking comes mainly from ϕ_{m_0}, with small admixtures of ϕ_{m_p} suppressed by $(\phi_S/M)^P$. The same small quantities will appear in the fermion mass matrices.

Obviously, the suppression factors $(\phi_S/M)^P$ are a pure group theoretical consequence of G symmetry. Their general structure has been discussed extensively in ref. [9]. Here we only note that the graphical representation with tree exchange of heavy $(\mu^2 \approx M^2)$ doublets (see fig. 1) can be extended to exchange of arbitrary particles with mass M. Also, terms higher than quartic can easily be included in the effective scalar potential, with coefficients scaled by inverse powers of M. The generalization to several scales χ_i for G symmetry breaking with associate factors $\prod_i(\chi_i/M)^{P_i}$ is straightforward. Finally, the same scenario could be obtained in models with only one scalar doublet ϕ_{m_0}, where the other ϕ_m can be identified with effective scalar operators generated by a loop expansion (radiative mass generation [20]).

3. A model for fermion mass matrices without small couplings

In this section we realize the ideas discussed previously in an example with abelian symmetry. A field with charge n transforms

$$\phi_n \to \exp(i\alpha n)\phi_n\,, \tag{3.1}$$

where α may be continuous $0 \leqslant \alpha < 2\pi$ (G = U(1)) or discrete $\alpha_k = 2\pi k/N$ with k, N integers (G = Z_N). We assume one scalar singlet χ with charge $n = 1$. The fermion content of our model is consistent with SU(5) and we consider a three-generation case. The three 10-plets (u, d, u^c, e^c) have charges $n = 0, 1, 2$ whereas the three $\bar{5}$ (d^c, ν, e) have charges $n = 1, 1, 2$. The continuous U(1) symmetry would be anomalous for this fermion content. We could either add $SU(3)_C \times SU(2)_L \times U(1)_Y$ non-chiral fermions to cancel the anomalies and consider $U(1)_G$ as a local generation group, or we could treat $U(1)_G$ as an anomalous global Peccei-Quinn-type symmetry. Alternatively, we can consider a discrete symmetry Z_N with $N \geqslant 8$.

We allow for all possible scalar doublets ϕ_n, but only those with $|n| \leqslant 4$ can have Yukawa couplings to chiral fermions and are of interest for us. We arrange the potential so that the leading doublet is ϕ_0

$$\langle\phi_0\rangle \simeq \phi_L \simeq 174\ \text{GeV} \tag{3.2}$$

and

$$\langle\chi\rangle = \phi_S \simeq \tfrac{1}{10}M\,. \tag{3.3}$$

These two relations, together with the G quantum numbers, are the essential ingredients of our model. As described before, all remaining dimensionless parameters in the scalar potential are of order g^2 and all Yukawa couplings of order g. The scalar couplings obey (2.5) and

$$\begin{aligned} \alpha_{mn} &= \alpha_m \delta_{m,\,n+1}\,, \\ \beta_{mn} &= \beta_m \delta_{m,\,n+2}\,, \end{aligned} \tag{3.4}$$

and similar for Yukawa couplings. In case of Z_N symmetry all Kronecker δ_{mn} are modulo N. Neglecting for a moment differences between Yukawa couplings, the fermion mass matrices have the form

$$M_U \simeq g \begin{pmatrix} \phi_{-4} & \phi_{-3} & \phi_{-2} \\ \phi_{-3} & \phi_{-2} & \phi_{-1} \\ \phi_{-2} & \phi_{-1} & \phi_0 \end{pmatrix},$$

$$M_D \simeq g \begin{pmatrix} \phi_4^* & \phi_3^* & \phi_2^* \\ \phi_3^* & \phi_2^* & \phi_1^* \\ \phi_3^* & \phi_2^* & \phi_1^* \end{pmatrix},$$

$$M_L \simeq g \begin{pmatrix} \phi_4^* & \phi_3^* & \phi_2^* \\ \phi_3^* & \phi_2^* & \phi_1^* \\ \phi_3^* & \phi_2^* & \phi_1^* \end{pmatrix}. \tag{3.5}$$

By a suitable redefinition of the last two lines in M_D and the last two rows in M_L we obtain $(M_D)_{23} = 0$, $(M_L)_{32} = 0$. (There are two d^c and two e with identical G quantum numbers.)

The leading doublet ϕ_0 only couples to one element in M_U. Therefore only the top quark mass is predicted of order M_W (the mass of the weak vector boson). For definiteness, we take $m_t \approx 40$ GeV. All other VEV's of doublets are suppressed and one finds from (3.4), (2.16)

$$\langle \phi_n \rangle \simeq \left(\frac{\phi_S}{M} \right)^{|n|} \langle \phi_0 \rangle. \tag{3.6}$$

At the level $|n| = 1$ we obtain from ϕ_1^* the mass for bottom and tau in M_D and M_L. A value $m_b \approx 4.5$ GeV is well consistent with (3.3). Since we assume here M near the Planck scale ($M \approx 10^{17-18}$ GeV) we account for the different normalization of quarks and leptons by multiplying measured lepton masses with a correction factor ≈ 2.5–3. As is well known, the relation $m_\tau \approx m_b$ is valid in this case. Also, ϕ_{-1} couples to $(M_U)_{32}$ and $(M_U)_{23}$. For suitable Yukawa couplings one may obtain $(M_U)_{32} \approx 5$ GeV, $(M_U)_{23} \approx 8$ GeV again consistent with (3.3). This gives a contribution to the charm quark mass of about 1 GeV. A second contribution to m_c of a few hundred MeV comes from ϕ_{-2}. Also, values $m_s \approx 150$ MeV, $m_\mu \approx 300$ MeV from ϕ_2^* are consistent with a suppression $(\phi_S/M)^2 \approx 10^{-2}$. Both $(M_U)_{32}$ and $(M_D)_{32}$ give a mixing of order 10% between the second and third generation. A resulting mixing angle $\theta_{23} \approx 5\%$ appears rather naturally as the difference of mixing in M_U and M_D. The contribution to θ_{13} from ϕ_{-2} in $(M_U)_{31}$ is of order ϕ_S^2/M^2 and a value of 0.5% is again within the orders of magnitude. The VEV ϕ_3^* gives a similar contribution to θ_{13} in M_D. With a relative minus sign between both contributions the measured smallness of θ_{13} can well be explained. At the level $|n| = 3$ the VEV ϕ_3^* gives a typical contribution of about 30 MeV to $(M_D)_{21}$ and $(M_D)_{12}$. This both accounts for the Cabibbo angle θ_{12} and the successful relation $m_d \approx \sin^2(\theta_{12}) m_s$. Similarly, a value $(M_L)_{21} \approx (M_L)_{12} \approx 20$ MeV would give a contribution of about 1.5 MeV to m_e. On the other hand, the off-diagonal contributions to $(M_U)_{21}$ and $(M_U)_{12}$ from ϕ_{-3} are also expected of a few ten MeV. They are too small to give a sizeable contribution to θ_{12} and m_u. The up quark mass is finally generated by ϕ_{-4} which contributes a typical order of magnitude of a few MeV as it should be. Contributions ≈ 1 MeV from ϕ_4^* to m_d and m_e would again be consistent with observation, especially if the diagonal (from ϕ_4^*) and off-diagonal (from ϕ_3^*) contributions to m_e have a relative minus sign.

We conclude that our assignment of abelian quantum numbers together with the scale ratio (3.3) is sufficient to account for the structure of all fermion mass matrices including the mixing angles. This scenario is consistent with all Yukawa couplings given by the weak gauge coupling within factors of three. Such factors could well be explained by group theoretical Clebsch-Gordan coefficients or other dynamical details of a fundamental theory. No need for small Yukawa couplings or additional fine tuning of parameters in the scalar potential arises.

What about the predictive power of our model? The fermion mass matrices involve nine different scalar VEV's ϕ_m which can be considered as parameters whose order of magnitude is known. This predicts the order of magnitude of all fermion masses and mixings correctly. More quantitative predictions, however, depend in addition on the various Yukawa couplings of the model and there are uncertainties up to a factor five or so. Nevertheless, the model has some particular features going beyond the most general possible form of mass matrices. For example, the Cabibbo mixing is essentially generated in M_D with very little contribution from M_U and the model explains qualitatively why $m_d > m_u, m_e$.

We could also enlarge the symmetry and consider a model based on $SU(5) \times G$ where χ belongs to a 24 (or 75) of SU(5) and breaks simultaneously SU(5) and G. The scalars ϕ_n coupling to quarks and leptons belong to 5 or 45 of SU(5). This imposes additional relations between Yukawa couplings. We note that a scale ratio $\phi_S/M \approx \frac{1}{10}$ is in the right order of magnitude to account for corrections to the gauge couplings making their renormalization group evolution consistent [15] with proton stability (at the present level) and the observed value of the Weinberg angle.

4. Symmetry scanning

Are there other symmetries G equally successful as the particular abelian symmetry of the last section, or perhaps even more predictive? Since orders of magnitude for fermion masses and mixings only depend on G and one (or several) scale ratio ϕ_S/M, a systematic scan of acceptable symmetries G should be possible. In this section we describe a computerized scan [16] for three generations and G abelian. For simplicity, we will consider only continuous abelian symmetries. The discrete case Z_N can be obtained by requiring additivity of charges only modulo N and will yield results identical to the continuous case for sufficiently high N.

Conceptionally, a scan for realistic models can be performed at different stages with increasingly restrictive requirements. At the first stage we only need to know which scalar ϕ_m couples to which elements in the fermion mass matrices. We impose as a necessary criterion that there must be at least one possible assignment of scales to the different ϕ_m which reproduces correctly all orders of magnitude for fermion masses and mixings. (For example, a symmetry G allowing that the doublet responsible for m_b also couples to the up quark would predict $m_b \approx m_u$ and will be discarded.) At this first stage we do not yet attempt to calculate the different scales in powers of ϕ_S/M.

Consider $G = U(1)^n$. We denote by $Q_k(q_i)$ $(k=1\dots n)$ the charges for the ith quark doublet and similarly by Q_k (u_i^c), Q_k (d_i^c), the charges for u_i^c, d_i^c etc. The leading Higgs doublet, responsible for the top quark mass, must have charge

$$Q_k(\phi_{t^c t}) = -Q_k(t^c) - Q_k(t). \tag{4.1}$$

More generally, doublets with charges

$$Q_k = -Q_k(u_i^c) - Q_k(q_j) \tag{4.2}$$

couple to the (ij)-element in the up quark mass matrices M_U and doublets with

$$Q_k = Q_k(d_i^c) + Q_k(q_j),$$

or

$$Q_k = Q_k(e_i^c) + Q_k(L_j) \tag{4.3}$$

couple to the (ij)-element in M_D or M_L. Note the difference in sign between (4.2) and (4.3) due to the opposite hypercharge of the corresponding fermion bilinears.

For the systematics of our scanning we require that all small quantities below an order of magnitude in the fermion mass matrices should be explained by a symmetry G and different scales $\langle\phi_m\rangle$. This concerns small ratios of mass eigenvalues as well as the small mixing angles. We allow for the possibility that factors up to 5 are attributed to group theoretical Clebsch-Gordan coefficients or other details of dynamics. There is of course some arbitrariness in the choice of a boundary for small quantities not explained by G. Given the fact that quantities as small as 10^{-5} (m_e/m_t) appear, this does not change the overall scheme.

Our scanning program is based on the observation that for the three-generation case the structure of fermion mass matrices is well described by four (or five) scales.

These scales themselves are separated by about an order of magnitude. The highest scale is the top quark mass, which we assume to be several ten GeV. The second scale is a few GeV. At this level we have the masses for b-quark, τ-lepton and c-quark, with the latter a factor three smaller than the others. The masses for strange quark and muon constitute the third level of a few hundred MeV. The fourth scale is a few ten MeV or below and may be responsible for Cabibbo mixing in M_{D} and for the lowest generation masses. If we are more severe we require a separate fifth scale of a few MeV or below for the electron mass and possibly also for m_{u}. The only other information about the mass matrices comes from measured mixing angles. The Cabibbo angle is fairly large ($\sim 20\%$) whereas the mixing between the second and third generation is about 4%. The limit on mixing between the first and third generation is somewhat less than 1% [17]. Nothing is known about lepton mixing angles.

We will denote the different scales by n_{s}, every scale being a few times $10^{n_{\mathrm{s}}}$ MeV. For the example of sect. 3 the scales n_{s} are

$$M_{\mathrm{U}}=\begin{pmatrix}0&1&2\\1&2&3\\2&3&4\end{pmatrix},\qquad M_{\mathrm{D}}=\begin{pmatrix}0&1&2\\1&2&3\\1&2&3\end{pmatrix},\qquad M_{\mathrm{L}}=\begin{pmatrix}0&1&1\\1&2&2\\2&3&3\end{pmatrix}. \tag{4.4}$$

Consistency with observation requires the following upper bounds on elements in M_{U}, M_{D} and M_{L}:

$$M_{\mathrm{U}}\leqslant\begin{pmatrix}1(0)&3&4\\2&3&4\\2&3&4\end{pmatrix},$$

$$M_{\mathrm{D}}\leqslant\begin{pmatrix}1&2&3\\1&2&3\\1&2&3\end{pmatrix},$$

$$M_{\mathrm{L}}\leqslant\begin{pmatrix}1(0)&2&3\\2&2&3\\2&2&3\end{pmatrix}\quad\text{or}\quad M_{\mathrm{L}}\leqslant\begin{pmatrix}1(0)&2&2\\2&2&2\\3&3&3\end{pmatrix}. \tag{4.5}$$

The mass matrices (4.5) have been ordered here in the standard way. The bounds on the M_{33} elements come from the maximal size of mass eigenvalues. This entry is always required to generate the largest mass. The bounds on M_{32} and M_{31} come from the observed small values of the mixing with the third generation. This does not provide a bound on these elements in M_{L} though. The bounds on M_{23} and M_{13} come from the size of the largest mass since these elements can always be removed by a left multiplication of a unitary matrix which is unobservable. The bound on M_{22} and M_{12} is from the 2nd eigenvalue and the one on M_{21} comes from the smallness of the Cabibbo angle*. In the lepton matrix limits only come from the eigenvalues. Hence, whenever M_L is acceptable, M_L^T (transposed) is too. Finally the bounds on M_{11} reflect the smallness of first generation masses. The lower masses could also be generated by paired off-diagonal elements [18] and we have to impose additional "quadratic" constraints. In terms of n_s they read

$$\begin{aligned}
&(M_U)_{13} + (M_U)_{31} \leqslant 5(4), \\
&(M_U)_{21} + (M_U)_{12} \leqslant 4(3), \\
&(M_D)_{21} + (M_D)_{12} \leqslant 3, \\
&(M_L)_{13} + (M_L)_{31} \leqslant 4(3), \\
&(M_L)_{12} + (M_L)_{21} \leqslant 3(2).
\end{aligned} \tag{4.6}$$

The sizes in brackets are those where we require a fifth scale for the electron mass or the up quark mass**.

We now can describe our scanning procedure: The fermion masses are generated level by level. At each step we try all possible assignments of the required scales n_s to suitable entries*** in the corresponding mass matrix. If the same doublet ϕ_m is allowed to couple to more than one entry in M_U, M_D or M_L we assign the same scale n_s. Consistency is then checked by comparing the scale pattern with the bounds (4.5)–(4.6). A model is rejected if at some level no consistent assignment is found. We note that we arbitrarily can permute all rows in fermion mass matrices as well as columns in M_L in order to bring them to the standard form. For the quark mass matrices, permutations of columns have to be done simultaneously in M_U and M_D in order to keep track of mixing angles.

At the first level we look for an entry only appearing in one column of M_U and not in M_D or M_L. This defines the top mass with $n_s = 4$. At level two we first assign a candidate for m_b with $n_s = 3$. We veto if the level $n_s = 3$ appears in more than one column in M_D or M_L or if it appears in more than one column outside the top column in M_U. If m_τ is not generated by the m_b entry, we try additional $n_s = 3$ entries in M_L. The same procedure then applies to m_c which can be generated either by diagonal or paired off-diagonal $n_s = 3$ entries. The combined set of all $n_s = 3$

* This is however a limiting case. If we would allow the Cabibbo angle to be of order 1 these elements could be an order of magnitude higher.

** As a limiting case we could allow $(M_L)_{33} \approx (M_L)_{32} \approx (M_L)_{23} = 3$ in the lepton mass matrix. This would require 3 factors of O(1) to conspire to produce a difference in order of magnitude. A similar argument would increase all the bounds in eq. (4.6) by 1. In this case no new information (assuming the Cabibbo angle $O(10^{-1})$) is contained in the quadratic constraints. This, however, requires a set of O(1) factors to go all in the right direction. We discard this possibility in our scanning.

*** In the remainder of this paper we use the word element to refer to a specific fermion bilinear. We use entry for all fermion bilinears coupling to a specific vacuum expectation value.

entries is subject to the consistency veto described for m_b. Here the last column and row has been determined for $M_{U,D,L}$. At the third level we first generate m_μ by an $n_s = 2$ entry. We again allow for diagonal or paired off-diagonal entries. We veto if this entry appears in the first column of M_D or in $(M_U)_{11}$ or $(M_L)_{11}$ or if one of the quadratic bounds (4.6) is violated. If m_s is not yet generated, we assign additional $n_s = 2$ entries in M_D with the same veto. At the end of level three all generation numbers u, c, t etc. are assigned to the various rows and columns. It is now easy to check by inspection of the various levels 4, 3 and 2 in the mass matrices if sufficient mixings θ_{23} and θ_{12} are already generated. If not, we have to assign for θ_{23} an appropriate $n_s = 2$ or 3 entry in M_U or an $n_s = 1$ or 2 entry in M_D. The same holds for θ_{12} with $n_s = 2$ or 1 for M_U and M_D, respectively. Of course, possible $n_s = 3$ or 2 entries are subject to the appropriate consistency vetos of level two or three, respectively. Finally, we check if all first-generation masses can be generated by $n_s = 1$ entries. This will always be the case unless "topological reasons" enforce the absence of certain doublets coupling to the first generation bilinears. We can account for such topological restrictions by setting appropriate entries to zero ($n_s = -10$).

The scanning just described uses rather mild consistency criteria. In particular, it does not differentiate between entries below a few ten MeV, thus accounting for the theoretical uncertainty for the lowest masses, which involve a high order of G symmetry breaking and therefore products of many dimensionless couplings. We will refer to this criterion as (a). We also have implemented two additional stronger criteria: Criterion (b) requires that the mixing angle θ_{23} is of the same order as m_S/m_b. This requires either an $n_s = 3,2$ entry in $(M_U)_{32}$ or an $n_s = 2$ entry in $(M_D)_{32}$. Indeed, a generation of θ_{23} by $(M_D)_{32} = 1$ for case (a) is at the borderline where mixing between the second and third generation becomes unacceptably small. For criterion (c) we require in addition that the electron mass is at a fifth level $n_s = 0$. Consistency then requires the values in the brackets for M_L in (4.5) and (4.6). A similar $n_s = 0$ requirement for the up quark mass can be advocated, but we did not implement this additional restriction yet. It is easy to add other restrictions in this scanning process motivated by certain observed mass relations. For example, we can require that m_b and m_τ are generated by the same $n_s = 3$ entry so that suitable symmetries for Yukawa couplings could predict $m_b = m_\tau$. Another possibility motivated by the successful relation $m_d/m_s \approx \sin^2\theta_{12}$ would require that m_d is generated by paired off diagonal $n_s = 1$ elements in $(M_D)_{21}$ and $(M_D)_{12}$ and that $(M_U)_{21}$ is at a level $n_s \leqslant 1$. This concludes our discussion of the first stage of scanning which will be referred to as I.

Stage two of our scanning process (II) accounts for the various powers $(\phi_S/M)^P$ appearing in doublet VEV's as described in sect. 2. In leading order ϕ_S/M * the

* Actually, the leading order could also be $(\phi_S/M)^2$ if cubic couplings play no role. This replacement does not change our argument.

doublet $\phi_{b^c b}$ coupling to m_b should mix with the leading doublet $\phi_{t^c t}$. The charges $Q_k(\chi)$ for the scalar χ breaking G must therefore be

$$\pm Q_k(\chi) = Q_k(\phi_{t^c t}) - Q_k(\phi_{b^c b})$$
$$= -Q_k(t^c) - Q_k(t^c) - 2Q_k(t). \tag{4.7}$$

Here the same overall sign must hold for all k (note $Q_k(t) \equiv Q_k(b)$). The charges $Q_k(\chi)$ of the G symmetry breaking operator being known, we are now able to determine the "chain" of doublet VEV's produced by this operator. This is particularly easy in the case of $U(1)^n$: Any doublet with charges

$$Q_k(\phi_m) = Q_k(\phi_{t^c t}) \pm P Q_k(\chi) \tag{4.8}$$

will be suppressed by a factor $(\phi_S/M)^P$. (Again, (4.8) must hold with the same sign for all k. Doublets not obeying (4.8) for some P do not get VEV's within the chain of χ.)

The chains of operators are easily implemented in our scanning procedure. After the assignment of $\phi_{b^c b}$ we calculate $Q_k(\chi)$. Whenever a scalar doublet ϕ_m appears in the chain of χ in order P (suppressed by $(\phi_S/M)^P$) we assign a value $n_s = 4 - P$ to the corresponding elements in the fermion mass matrices. Values assigned via a chain have then to be taken into account for checks of consistency with (4.5)–(4.6). If the tau lepton does not acquire a mass at $n_s = 3$ in the chain of χ, we assign an independent $n_s = 3$ element from $\phi_{\tau^c \tau}$. Charges of additional operators χ' are calculated and the combined chain of χ and χ' will be determined as above. (For example, an $n_s = 2$ element can be generated by one step in the chain of χ and one step in the chain of χ'.) This procedure can be repeated for arbitrary $n_s = 3$ elements. We also could assign chains for operators essentially needed to generate additional $n_s = 2$ elements not appearing in the chains of previous operators. So far we did not yet implement this in our program.

So far we have allowed for an arbitrary number of G symmetry breaking scalar VEV's χ_i. At the third stage of scanning (III) we only consider a limited number of symmetry breaking operators. For the most extreme case III_1 we will allow only one χ and require that *all* elements in fermion mass matrices are generated by the chain of χ. This is perhaps the most interesting case, since fermion mass matrices are described in terms of only one small scale ratio generated by one symmetry breaking operator. Especially for the case of G having several U(1) factors, this requirement may be too restrictive. A single χ will not completely break G and certain fermions may not be allowed to acquire masses unless G is completely broken. We therefore consider also the cases III_2 and III_3 where G is broken by two or three operators χ_i inducing $n_s = 3$ elements in lowest order in their chains. Finally we consider cases III_1' and III_1'' where only one chain is allowed which generates $n_s = 3$ elements in lowest order, but arbitrary doublets not in the chain of χ can acquire VEV's at level $n_s \leqslant 2$ or $n_s \leqslant 1$, respectively. This accounts for the case of two symmetry breaking operators with different scales.

5. Scanning for G = U(1)

In this section we discuss the case G = U(1). (The same patterns can also always be produced by choosing an abelian discrete symmetry Z_N, with high enough N.) We first consider arbitrary U(1) charges for the fermions and start with a short description of the sample we want to scan. We only include the fermions present in the standard model with three generations. We assign arbitrary integer charges to all fermions and we denote here the quantum number by the same symbol as the fermion, i.e. u_i^c instead of $Q(u_i^c)$. There are 15 independent SU(3) × SU(2) × U(1) representations and a corresponding number of charges to be assigned – $u_i^c, d_i^c, q_i, e_i^c, L_i \quad (i = 1,2,3)$.

If we consider a sample with all quantum numbers smaller in absolute value than n, there are $(2n+1)^{15}$ possible assignments of the quantum numbers. It is, however, impossible to distinguish the three u_i^c, the three d_i^c, etc. so that the real number of distinct assignments per particle type is not $(2n+1)^3$ but rather all possible ways to have three numbers of absolute value less than n. This is

$$N = \binom{2n+1}{3} + 2\binom{2n+1}{2} + \binom{2n+1}{1}, \tag{5.1}$$

where $\binom{a}{b} = a!/(b!(a-b)!)$. The number of charge assignments is now given by

$$n_{\mathrm{U}(1)} = N^5. \tag{5.2}$$

Both N and N^5 are listed in table 1 for $n = 1, 3$. The overall sign of charges is also irrelevant, this almost halves the number of distinct charge assignments. The number left is slightly more than half the previous one because assignments of the type

$$\begin{aligned} &(u_1^c, u_2^c, u_3^c) = (-u, 0, u), \qquad (d_1^c, d_2^c, d_3^c) = (-d, 0, d), \\ &(q_1, q_2, q_3) = (-q, 0, q), \qquad (e_1^c, e_2^c, e_3^c) = (-e, 0, e), \\ &(L_1, L_2, L_3) = (-L, 0, L) \end{aligned} \tag{5.3}$$

TABLE 1

The number of distinct U(1) charge assignments for a given maximum size of the quantum numbers

n	N	N^5			
1	10	100 000	49 984	10 816	1 808
2	35	52 521 875	26 260 816	2 067 309	826 956
3	84	4 182 119 424	2 091 059 200	77 447 168	41 489 664

For an explanation of the columns see the text.

were not double counted. So the number of distinct charge assignments is

$$N_{U(1)} = \tfrac{1}{2}\left[N^5 + (n+1)^5\right]. \tag{5.4}$$

This is listed in the 3rd column of table 1. This number is still quite large and a scan as described in sect. 4 would still be very time consuming for the case $n = 2$. Let us try to diminish this number further by eliminating equivalent assignments and assignments which obviously do not lead to acceptable mass patterns. What matters is whether couplings $\phi_i u_j^c q_k$, $\phi_i^* d_j^c q_k$, $\phi_i^c e_j^c L_k$ are non-zero or not. They only depend on the relative charges of the fermion bilinears. The charges of these bilinears are

$$\begin{aligned} (Q_U)_{ij} &= u_i^c + q_j, \\ (Q_D)_{ij} &= d_i^c + q_j, \\ (Q_L)_{ij} &= e_i^c + L_j. \end{aligned} \tag{5.5}$$

We can now change the quantum numbers without changing the structure of the fermion bilinears if

$$\begin{aligned} \delta\big((Q_U)_{33} - (Q_U)_{ij}\big) &= 0, \\ \delta\big((Q_U)_{33} + (Q_D)_{ij}\big) &= 0, \\ \delta\big((Q_U)_{33} + (Q_L)_{ij}\big) &= 0. \end{aligned} \tag{5.6}$$

The changes in quantum numbers obviously have to be generation symmetric, i.e.

$$\delta u_1^c = \delta u_2^c = \delta u_3^c = \delta u, \qquad \delta d_1^c = \delta d_2^c = \delta d_3^c = \delta d^c, \dots . \tag{5.7}$$

The relations (5.6) are then satisfied if

$$\begin{aligned} \delta u^c &= -\delta_1 + \delta_3, \\ \delta d^c &= -\delta_1 - \delta_3, \\ \delta q &= \delta_1, \\ \delta e^c &= -\delta_2 - \delta_3, \\ \delta L &= \delta_2. \end{aligned} \tag{5.8}$$

From (5.8) it is obvious that we can always choose

$$u_1^c = q_1 = L_1 = 0. \tag{5.9}$$

We then still have 12 free charges (instead of the 15 earlier ones). Requiring these charges to be integer and of absolute value smaller than or equal to n leaves a significantly lower set of distinct assignments. This number can be determined using similar reasoning as before and is listed in the 4th column of table 1.

One last easy requirement can be made on the quantum numbers. From the discussion in sect. 4 it is obvious that all generations of the left-handed quark doublets need to have different quantum numbers. This reduces the numbers further to those mentioned in the last column of table 1. We have performed a scan for $n = 2$ for the case with only one $SU(3) \times SU(2) \times U(1)$ scalar singlet χ, i.e. for the case where all masses are generated by a single chain.

Each scan (III_1) of this sample required on the order of 40 minutes CPU time on an IBM mainframe. Scans under I, III_2 or III_3 conditions are more CPU time consuming. There were 301 different charge assignments possible using the (a) requirement on the sizes of mass matrix elements. Using the (b) requirement this number was unchanged and using the (c) requirement, the most natural ones, there were 133 different charge assignments giving an acceptable mass pattern. A complete description of these solutions is given in appendix A. These are the only charge assignments reproducing the observed mass pattern with a single small scale difference ϕ_S/M within this range of quantum numbers.

In the case of continuous symmetries we can impose stronger restrictions on the allowed U(1) charge assignments. A local $U(1)_G$ symmetry requires for consistency that there are no mixed anomalies with $SU(3) \times SU(2) \times U(1)$ and no pure $U(1)_G$ anomalies or mixed gravitational anomalies. The pure $U(1)_G$ anomalies and mixed gravitational anomalies could be cancelled by adding $SU(3) \times SU(2) \times U(1)$ singlet fermions which are chiral with respect to $U(1)_G$. These would acquire a mass at the scale ϕ_S at which $U(1)_G$ is broken and not change the physics at the weak scale. Similarly, mixed anomalies could be cancelled by adding fermions that are chiral with respect to $U(1)_G$ and belong to non-trivial vector-like reducible representations of $SU(3) \times SU(2) \times U(1)$. After $U(1)_G$ breaking they can also require a heavy mass. This leads us back to the general case already discussed earlier. If we only allow for additional heavy $SU(3) \times SU(2) \times U(1)$ singlet fermions (as the right-handed neutrino), we still have to require the absence of all mixed anomalies.

For a global U(1) it is also often undesirable to have mixed anomalies with the $SU(3) \times SU(2) \times U(1)$ gauge interactions. These would lead upon spontaneous breakdown of the global symmetry to an almost Goldstone boson. When mixed anomalies with respect to SU(3) are present (a Peccei-Quinn U(1) symmetry) there are (very) strong bounds for the existence of such a particle (the axion) [19].

The absence of mixed anomalies leads to the following requirements on the fermion quantum numbers:

$$
\begin{aligned}
\mathrm{SU}(3)^2\times \mathrm{U}(1)_{\mathrm{G}}:&\quad \sum_{i=1}^{3} u_i^{\mathrm{c}}+d_i^{\mathrm{c}}+2q_i=0,\\
\mathrm{SU}(2)^2\times \mathrm{U}(1)_{\mathrm{G}}:&\quad \sum_{i=1}^{3} 3q_i+L_i=0,\\
\mathrm{U}(1)_Y^2\times \mathrm{U}(1)_{\mathrm{G}}:&\quad \sum_{i=1}^{3} 8u_i^{\mathrm{c}}+2d_i^{\mathrm{c}}+q_i+6e_i^{\mathrm{c}}+3L_i=0,\\
\mathrm{U}(1)_Y\times \mathrm{U}(1)_{\mathrm{G}}{:}^2&\quad \sum_{i=1}^{3} -2(u_i^{\mathrm{c}})^2+(d_i^{\mathrm{c}})^2+(q_i)^2+(e_i^{\mathrm{c}})^2-(L_i)^2=0.
\end{aligned}
\tag{5.10}
$$

In the case of one generation there are two solutions, $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_{B-L}$. The latter doesn't satisfy the pure $\mathrm{U}(1)_{\mathrm{G}}$ anomalies without adding $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ singlet fermions (i.e. right-handed neutrinos). The extra conditions for completely anomaly free $\mathrm{U}(1)_{\mathrm{G}}$ symmetries are

$$
\begin{aligned}
\mathrm{U}(1)_{\mathrm{G}}^3:&\quad \sum_{i=1}^{3} 3(u_i^{\mathrm{c}})^3+3(d_i^{\mathrm{c}})^3+6(q_i)^3+(e_i^{\mathrm{c}})^3+2(L_i)^3=0,\\
\mathrm{U}(1)_{\mathrm{G}}:&\quad \sum_{i=1}^{3} 3u_i^{\mathrm{c}}+3d_i^{\mathrm{c}}+6q_i+e_i^{\mathrm{c}}+2L_i=0.
\end{aligned}
\tag{5.11}
$$

Eq. (5.11b) is the requirement that the anomalous $\mathrm{U}(1)_{\mathrm{G}}$-graviton-graviton coupling vanishes since that would spoil local gauge invariance.

Removing anomalies by shifts of charges without changing the mass pattern is in general not possible. The change of the $\mathrm{U}(1)_{\mathrm{G}}^3$ and $\mathrm{U}(1)_{\mathrm{G}}^2\times\mathrm{U}(1)_Y$ anomaly under the δ_1, δ_2 and δ_3 transformation described earlier (5.8) will in general only make them vanish using non-integer δ_i. The change of the anomalies linear in $\mathrm{U}(1)_{\mathrm{G}}$ is given by:

$$
\begin{aligned}
\delta\big(\mathrm{SU}(3)^2\times\mathrm{U}(1)_{\mathrm{G}}\big)&=0,\\
\delta\big(\mathrm{SU}(2)^2\times\mathrm{U}(1)_{\mathrm{G}}\big)&=3\delta_1+\delta_2,\\
\delta\big(\mathrm{U}(1)_Y^2\times\mathrm{U}(1)_{\mathrm{G}}\big)&=-3(3\delta_1+\delta_2),\\
\delta\big(\mathrm{U}(1)_{\mathrm{G}}\big)&=\delta_2-\delta_3.
\end{aligned}
\tag{5.12}
$$

We performed an additional scan for (mixed) anomaly free U(1) symmetries with higher allowed charges for the fermions. The different samples of U(1) charge assignments we used were:

Sample 1. The absolute value of all charges for quarks was less than or equal to 4 and for leptons less than or equal to 6. We require absence of all anomalies.

Sample 2. Here we have $|u_i^c|, |d_i^c|, |q_i| \leqslant 2$ and $|e_i^c|, |L_i| \leqslant 4$ and no mixed anomalies.

Sample 1 contained 1006 different U(1) charge assignments while sample 2 contained 4504. Our strongest result concerning these two samples is that there were no solutions for the scanning at stage III_1, i.e. with one chain producing all entries in the mass matrices. To extend this result we have also done searches for solutions of this type using samples containing larger values of the quantum numbers but more restrictive in some other sense. These are:

Sample 3. No mixed $SU(3) \times SU(2) \times U(1), U(1)_G$ anomalies but with an SU(5) type symmetry;

$$u_i^c = q_i = e_i^c , \qquad d_i^c = L_i ,$$

with all quantum numbers less than or equal to 4.

Sample 4. The same as above, but with all quantum numbers less than or equal to 6.

Sample 5. No mixed $SU(3) \times SU(2) \times U(1), U(1)_G$ anomalies with SO(10) symmetry. There is only one generation quantum number of size less than or equal to 20.

In table 2 we have listed the size of these various samples and the size of the subset that is totally anomaly free. The 3rd column in the size of the sample after removing the sign ambiguity, those charges that are pure multiples of previous ones and those where all q_i are not different. It is this last number that is relevant when comparing the number of U(1) charge assignments that have acceptable solutions given in

TABLE 2
Size of the different U(1) samples used

Sample	No. mixed anomalies	No. anomalies	After first reduction
1		1623	517
2	4504*	44*	2235
3	687	27	318
4	3659	51	1252
5	221	21	65

* This already requires the q_i to be all different.

TABLE 3
The number of distinct U(1) charge assignments with acceptable mass patterns

Sample	I(a)	I(b)	I(c)	III_1''(a)	III_1''(b)	III_1''(c)	III_1'(a)	III_1'(b)	III_1'(c)
1	294	266	258	139	110	100	3	2	2
2	706	428	399	335	161	135	1	1	0
3	160	135	135	85	56	55	4	2	0
4									
5	63	63	63	2	2	2	2	1	1

Sample	III_3(a)	III_3(b)	III_3(c)	III_2(a)	III_2(b)	III_2(c)	III_1^*(a)
1	37	25	5	3	1	0	0
2	29	28	1	17	16	1	0
3	36	35	12	15	15	4	0
4							0
5	4	4	4	4	4	4	0

*Obviously, also all zero for III_1(b) and III_1(c).
We use the different scanning conditions described in sect. 4 and the different (mixed) anomaly free U(1) samples described in sect. 5.

table 3. All of these numbers only quote the relevant number for sets of indistinguishable quantum numbers as described earlier for the general case.

An overview of the results is given in table 3. Our most significant result is that none of the samples described allows for a solution with only one SU(3) × SU(2) × U(1) singlet scalar being responsible for all the different scales in the observed pattern of masses and mixing angles.

At level I (columns 1–3) the number of charge assignments consistent with sufficient differentiation in the fermion mass matrices is a drastic reduction of the number of all possible U(1) with a given set of constraints on the quantum numbers. The requirement (b) of a fairly large mixing between the 2nd and 3rd generation provides a strong supplementary constraint on the number of solutions. Requiring the stronger criterion (c) on the electron mass essentially halves the number of possible charge assignments at this level of scanning.

At level III_1'' where we allow for one chain (starting at $n_s = 3$) and for additional entries of size $n_s = 2, 1$ for those elements of the mass matrices that did not already receive a nonzero value within the chain there is still a fairly large number of possible charge assignments. Increasing the strength of the procedure to the case where only new values of $n_s = 1$ are allowed to be assigned, case III_1', we see that the number of solutions becomes very limited. In the strongest case, where all entries are generated by chains, starting at $n_s = 3$, the number of solutions decreases quite dramatically. There are no solutions for only one chain, case III_1 and only a few for cases III_2 and III_3. As an example we list the solutions found for the III_1'(a) scan of sample 2. The quantum numbers are

$$\begin{aligned}
(u_1^c, u_2^c, u_3^c) &= (-2, -2, 2), \\
(d_1^c, d_2^c, d_3^c) &= (0,0,0), \\
(q_1, q_2, q_3) &= (-1,0,2), \\
(e_1^c, e_2^c, e_3^c) &= (-4,4,4), \\
(L_1, L_2, L_3) &= (-4, -2, 3).
\end{aligned} \tag{5.13}$$

The quantum numbers ($Q_{\mathrm{U,D}}$) and sizes of the mass matrix (in n_s) elements are

$$Q_{\mathrm{U}} = \begin{pmatrix} -3 & 0 & -2 \\ -3 & 0 & -2 \\ 1 & 4 & 2 \end{pmatrix}, \qquad M_{\mathrm{U}} = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 4 \end{pmatrix},$$

$$Q_{\mathrm{D}} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix}, \qquad M_{\mathrm{D}} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$Q_{\mathrm{L}} = \begin{pmatrix} -1 & -6 & -8 \\ 7 & 2 & 0 \\ 7 & 2 & 0 \end{pmatrix}, \qquad M_{\mathrm{L}} = \begin{pmatrix} 1 & 2 & 1 \\ * & 2 & 3 \\ * & 2 & 3 \end{pmatrix}, \tag{5.14}$$

and there is another possibility allowed for the leptons by rearranging rows.

$$Q_{\mathrm{L}} = \begin{pmatrix} 7 & 2 & 0 \\ -1 & -6 & -8 \\ 7 & 2 & 0 \end{pmatrix}, \qquad M_{\mathrm{L}} = \begin{pmatrix} * & 2 & 3 \\ 1 & 2 & 7 \\ * & 2 & 3 \end{pmatrix}. \tag{5.15}$$

In (5.14) $*$ means a corresponding element of size -1 or smaller. In conclusion, we described in this section the samples of U(1) assignments we have searched for allowed mass patterns and we found that using the most "natural" mechanism with only one U(1) breaking operator there are no solutions without mixed anomalies with the standard model. For unconstrained U(1)'s we found a limited set of solutions described in appendix A. Using a slight extension of our mechanism, i.e. allowing for several $SU(3) \times SU(2) \times U(1)$ singlet scalar fields we found possible U(1) quantum numbers compatible with the observed mass pattern and having no mixed anomalies. In the next section we describe a similar search for a particular higher dimensional model.

6. Scan of a higher dimensional model

The six-dimensional SO(12) model studied here was described in detail in ref. [9] and a short description together with some partial results of our scanning procedure can be found in ref. [16]. The relevant features of the model here are that it has a symmetry group

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y \times \mathrm{SU}(2)_I \times \mathrm{U}(1)_q, \tag{6.1}$$

where $\mathrm{SU}(2)_I \times \mathrm{U}(1)_q$ is interpreted as a local generation group. The fermions in this model have quantum numbers (using the usual names for given $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ transformation properties):

$$\begin{aligned}
q&: \quad \left[\tfrac{1}{2}(n+p)\right]_{1/2} + \left[\tfrac{1}{2}(n-p)\right]_{-1/2}\\
u^\mathrm{c}&: \quad \left[\tfrac{1}{2}(n-p+2m)\right]_{1/2} + \left[\tfrac{1}{2}(n+p-2m)\right]_{-1/2}\\
d^\mathrm{c}&: \quad \left[\tfrac{1}{2}(n-p-2m)\right]_{1/2} + \left[\tfrac{1}{2}(n+p+2m)\right]_{-1/2}\\
L&: \quad \left[\tfrac{1}{2}(n-3p)\right]_{1/2} + \left[\tfrac{1}{2}(n+3p)\right]_{-1/2}\\
e^\mathrm{c}&: \quad \left[\tfrac{1}{2}(n+3p-2m)\right]_{1/2} + \left[\tfrac{1}{2}(n-3p+2m)\right]_{-1/2}.
\end{aligned} \tag{6.2}$$

The number in brackets is the $\mathrm{SU}(2)_I$ representation and the subscript is the corresponding q quantum number. If the number in a bracket is negative it means that it is a mirror particle $\bar{\bar{q}}, \bar{\bar{u}}^\mathrm{c}$, etc. with charge q opposite to the indicated properties and the $\mathrm{SU}(2)_I$ representation given by the absolute value of the bracket. The symmetry group G used in our analysis is the subgroup

$$\mathrm{G} = \mathrm{U}(1)_I \times \mathrm{U}(1)_q, \tag{6.3}$$

where $\mathrm{U}(1)_I$ is the third component of the I-spin $\mathrm{SU}(2)_I$. The $\mathrm{U}(1)_I$ charge for q is given by $(n+p>0,\ n-p \geqslant 0)$

$$Q_I(q) = \begin{cases} \frac{1}{4}(n+p)-\frac{1}{2}, \frac{1}{4}(n+p)-\frac{3}{2}, \ldots, -\frac{1}{4}(n+p)+\frac{1}{2} & \left(q=\frac{1}{2}\right) \\ \frac{1}{4}(n-p)-\frac{1}{2}, \frac{1}{4}(n-p)-\frac{3}{2}, \ldots, -\frac{1}{4}(n-p)+\frac{1}{2} & \left(q=-\frac{1}{2}\right) \end{cases}, \tag{6.4}$$

and correspondingly for the mirrors and other fermions. The mirrors and standard fermions have the same $\mathrm{U}(1)_q$ quantum number. Therefore, the mirrors can only acquire mass if $\mathrm{U}(1)_q$ is spontaneously broken.

Mirror masses are generated by $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ singlet VEV's. It would be most natural if the VEV's responsible for the mirror masses were also those responsible for the different scales in the standard mass matrices [9]. We have not yet imposed this restriction and only assumed that the mirrors become heavy by an unspecified mechanism. If for a given particle type there were n standard fermions and $n-3$ mirror fermions we pick arbitrarily three of the n as the standard fermions and assume that the remaining $n-3$ combine with the mirrors to become heavy.

We have considered this model for the three-generation case with "monopole numbers" n, m, p:

$$n = 3,$$

$$p = 1, 3,$$

$$m = -3, -2, \ldots, 3. \tag{6.5}$$

In table 4 we have listed the set of mirrors for each particle type and the number of distinct G quantum assignments corresponding to an arbitrary choice of the "surviving" chiral fermions. Given the size of the $(m, p) = (-2, 3)$ and $(-3, 3)$ samples (because of the large numbers of mirrors) we have not performed a complete scan for all our different scanning requirements.

The results of the scanning are listed in table 5. First, we again have the result that there are no solutions with only one SU(3) × SU(2) × U(1) singlet VEV but there are also no solutions for the case where there are two scalar singlet VEV's. Even with 3 singlet VEV's the number of solutions is very limited. The number of solutions listed in table 5 is not only the number of distinct G quantum numbers

TABLE 4

The number of mirrors of each type for a given $n = 3$, m, p and the total number of distinct G quantum number assignment for the standard fermions for each case

m	P	$\bar{\bar{u}}^c$	$\bar{\bar{d}}^c$	$\bar{\bar{q}}$	$\bar{\bar{e}}^c$	$\bar{\bar{L}}$	n_G
+3	1	1	2	0	0	0	40
+2	1	0	1	0	0	0	4
+1	1	0	0	0	0	0	1
0	1	0	0	0	0	0	1
−1	1	0	0	0	1	0	4
−2	1	1	0	0	2	0	10
−3	1	2	1	0	3	0	800
3	3	0	3	0	0	3	400
2	3	0	2	0	1	3	40
1	3	0	1	0	2	3	80
0	3	0	0	0	3	3	400
−1	3	1	0	0	4	3	2800
−2	3	2	0	0	5	3	11200
−3	3	3	0	0	6	3	33600

TABLE 5

m	P	I(a)	I(b)	I(c)	III_1''(a)	III_1''(b)	III_1''(c)	III_1'(a)*	III_3(a)*	III_2(a)*	III_1(a)*
3	1	116	72	16	102	72	16	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0
1	1	4	0	0	4	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
−1	1	0	0	0	0	0	0	0	0	0	0
−2	1	8	0	0	0	0	0	0	0	0	0
−3	1	7034	4330	3652	624	400	356	0	0	0	0
3	3	15744	12440	10836	0	0	0	0	92	0	0
2	3	26414	25410	21002	1344	1036	628	0	0	0	0
1	3	3162	3162	2252	1684	1684	1224	0	0	0	0
0	3	0	0	0	0	0	0	0	0	0	0
−1	3	38356	26776	23556	2112	0	0	0	0	0	0
−2	3	⩾150000	NC**	NC**	22327	19196	15987	0	95	0	0
−3	3	⩾150000	NC**	NC**	0	0	0	0	NC	0	0

* Whenever (a) has no solutions, so of course (b) and (c). The number solutions for III_3(b) was 92 for $(m, p) = (3,3)$ and 95 for $(-2,3)$. The number solutions for III_3(c) was for $(m, p) = (3,3)$ 8 and for $(-2,3)$ 32.

**Not computed.

allowing a solution, but for each assignment of G quantum numbers we have also counted as different solutions the different ways of choosing the top quark, tau lepton, bottom quark and charm quark, strange quark and muon. Included also are the two different ways for the three latter fermions for their mass to be generated, via a diagonal element or two paired off-diagonal elements. We have not counted as different the different ways in which the first-generation masses can be generated or in which sufficient mixing between the generations can be induced.

We can estimate the number of solutions for one given charge assignment when all elements of the mass matrices are independent. There are 6 ways to order the rows according to generations for the up, down quarks and leptons. And similarly for the columns, however, the up and down quark columns are not independent because of the mixing angles. This gives a factor 6^5. In addition for the 2nd generation there are 2 ways to generate each mass and whenever M_L is acceptable so is M_L^T. The number of solutions for the case of maximal differentiation is

$$n_{sol} = \mathcal{O}(6^5 \cdot 2^4) = \mathcal{O}(10^5). \tag{6.6}$$

That this number is heavily reduced by even some quantum numbers of fermion bilinears being the same can be seen easily in table 5.

As mentioned before our strongest result here is that there are no solutions with one chain, even if we allow for arbitrary additional $n_s \leqslant 1$ entries. There are also no solutions for two chains and only a very limited set of those with 3 chains. A few of

TABLE 6
Solutions for a III_3(c) scan for $n = m = p = 3$

$2Q_U$	$2Q_D$	$2Q_L$	M_U	M_D	M_L
$\begin{pmatrix} 2;4 & 2;\ 2 & 2;\ 0 \\ 2;2 & 2;\ 0 & 2;-2 \\ 2;0 & 2;-2 & 2;-4 \end{pmatrix}$	$\begin{pmatrix} 0;7 & 0;5 & 0;\ 3 \\ 0;5 & 0;3 & 0;\ 1 \\ 0;3 & 0;1 & 0;-1 \end{pmatrix}$	$\begin{pmatrix} 0;\ 7 & 0;\ 5 & 0;-3 \\ 0;\ 5 & 0;\ 3 & 0;-5 \\ 0;\ 3 & 0;\ 1 & 0;-7 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & [illegible] \\ 1 & 1 & [illegible] \\ 1 & 2 & [illegible] \end{pmatrix}$
$\begin{pmatrix} 2;4 & 2;\ 2 & 2;\ 0 \\ 2;4 & 2;\ 0 & 2;-2 \\ 2;0 & 2;-2 & 2;-4 \end{pmatrix}$	$\begin{pmatrix} 0;7 & 0;5 & 0;\ 3 \\ 0;5 & 0;3 & 0;\ 1 \\ 0;3 & 0;1 & 0;-1 \end{pmatrix}$	$\begin{pmatrix} 0;7 & 0;\ 5 & 0;\ 3 \\ 0;\ 5 & 0;\ 3 & 0;\ 1 \\ 0;-3 & 0;-5 & 0;-7 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & [illegible] \\ 1 & 1 & [illegible] \\ 1 & 2 & [illegible] \end{pmatrix}$
$\begin{pmatrix} 2;4 & 2;\ 2 & 2;\ 0 \\ 2;2 & 2;\ 0 & 2;-2 \\ 2;0 & 2;-2 & 2;-4 \end{pmatrix}$	$\begin{pmatrix} 0;7 & 0;5 & 0;\ 3 \\ 0;5 & 0;3 & 0;\ 1 \\ 0;3 & 0;1 & 0;-1 \end{pmatrix}$	$\begin{pmatrix} 0;\ 7 & 0;\ 5 & 0;\ 3 \\ 0;\ 3 & 0;\ 1 & 0;-1 \\ 0;-3 & 0;-5 & 0;-7 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & [illegible] \\ 1 & 2 & [illegible] \\ 2 & 2 & [illegible] \end{pmatrix}$

The last solution is counted twice since the muon mass is generated by both diagonal and paired off-diagor elements in M_L. The other four solutions are the ones with $U(1)_f$ quantum number opposite to the ones listed here. T columns $2\,Q_{U,D,L}$ list twice the charges of the corresponding fermion bilinears. The first number is the $U(1)_q$ and t second the $U(1)_f$ quantum number. The columns labelled $M_{U,D,L}$ list the sizes in n_s (see sect. 4) of the correspondi elements in the mass matrices. Note the relatively large value for m_u which is a limiting case still allowed requirements (c).

the solutions in the case III_3(c) are given in table 6 for $m = p = 3$. This particular solution has SU(5) symmetry. Even for much looser constraints on how masses are generated there are no solutions for a fairly wide range of m and p.

Appendix A

In this appendix we exhibit all solutions found by our scanning program for the case $G = U(1)$ with charge assignments $u_1^c = q_1 = L_1 = 0$ and all other charges smaller than or equal to 2. First we discuss the case where the charge of the top bilinear is zero and then those where the charge of the top bilinear is ± 1. There were no solutions where the charge of the top bilinear was ± 2. We only treat the case for one $SU(3) \times SU(2) \times U(1)$ scalar singlet that breaks G symmetry. In all cases the charge of the singlet scalar is $Q(\chi) = \pm 1$.

A.1. $Q(t t^c) = 0$

All matrices here are ordered in order of increasing mass. We list first the fermion charges, then the quantum numbers of fermion bilinears and the size of the corresponding elements in M_L using the n_s notation of sect. 4. There are 10 possible charge assignments for the leptons. We list in table 7a only 5. The remaining 5 are those with opposite signs for the quantum numbers. Our example of sect. 3 is the last in this list. There are 10 possibilities for the assignments of the quark quantum numbers. They all have $(q_1, q_2, q_3) = (-2, -1, 0)$.

TABLE 7a
The solutions for the leptons when $Q(tt^c)=0$

(L_1, L_2, L_3)	(e_1^c, e_2^c, e_3^c)	Q_L	M_L
$(-2,-1,0)$	$(-2,-2,-1)$	$\begin{pmatrix} -4 & -3 & -2 \\ -4 & -3 & -2 \\ -3 & -2 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$
$(-2,0,0)$	$(-2,-2,-1)$	$\begin{pmatrix} -4 & -2 & -2 \\ -4 & -2 & -2 \\ -3 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}$
$(-2,0,1)$	$(-2,-2,-2)$	$\begin{pmatrix} -4 & -2 & -1 \\ -4 & -2 & -1 \\ -4 & -2 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{pmatrix}$
$(-2,0,-2)$	$(-2,-2,-1)$	$\begin{pmatrix} -4 & -2 & -4 \\ -4 & -2 & -4 \\ -1 & +1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$
$(-2,-1,0)$	$(-2,-1,-1)$	$\begin{pmatrix} -4 & -3 & -2 \\ -3 & -2 & -1 \\ -3 & -2 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

There are 3 different d_i^c assignments.

$$\begin{aligned} (d_1^c, d_2^c, d_3^c) &= (-1,-1,-1), \quad Q_D = \begin{pmatrix} -3 & -2 & -1 \\ -3 & -2 & -1 \\ -3 & -2 & -1 \end{pmatrix}, \quad M_D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \\ &= (-2,-1,-1), \quad Q_D = \begin{pmatrix} -4 & -3 & -2 \\ -3 & -2 & -1 \\ -3 & -2 & -1 \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \\ &= (-2,-2,-1), \quad Q_D = \begin{pmatrix} -4 & -3 & -2 \\ -4 & -3 & -2 \\ -3 & -2 & -2 \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}. \end{aligned} \tag{A.1}$$

In the case of the 2nd solution a different ordering of the rows is also allowed.

$$(d_1^c, d_2^c, d_3^c) = (-1,-2,-1), \qquad Q_D = \begin{pmatrix} -3 & -2 & -1 \\ -4 & -3 & -2 \\ -3 & -2 & -1 \end{pmatrix},$$

$$M_D = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}. \tag{A.2}$$

The last one of (A.1) has a too small contribution to the mixing of the first and second generation to be responsible for its observed value. Hence, it is only a solution for the last two of the following possibilities for the up-quark mixing matrix.

$$\begin{aligned}
(u_1^c, u_2^c, u_3^c) &= (-2,-1,0), \quad Q_U = \begin{pmatrix} -4 & -3 & -2 \\ -3 & -2 & -1 \\ -2 & -1 & 0 \end{pmatrix}, \quad M_U = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}, \\
&= (-1,-1,0), \quad Q_U = \begin{pmatrix} -3 & -2 & -1 \\ -3 & -2 & -1 \\ -2 & -1 & 0 \end{pmatrix}, \quad M_U = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}, \\
&= (-2,0,0), \quad Q_U = \begin{pmatrix} -4 & -3 & -2 \\ -2 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}, \quad M_U = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}, \\
&= (-1,0,0), \quad Q_U = \begin{pmatrix} -3 & -2 & -1 \\ -2 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}, \quad M_U = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}.
\end{aligned} \tag{A.3}$$

So far we have accounted for 100 of the acceptable assignments found. The model discussed in sect. 3 is the only solution consistent with SU(5) symmetry in this class of solutions.

A.2. $Q(tt^c) = +1$

In this case there are eleven solutions for the lepton charges. The results (using the same notation as before) are given in table 7b. In some cases (all those where one of the e_i^c was zero) a valid solution is with e_i^c and L_i interchanged. Some quantum numbers in table 7b therefore correspond to two solutions by making these changes. These quantum numbers have a superscript $*$.

The possibilities for leptonic charges go together with only one solution for the quark charges:

$$\begin{aligned}
(u_1^c, u_2^c, u_3^c) &= (-1,0,1), \\
(d_1^c, d_2^c, d_3^c) &= (-2,-2,-2), \\
(q_1, q_2, q_3) &= (-2,-1,0).
\end{aligned} \tag{A.4}$$

TABLE 7b
The solutions for the leptons when $Q(tt^c) = 1$

(L_1, L_2, L_3)	(e_1^c, e_2^c, e_3^c)	Q_L	M_L
(2,0,−1)	(1,1,1)	$\begin{pmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{pmatrix}$
(2,0,−1)	(1,2,1)	$\begin{pmatrix} 3 & 1 & 0 \\ 4 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 3 \\ -1 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$
(1,0,−1)	(2,1,1)	$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
(1,0,−1)	(2,2,1)	$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$
(2,0,2)	(1,1,−2)	$\begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 3 \\ 0 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$
(2,1,0)	(1,0,0)*	$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
(1,1,0)	(2,0,0)*	$\begin{pmatrix} 3 & 3 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$
(2,1,0)	(1,1,0)*	$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

This leads to

$$Q_U = \begin{pmatrix} -3 & -2 & -1 \\ -2 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad M_U = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix},$$

$$Q_D = \begin{pmatrix} -4 & -3 & -2 \\ -4 & -3 & -2 \\ -4 & -3 & -2 \end{pmatrix}, \qquad M_D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}. \tag{A.5}$$

This accounts over all for 11 solutions.

A.3. $Q(tt^c) = -1$

The solutions for the lepton charges are those of the previous section with opposite sign while there is one possible assignment for the charges of down quarks and the left-handed quark doublet.

$$(q_1, q_2, q_3) = (2,1,0),$$

$$(d_1^c, d_2^c, d_3^c) = (2,2,2),$$

$$Q_D = \begin{pmatrix} 4 & 3 & 2 \\ 4 & 3 & 2 \\ 4 & 3 & 2 \end{pmatrix}, \qquad M_D = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}. \tag{A.6}$$

There are two possible assignments for the up quark charges:

$$(u_1, u_2, u_3) = (0,-1,-1), \qquad Q_U = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \qquad M_U = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix},$$

$$(u_1, u_2, u_3) = (0,0,-1), \qquad Q_U = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad M_U = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}. \tag{A.7}$$

This adds another 22 possible assignments completing the 133 possible assignments of U(1) quantum numbers. These (and those with all charges sign reversed) are the only possible assignments within the stated limits on the sizes of quantum numbers and the constraints on the order of magnitude of the elements of the mass matrices as stated in sect. 4.

References

[1] S. Glashow, Nucl. Phys. 22 (1961) 571;
A. Salam, *in* Elementary particle theory, ed. N. Svartholm (Almquist and Wiksell, Stockholm, 1968) p. 367;
S. Weinberg, Phys. Rev. Lett. 19 (1967) 1264

[2] M.S. Chanowitz, J. Ellis and M.K. Gaillard, Nucl. Phys. B128 (1977) 506

[3] E. Davidson et al., Phys. Rev. Lett. 48 (1981) 11, Phys. Rev. 29D (1985) 1504;
G. Segre et al., Phys. Lett. 83B (1979) 351;
S. Pakvasa and H. Sugawara, Phys. Lett. 82B (1979) 105;
H. Georgi and D. Nanopoulos, Nucl. Phys. B155 (1979) 52, B159 (1979) 16;
D. Grosser, Phys. Lett. 83B (1979) 355, 136B (1984) 437;
R. Barbieri and D. Nanopoulos, Phys. Lett. 91B (1980) 369;
S.L. Glashow, Phys. Rev. Lett. 45 (1980) 1914;
S. Dimopoulos and H. Georgi, Phys. Lett. 140B (1984) 67;
G. Ecker, Z. Phys. C24 (1984) 353; Phys. Rev. D33 (1986) 2051;
H. Georgi, Phys. Lett. 151B (1985) 57

[4] M. Magg and C. Wetterich, Phys. Lett. 94B (1980) 61

[5] M. Gell-Mann, P. Ramond and R. Slansky, *in* Supergravity, eds. D. Freedman and P. van Nieuwenhuizen (North-Holland, 1980);

T. Yanagida, Proc. Workshop on the Unified theory and the baryon number in the universe (KEK, 1979)
[6] G. Lazarides, Q. Shafi and C. Wetterich, Nucl. Phys. B181 (1981) 287;
C. Wetterich, Nucl. Phys. B187 (1981) 343
[7] C. Wetterich, *in* Perspectives in weak interactions, ed. I. Tran Thanh Van (Frontières, 1985)
[8] S. Dimopoulos, Phys. Lett. 129B (1983) 417;
S. Dimopoulos and H. Georgi, Phys. Lett. 140B (1984) 67;
J. Bagger and S. Dimopoulos, Nucl. Phys. B244 (1984) 247;
J. Bagger, S. Dimopoulos, H. Georgi and S. Raby, 5th Workshop on Grand Unification, Providence, *in* Providence Grand Unif. (1984) p. 95
[9] C. Wetterich, Nucl. Phys. B261 (1985) 461; and Nucl. Phys. B279 (1987) 711
[10] A. Strominger and E. Witten, Comm. Math. Phys. 101 (1985) 341;
A.N. Schellekens, CERN preprint, TH-4295/85;
R. Holman and D.R. Reiss, Phys. Lett. 166B (1986) 305; and University of Minnesota preprint UMN-TH-556/86
[11] R. Peccei and H. Quinn, Phys. Rev. D16 (1977) 1791
[12] E. Witten, Nucl. Phys. B258 (1985) 75, B268 (1986) 253, B276 (1986) 291
[13] C. Wetterich, Nucl. Phys. B223 (1983) 109;
E. Witten, Proc. Shelter Island II Conf., ed. N. Khuri (MIT press, 1984)
[14] L. Wolfenstein, Phys. Rev. Lett. 51 (1984) 1945
[15] Q. Shafi and C. Wetterich, Phys. Rev. Lett. 52 (1984) 875;
C.T. Hill, Phys. Lett. 135B (1984) 47
[16] J. Bijnens and C. Wetterich, Phys. Lett. B176 (1986) 431
[17] K. Kleinknecht and B. Renk, Phys. Lett. 130B (1983) 459; Review of Particle Properties, Phys. Lett. 170B (1986) 1
[18] H. Fritzsch, Nucl. Phys. B155 (1979) 189
[19] F. Wilczek, Erice Lectures 1983, NSF/ITP 84-14
[20] S.M. Barr, Phys. Rev. D21 (1979) 1424; D24 (1981) 1895;
R. Barbieri and D.V. Nanopoulos, Phys. Lett. 95B (1980) 43;
M.J. Bowick and P. Ramond, Phys. Lett. 103B (1981) 338