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CLASSICAL STABILITY FOR SPONTANEOUS COMPACTIFICATION IN HIGHER DERIVATIVE GRAVITY

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Quantum theories of gravity (e.g. string theories) lead to higher derivative terms in the effective gravitational action. We present a general discussion of classical stability for compactification solutions of higher derivative gravity (also applicable for other bosonic fields and in four dimensions). In general there is no need for higher derivative terms to appear as generalized Euler forms.

As a specific example we discuss solutions $\mathscr{M}^4 \times S^D$ for the most general four-derivative approximation of pure higher dimensional gravity. For a certain range of parameters, instabilities for low momentum fluctuations may be absent. For the allowed parameter range, however, higher poles of propagators appear at momenta only slightly larger than the compactification scale, indicating insufficiency of the four-derivative approximation at this scale.

1. Introduction

Since the possible existence of more than four spacetime dimensions has been taken seriously, questions about the ground state of gravitational theories have gained new importance: One has to explain why we live in a world where we can only "see" the existence of three space dimensions, whereas all other space dimensions have a very small characteristic length scale making them inaccessible for direct observation. In our context we mean by "ground state" a state of the universe where the relevant local physics can be described successfully by excitations around this ground state. This state may be a unique property of the underlying theory or only a (partly accidental) product of previous evolution – in any case a realistic ground state must fulfil a requirement of stability: It should be static in time (or at least evolve with a time scale much longer than the time scale of our experiments and observation) and this must also hold in presence of arbitrary small excitations.

The appropriate tool for studying the ground state in gravitational theories is the effective action as functional of a background metric field, $S[g_{\mu\nu}]$, which is obtained as usual by evaluating a functional integral in the presence of external sources and performing a Legendre transformation. The effective action includes all quantum effects and the ground state metric $\mathring{g}_{\mu\nu}$ obeys

$$\frac{\delta S}{\delta g_{\rho\sigma}}\left[\mathring{g}_{\mu\nu}\right]=0\,. \qquad (1.1)$$

Other background fields beyond the metric will often be needed for a characterization of the ground state and (1.1) has to be generalized appropriately.

Solutions of (1.1) with four-dimensional Poincaré symmetry P_4 * (or a corresponding approximate symmetry as for the Friedmann universe) have the necessary static property. However, they are not unique solutions. Nearby solutions – interpreted as excitations – may lead to destabilization. Any small excitation which becomes large at time scales smaller than the scale of evolution of the ground state indicates that this "ground state" lacks the required stability. It is therefore a *necessary* condition that the ground state is classically stable with respect to small fluctuations in the above sense. Beyond (1.1), this imposes conditions on the behaviour of the second functional derivative $\delta^2 S/\delta g_{\mu\nu}\delta g_{\rho\sigma}$ evaluated at the ground state metric.

The effective action for gravity will contain terms with more than two derivatives of $g_{\mu\nu}$, like R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$ and terms with even more derivatives. Such terms appear in any quantum theory of gravity and are present even at the tree level in string theories. The appropriate gravitational equations of motion are those derived from the effective action (eq. (1.1)) and we expect in general a modified version of gravity – more complicated than Einstein gravity.

In four-dimensional gravity the contribution of higher derivative terms to classical field equations can be neglected for most purposes – they are suppressed by powers E_0^2/M^2 or R_0/M^2 where $E_0(R_0)$ is a characteristic energy (curvature) of the process considered and M^2 a very high mass scale (of the order of the Planck mass M_P^2) appearing in the ratio of the coefficients of terms with a different number of derivatives. (We note, however, that higher derivative contributions play a crucial role near black hole singularities or in very early cosmology where R_0 is of the same order as M^2.)

In higher dimensional theories spontaneous compactification involves a characteristic length scale L_0 not necessarily large compared to M^{-1} and the higher derivative terms of the effective action cannot be neglected a priori. They may even play an important role [1]: There are several known cases of spontaneous compactification where the presence of higher derivative terms is necessary. No solutions of the type $\mathscr{M}^4\times$ compact space are possible in pure Einstein gravity (including a cosmological constant) but they exist [1,2] taking R^2-type terms in the effective action into account. For the heterotic string, terms with four derivatives [3] are needed for a possible realization of Calabi-Yau spaces as an approximate ground

* Note that this does not mean that spacetime is flat since we consider more than four dimensions.

state*. In conclusion, higher derivative terms seem not only unavoidable, but may even prove to be useful. This includes their possible important role for the realization of an inflationary universe in higher dimensional theories [5].

On the other hand there is a widespread prejudice that higher derivatives necessarily lead to classical instabilities (often referred to as tachyons and ghosts). As we will see, this is by no means always the case. So far, the discussion of higher derivative theories has mainly concentrated on problems defining consistently a quantum field theory in perturbation theory for actions with four or more derivatives [6–8]. The problems appearing in this context are related to the classical instability of the "bare" action**.

We can learn from previous discussions of four-derivative bare actions which general type of instabilities can appear in the higher derivative effective action. However, we should emphasize at this place one important difference between the bare action and the effective action: For reasons of locality the bare action usually involves a finite power of derivatives, whereas the full effective action is not expected to have a simple polynomial form.

Furthermore, all discussions of classical stability for higher derivative actions were restricted so far to flat space. These results cannot be easily generalized to the curved background of spontaneous compactification. Stability properties for small fluctuations around flat space may be very different from stability properties around some compactification solution. Concerning classical stability, no general conclusions can be drawn independent of the ground state – the question is always about stability of a given ground state and not about "stability of a theory". (There is only one sort of fluctuations for which the stability discussion becomes independent of specific details of the ground state: fluctuations with characteristic momentum much larger than all inverse characteristic length scales (L_0^{-1}) of the ground state will not "feel" L_0. Unfortunately, these high momentum fluctuations are very often outside the range where the effective action can be calculated or estimated reliably.)

The purpose of this paper is twofold. First we want to give a general discussion of the stability problem for ground states of higher dimensional theories exhibiting

* Terms of order R^4 indicate that these spaces are indeed not exact solutions [4]. We suspect the existence of many classical solutions with various topologies, and higher derivative terms will play a role unless the compactification radius is very large.

** The connection between "bare" action and effective action is not immediate and even a bare action with instabilities from higher derivatives could lead to an acceptable effective action guaranteeing classical stability for the ground state [9]. Properties of the bare action S_0 are relevant for stability only if there exists some well behaved expansion series (e.g. perturbation theory) where the zeroth order effective action is given by S_0 and higher order terms lead to small quantitative corrections of the ground state without changing its qualitative features. Typically, convergence properties of such an expansion series are momentum dependent and the expansion breaks down above the scale M. Instabilities from the bare action for momenta larger than M should therefore not be taken seriously.

spontaneous compactification. Then we demonstrate these ideas by an explicit calculation of the linearized fluctuations around a ground state $\mathscr{M}^4 \times S^D$ and an investigation of their stability properties. We use the most general approximation up to four derivatives for the effective action of pure gravity. Specific features of the model under consideration are put in step by step, so that the results of the first steps can be used in a very general context.

2. Classical stability for small fluctuations

In this section we assume that the full effective action is known and admits a solution of the classical field equations with four-dimensional Poincaré symmetry*. To perform a stability analysis for small fluctuations, one first expands the effective action around this assumed ground state up to terms quadratic in the excitations. Next we have to eliminate all unphysical degrees of freedom by imposing suitable gauge conditions or considering physical sources to implement Bianchi's identities. We remain with a quadratic action for the physical degrees of freedom ϕ_i, which, by Poincaré invariance, has the following form in momentum space:

$$S^{(2)}_{\text{phys}} = \int \frac{d^4k}{(2\pi)^4} \phi_i(-k) F_{ij}(k^2) \phi_j(k)$$

$$= \int \frac{d^4k}{(2\pi)^4} \phi_i(-k) \left[A_{ij} + B_{ij}k^2 + C_{ij}k^4 + \cdots \right] \phi_j(k). \tag{2.1}$$

(Here we have assumed that the inverse propagator $F_{ij}(k^2)$ can be expanded in a power series of the four-dimensional momentum squared $k^2 = k_\mu k^\mu$.) The physical fields ϕ_i carry different helicities and internal quantum numbers. There may be a finite number of ϕ_i, as in four-dimensional field theory, or infinitely many as obtained by dimensional reduction of higher dimensional field theories or in string theories. (Any higher dimensional theory has first to be expanded on internal space and to be integrated over internal coordinates. This is necessary since we consider fluctuations which are local in four-dimensional spacetime, but not in internal space.) One chooses a basis for the ϕ_i reflecting the symmetries of the ground state so that A_{ij}, B_{ij} etc. become block diagonal for different representations.

We first concentrate on a single mode ϕ decoupled from all other modes at the linearized level. The field equations are

$$F(k^2)\phi(k) \equiv (A + Bk^2 + Ck^4 + \cdots)\phi(k) = 0,$$

$$(A - B\partial_\mu \partial^\mu + C\partial_\mu \partial^\mu \partial_\nu \partial^\nu - \cdots)\phi(x) = 0. \tag{2.2}$$

Solutions for propagating excitations $\phi(x)$ correspond to zeros of $F(k^2)$. Possible

* Generalization to more than four flat dimensions is straightforward.

classical instabilities of (2.2) have been discussed extensively by Pais and Uhlenbeck [6] and we recall here the main features:

(i) If $F(k^2)$ vanishes for some real negative k^2 one has an unbounded solution

$$\phi = \phi_0 \exp(\bar{k}t), \qquad k^2 = -\bar{k}^2, \qquad \bar{k} > 0. \tag{2.3}$$

This instability is usually called a tachyon. Similar unbounded solutions exist for excitations localized in space.

(ii) Zeros of $F(k^2)$ in the complex k^2-plane outside the real axis appear in complex conjugate pairs. Neglecting spatial variations, such zeros correspond to $k_0 = k_{0R} + ik_{0I}$ with unstable excitations of the form

$$\phi = \phi_0 \exp(|k_{0I}|t)\cos(k_{0R}t + \chi). \tag{2.4}$$

(iii) Double zeros (or multiple zeros) of $F(k^2)$ lead to unstable solutions (for positive real k^2)

$$\phi = \phi_0 t \cos(k_0 t + \chi). \tag{2.5}$$

So far we are still left with the possibility of several distinct zeros of F for real positive $k^2 = \bar{k}^2$. They do not lead to classical instabilities at the linearized level. Indeed, at the linearized level excitations with different $\bar{k}^2$ are decoupled and can be regarded as independent particles. Without interactions, the residue of poles of $F^{-1}(k^2)$ has no importance since a separate conserved energy can be associated with every excitation and the sign can be freely chosen so that it is bounded from below. This situation changes drastically for the interacting system beyond the linear approximation. Total energy is still conserved, but it is bounded from below only if the residues of all poles of $F^{-1}(k^2)$ for all excitations ϕ have the same sign. (Kinetic energy must be positive for all particles.) Any conserved quantity which is bounded from below and takes its minimum value only for the ground state is sufficient to guarantee classical stability*. For generic interacting systems, the absence of such a quantity usually leads to classical instability. Although other conserved quantities than energy could in principle assure stability, we will require in this paper that energy is bounded from below. This implies additional requirements** of the form of $F(k^2)$:

(iv) For any zero of F at real $\bar{k}^2 \geqslant 0$ the positivity of kinetic energy requires (for all ϕ)

$$\frac{\mathrm{d}F}{\mathrm{d}k^2}(\bar{k}^2) > 0. \tag{2.6}$$

* For small fluctuations, a relative minimum in configuration space of all local excitations is sufficient.

** Failure of these requirements is not a proof of classical instability, but it indicates the possibility that unstable solutions can be found for the nonlinear field equations.

Excitations violating (2.6) are often called ghosts – in our classical context this means negative kinetic energy particles. We note that the sign of $\mathrm{d}F/\mathrm{d}k^2$ cannot be found from the field equations (2.2) since they are insensitive to a change of sign. We can determine this quantity either from the effective action (2.1) or alternatively from field equations coupled to sources (compare ref. [10]). The requirement (2.6) immediately excludes the existence of more than one zero of $F(k^2)$ if F is continuous for all positive k^2. (For several distinct zeros, $\mathrm{d}F/\mathrm{d}k^2$ necessarily alters its sign.)

(v) Finally $F(k^2)$ should not have essential singularities for $|k|^2 \to \infty$. Such singularities can be viewed as degenerate zeros which could appear at finite $|k^2|$ for the interacting system and then lead to instabilities [6].

We can now state our stability criterion as follows: For all physical excitations ϕ the inverse propagator $F(k^2)$ should have only one zero for real $k^2 \geqslant 0$ with $\mathrm{d}F/\mathrm{d}k^2 > 0$ and no essential singularity at $|k^2| \to \infty$. (The only entire function fulfilling this criterion is $F = A + Bk^2$ with $A \leqslant 0$, $B > 0$. However, the inverse propagator derived from the effective action of a realistic theory is not expected to be analytic in the whole k^2 plane. It typically has branch cuts – for example a logarithmic behaviour [11].) Especially, the constant term in F must always be negative or zero:

$$A = F(k^2 = 0) \leqslant 0. \tag{2.7}$$

This is equivalent to the boundedness of potential energy. Indeed, for $k^2 \equiv 0$ the quantity $-S_{\text{phys}} = V$ is the effective potential which should be (locally) positive semidefinite with minimum for the ground state at $V = 0$. For massless modes ($A = 0$) one needs

$$B = \frac{\mathrm{d}F}{\mathrm{d}k^2}(k^2 = 0) > 0. \tag{2.8}$$

To complete our general discussion we still have to consider the case where the inverse propagator F_{ij} in (2.1) involves several modes ϕ_j. The propagating excitations are now determined by a matrix equation

$$F_{ij}(\bar{k}^2)\phi_j(\bar{k}) = 0. \tag{2.9}$$

For any given zero eigenvalue of F at $\bar{k}^2$ we can diagonalize the system in the vicinity $k^2 \approx \bar{k}^2$, so that the propagating mode is decoupled from the other components in ϕ_j, and then apply condition (2.6). By similar arguments as before one finds that zeros should only exist for real $\bar{k}^2 \geqslant 0$. The appearance of more zeros of F than components ϕ_j again indicates instability (as in (iv) above). Also, the effective potential has to be bounded from below requiring A_{ij} to be a negative semidefinite matrix. A proof of these statements uses continuity arguments which are preserved by $\bar{k}^2$ dependent rotations and rescalings of ϕ_j by which F_{ij} $(k^2=\bar{k}^2)$ and $\mathrm{d}F_{ij}/\mathrm{d}k^2$ $(k^2=\bar{k}^2)$ can be simultaneously diagonalized. At this point we mention that a discussion of classical stability is not affected by arbitrary k^2 dependent field redefinitions of ϕ_j, as long as these transformations are invertible and contain no additional zeroes or poles in the complex k^2 plane.

In conclusion, higher derivative effective actions do not necessarily lead to classical instabilities [12]. First of all, not all higher derivative terms really induce higher powers of k^2 in the inverse propagator $F(k^2)$ for physical modes. A well known example are dimensionally continued Euler forms in gravity [13]. However, a restriction to such terms for the effective action is by no means necessary. In general, we expect $F(k^2)$ to deviate from the simple form $A + Bk^2$. This does not necessarily lead to additional poles. In contrast, higher poles appear if an expansion in powers of k^2 or in powers of derivatives in the effective action is cut after a finite number of terms. This happens even for well behaved propagators and should then be considered as an artefact of these expansions, to which we turn in the next section.

3. Expansions in powers of k^2 and in the number of derivatives in the effective action

For realistic models it will be impossible to calculate the full effective action and to perform the stability analysis outlined above. One will have to use approximations and typically consider only a finite number of derivatives in the effective action and/or a finite power of k^2 in the inverse propagator. What can one conclude from such expansions for the stability problem?

We first describe the expansion in powers of momentum. For this discussion, let us assume that the full effective action for physical small fluctuations $S^{(2)}_{\mathrm{phys}}$ is given and can be expanded in powers of k^2 around $k^2=0$. It is obvious that any approximation by a finite number of terms in this expansion series breaks down for sufficiently high k^2. A restriction of the power series to a finite number of terms up to $(k^2)^N$ will always lead to N zeros of $F(k^2)$. If such zeros occur outside the k^2 range where the approximation can be trusted, they should be interpreted as artefacts of an insufficient approximation rather than as an indication of classical instability.

In principle, the range of validity for a finite number of terms can be estimated roughly to be the overall mass scale $\overline{M}^2$ appearing in ratios of coefficients for various powers of k^2 in the series. The scale $\overline{M}^2$ may be related to a fundamental mass scale M^2 of the theory, like the string tension in string theories. The expansion parameter is $k^2/\overline{M}^2$. In practice, however, a quantitative estimate of the range of validity in k^2 requires the knowledge of several terms in the power series expansion which are often difficult to obtain. This means that it is often impossible to decide if "higher poles" appearing in $F^{-1}(k^2)$ are indications of instability or artefacts of the approximation.

There are special cases where the first zero of $F(k^2)$ appears for $|k^2| \ll \overline{M}^2$. This is the case for massless fields like graviton and gauge fields. It also can happen for the lowest massive modes from the harmonic expansion around a compactification solution, if the compactification "radius" L_0 fulfils

$$L_0^{-2} \ll \overline{M}^2. \tag{3.1}$$

For such "lowest zeros" we can approximate $F(k^2)$ by the first two terms $F = A + Bk^2$. Positive values for A or negative values for B imply classical instability. We call possible instabilities from zeros of F at $|k^2| \ll \overline{M}^2$ "low momentum instabilities". Any ground state with low momentum instabilities has to be discarded, whereas apparent "high momentum instabilities" from zeros at $|k^2| \gtrsim \overline{M}^2$ have a very questionable status and should not be taken too seriously.

We next turn to expansions in the number of derivatives in the effective action. Although related to the k^2 expansion, the latter approximation is of a different nature. In higher dimensional theories with spontaneous compactification, a term with p derivatives in the effective action does not only give contributions of order k^p to $F(k^2)$ in $S^{(2)}_{\rm phys}$, but also all sorts of terms $\sim L_0^{-R} k^{p-R}$ since derivatives can act on internal coordinates. In particular it contributes to the coefficients A and B in the k^2 expansion. As a consequence, the inclusion of higher derivative terms could in principle modify conclusions about low momentum (in)stability. For example, they could switch the sign of the kinetic term for the graviton. In theories with a fundamental mass scale M^2 the expansion of the effective action in the number of derivatives generates a double expansion of $F(k^2)$ in powers of $(L_0 M)^{-1}$ and in k^2/M^2. The contributions to A and B (relevant for low momentum stability analysis) only converge if

$$L_0^2 M^2 \gg 1. \tag{3.2}$$

If $L_0^2 M^2$ is too small, no reliable stability analysis can be based on the first few derivatives in the effective action.

The scale $\overline{M}$ appearing in the k^2 expansion and the scale M from the expansion in the number of derivatives are both related to the fundamental mass scale of the theory. However, the k^2 expansion and the $(L_0 M)^{-1}$ expansion may have very different coefficients. For example, if all higher derivative terms would be dimensionally continued Euler forms [3], no higher powers of k^2 would be generated and the range of validity of the k^2 expansion would extend to infinity. Nevertheless, the higher derivative terms would induce a nontrivial expansion of $F = A + Bk^2$ in powers of $(L_0 M)^{-1}$. Similarly, if all higher derivative terms are "near" dimensionally continued Euler forms, all $k^N (N > 2)$ coefficients in $F(k^2)$ are small and

$\overline{M}^2$ is therefore large. Approaching Euler forms, the apparent higher poles move to very high values of k^2. If the effective action is sufficiently "near" the Euler forms, a stability analysis for low momenta is valid whereas the higher poles can be neglected as artefacts of the k^2 expansion.

Unfortunately, it is not so easy to give to these statements a more quantitative character. An order of magnitude analysis becomes often rather involved due to the existence of small parameters inherent in spontaneous compactification, like $1/D$ or $1/N$ with D the number of internal dimensions and N a typical topological number like monopole number.

Let us finally briefly discuss the two lowest approximations based on two or four derivatives in the effective action. If the second derivative approximation leads to a ground state solution with $L_0^{-2} \ll M^2$, low momentum classical stability can be investigated reliably for all massless and massive particles. (Such an analysis was done for the six-dimensional Einstein-Maxwell system in ref. [10].) As long as M^2 is not known (its determination requires the knowledge of higher derivative corrections) classical stability analysis can be considered as self-consistent.

This changes for the fourth derivative approximation (which is sometimes necessary to obtain compactification). Unless the fourth derivative term is a pure dimensionally continued Euler form (in which case the status of stability analysis is similar to the second derivative approximation) we now have terms of order k^4 which necessarily lead to unwanted higher poles in all propagators. Let us denote by $|k_c^2|$ the location of the lowest higher pole of the system. If $L_0^{-2} \ll |k_c^2|$ it is still possible to make a low momentum stability analysis for particles with mass$^2 \ll |k_c^2|$. The neglected terms with more than four derivatives can modify the effective action strongly at $k^2 \approx |k_c^2|$ without altering qualitatively the coefficients A and B. (Contributions with $|k_c^2|/M^2$ of order one correspond to a small parameter $(L_0 M)^{-1}$.) This argument breaks down if $|k_c^2|$ is of the same order as L_0^{-2}.

A possible strategy for stability analysis, which we will follow in this paper, investigates first the parameter range for which low momentum stability is realized. Then the location of higher poles is established for this parameter range and one can check if $L_0^{-2} \ll |k_c^2|$ is fulfilled. If not, one would conclude that either the system is unstable or the fourth derivative approximation is unreliable.

However, even this conclusion may be too strong. First of all, possible contributions from neglected higher derivative terms could appear quite differently for various propagators such that a criterion based on a single $|k_c^2|$ may be questioned. A weaker criterion would only require that for any given propagator $F^{-1}(k^2)$ the (physical) lowest pole is sufficiently below the (unphysical) higher poles. Second, the use of some other function system $g_N(k^2)$ instead of $(k^2)^N$ to describe a systematic expansion will in general lead to a different location of $|k_c^2|$. (This is similar to the use of Padé approximants instead of a simple power expansion in statistical mechanics.) One could even imagine the existence of a system of functions for which no unphysical poles appear at any finite order. Certainly, a better check of consistency for a description of the ground state by a four-derivative approximation would include the next (six derivative) terms and verify if changes for the ground state solution as well as for location and residues of the lowest poles are small. This is, however, outside the scope of our paper.

4. Four-derivative approximation for gravity

To give a specific example for the points discussed above, we consider the most general pure-gravity effective action in $d = 4 + D$ dimensions with no more than 4 derivatives and vanishing torsion*:

$$S = -\frac{1}{N}\int \mathrm{d}^d x \sqrt{|g|}\left(\alpha R^2 + \beta R_{\hat{\mu}\hat{\nu}} R^{\hat{\mu}\hat{\nu}} + \gamma R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} R^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \delta R + \varepsilon\right). \tag{4.1}$$

Here N is a normalization constant which is chosen to be the volume of the D-dimensional compact manifold. The parameters α, β and γ are dimensionless, δ has dimension $(\text{mass})^2$ and the d-dimensional cosmological constant ε has $(\text{mass})^4$. The classical equations of motion derived from (4.1) have solutions which are a direct product of the form $\mathscr{M}^4 \times \mathrm{S}^D$ with the curvature tensor given by

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = \left(g_{\hat{\mu}\hat{\lambda}} g_{\hat{\nu}\hat{\rho}} - g_{\hat{\mu}\hat{\rho}} g_{\hat{\nu}\hat{\lambda}}\right)\bar{y}$$

for $\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}$ being S^D indices,

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = 0, \qquad \text{otherwise}. \tag{4.2}$$

For the ground state, the radius L_0 of the hypersphere S^D obeys

$$L_0^{-2} \equiv \bar{y} = \delta/2\zeta, \tag{4.3}$$

where

$$\zeta \equiv D(D-1)\alpha + (D-1)\beta + 2\gamma. \tag{4.4}$$

Obviously, $\delta\zeta^{-1}$ must be positive; in order to exclude that the potential of the scalar associated with the variation of L_0 is unbounded from below, which immediately would give rise to an instability, one even has to assume $\delta > 0$ and $\zeta > 0$ separately [1]. Furthermore, to obtain a vanishing 4-dimensional cosmological constant, ε has to fulfill the "fine tuning" condition

$$\varepsilon = \tfrac{1}{4}\delta^2 D(D-1)\zeta^{-1}. \tag{4.5}$$

* Our conventions are as follows: the signature of the metric is $(+ - - - \ldots -)$. Indices $\hat{\mu}, \hat{\nu}, \ldots$ run from 0 to $d-1$, whereas $\mu, \nu, \ldots$ are Minkowski space indices running from 0 to 3 and $\alpha, \beta, \ldots$ are internal indices running from 4 to $d-1$. The Riemann tensor is $R_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\hat{\lambda}} = \partial_{\hat{\nu}} \Gamma_{\hat{\mu}\hat{\rho}}{}^{\hat{\lambda}} + \cdots$ and the Ricci tensor $R_{\hat{\mu}\hat{\nu}} = R^{\hat{\lambda}}{}_{\hat{\mu}\hat{\lambda}\hat{\nu}}$. Furthermore, we use $\partial^2 = \partial_\mu \partial^\mu$, $\mathring{D}^2 = D_\alpha D^\alpha$, $D^2 = D_{\hat{\mu}} D^{\hat{\mu}}$ and $h = h^{\hat{\mu}}_{\hat{\mu}}$.

Hence the number of free dimensionless parameters in (4.1) is reduced to three. For later reference, we also note the 4-dimensional Planck mass M_P obtained by expanding around $\mathscr{M}^4 \times S^D$, which sets the overall mass scale:

$$M_P^2 = 16\pi\delta[\beta(D-1)+2\gamma]\zeta^{-1} \approx (10^{19}\ \mathrm{GeV})^2. \tag{4.6}$$

Note that for α, β, γ of order unity, the Planck mass squared is larger than the typical mass scale of the action, viz. δ.

The first step in the stability analysis of (4.1) with respect to $\mathscr{M}^4 \times S^D$ is to expand the metric as

$$g_{\hat{\mu}\hat{\nu}} = \mathring{g}_{\hat{\mu}\hat{\nu}} + h_{\mu\nu}, \tag{4.7}$$

where $h_{\hat{\mu}\hat{\nu}}$ describes small fluctuations around the background metric $\mathring{g}_{\hat{\mu}\hat{\nu}}$ given by

$$\mathring{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \mathring{g}_{\alpha\beta}(x^\gamma) \end{pmatrix}, \tag{4.8}$$

with $\eta_{\mu\nu}$ the metric of Minkowski space and $\mathring{g}_{\alpha\beta}(x^\gamma)$ the metric of the hypersphere with radius L_0. Inserting (4.7) into (4.1), the terms linear in $h_{\hat{\mu}\hat{\nu}}$ vanish due to the equations of motion. The bilinear terms describe the small fluctuations we are interested in. Higher terms, which we will ignore, would describe interactions between them. So we have to consider the quadratic action

$$S^{(2)} = -\frac{1}{2N}\int \mathrm{d}^d x \Big\{ \alpha\big(\sqrt{|g|}\,R^2\big)^{(2)} + \beta\big(\sqrt{|g|}\,R_{\hat{\mu}\hat{\nu}}R^{\hat{\mu}\hat{\nu}}\big)^{(2)}$$
$$+\gamma\big(\sqrt{|g|}\,R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}R^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\big)^{(2)} + \delta\big(\sqrt{|g|}\,R\big)^{(2)} + \varepsilon\big(\sqrt{|g|}\,\big)^{(2)} + 2\sqrt{|g|}\,h_{\hat{\mu}\hat{\nu}}Q^{\hat{\mu}\hat{\nu}}\Big\}, \tag{4.9}$$

where $(\dots)^{(2)}$ denotes the second variation of $(\dots)$. The explicit expressions for these second variations of the various invariants are listed in appendix A for a general background.

In (4.9) we have also included a coupling of the field $h_{\hat{\mu}\hat{\nu}}$ to an external source $Q_{\hat{\mu}\hat{\nu}}$. In order for the (linear) equations of motion derived from $S^{(2)}$ to be consistent, one must require this source to be covariantly conserved:

$$D_{\hat{\mu}}Q^{\hat{\mu}\hat{\nu}} = 0. \tag{4.10}$$

Instead of the fluctuations $h_{\hat{\mu}\hat{\nu}}$ one can use the sources $Q_{\hat{\mu}\hat{\nu}}$ to extract the propagating degrees of freedom. This is particularly useful if the gauge choice does not eliminate all unphysical degrees of freedom. Indeed, the fields $h_{\hat{\mu}\hat{\nu}}$ are related by Bianchi identities which have a rather complicated form, whereas the corresponding relation in terms of the sources takes the simple form (4.10). To extract the propagating degrees of freedom from the sources [8], one has to solve the equations of motion for $h_{\hat{\mu}\hat{\nu}}$ as a functional of $Q_{\hat{\mu}\hat{\nu}}$ and insert the result back in $S^{(2)}$:

$$S^{(2)}_{\text{phys}} = -\frac{1}{2N}\int \mathrm{d}^d x \sqrt{|g|}\, h^{\hat{\mu}\hat{\nu}}(Q) Q_{\hat{\mu}\hat{\nu}} . \tag{4.11}$$

After dimensional reduction, eq. (4.11) will define the propagators for all physical states of the theory, so that one can read off their masses and residues.

To proceed from (4.9), one chooses a gauge (or coordinate condition) which in the case at hand is most conveniently taken to be*

$$D_{\hat{\mu}} h^{\hat{\mu}\hat{\nu}} = 0 . \tag{4.12}$$

If one inserts the background metric (4.8) into (4.9) with the variations given in appendix A, it is possible to cast all terms containing covariant derivatives D_α with respect to internal coordinates into the form of the Laplace-Beltrami operator $\mathring{D}^2 \equiv D_\alpha D^\alpha$ on the sphere S^D or its square $\mathring{D}^4 \equiv D_\alpha D^\alpha D_\beta D^\beta$. To do so, one has to make extensive use of the commutation relations of the D_α's; these contain the Riemann curvature tensor and its contractions, which is explicitly given by the first line of (4.2). Since $\bar{y}$ has dimension (mass)^2, several types of terms can appear: pure derivatives as ∂^4, $\mathring{D}^2\partial^2$ or $\mathring{D}^4$, mixed terms like $\bar{y}\partial^2$ or $\bar{y}\mathring{D}^2$ and finally non-derivative terms proportional to $\bar{y}^2$. After a lengthy calculation one finds

$$\begin{aligned}
S^{(2)} = &-\frac{1}{2N}\int \mathrm{d}^d x \sqrt{|\mathring{g}|} \\
&\times\Big\{ h^{\alpha\beta}\big[A_1(\partial^4 + 2\partial^2\mathring{D}^2 + \mathring{D}^4) + A_2\partial^2 + A_3\mathring{D}^2 + A_4\big] h_{\alpha\beta} \\
&+ h^\alpha_\alpha\big[B_1(\partial^4 + 2\partial^2\mathring{D}^2 + \mathring{D}^4) + B_2\partial^2 + B_3\mathring{D}^2 + B_4\big] h^\beta_\beta \\
&+ h^\alpha_\alpha\big[C_1(\partial^4 + 2\partial^2\mathring{D}^2 + \mathring{D}^4) + C_2\partial^2 + C_3\mathring{D}^2 + C_4\big] h^\mu_\mu \\
&+ h^{\alpha\mu}\big[F_1(\partial^4 + 2\partial^2\mathring{D}^2 + \mathring{D}^4) + F_2\partial^2 + F_3\mathring{D}^2 + F_4\big] h_{\alpha\mu} \\
&+ h^{\mu\nu}\big[G_1(\partial^4 + 2\partial^2\mathring{D}^2 + \mathring{D}^4) + G_2\partial^2 + G_3\mathring{D}^2 + G_4\big] h_{\mu\nu} \\
&+ h^\mu_\mu\big[H_1(\partial^4 + 2\partial^2\mathring{D}^2 + \mathring{D}^4) + H_2\partial^2 + H_3\mathring{D}^2 + H_4\big] h^\nu_\nu \\
&+ L\big(h_{\alpha\mu}\partial^\mu\partial^\nu h^\alpha_\nu + h_{\rho\mu}\partial^\mu\partial^\nu h^\rho_\nu - h^\alpha_\alpha\partial_\mu\partial_\nu h^{\mu\nu} - h^\rho_\rho\partial_\mu\partial_\nu h^{\mu\nu}\big) \\
&+ 2\big(h_{\mu\nu}Q^{\mu\nu} + 2h_{\mu\alpha}Q^{\mu\alpha} + h_{\alpha\beta}Q^{\alpha\beta}\big)\Big\} .
\end{aligned} \tag{4.13}$$

* Another choice one could think of is the light-cone gauge, which is known to greatly simplify the stability analysis of 11-dimensional supergravity [14]. However, it turns out that because of the rather complex form of (4.9), the derivation of the constraint equations is prohibitively complicated.

The coefficients A_i, B_i,... are defined in appendix B; they are polynomials in D of maximal degree 4 for R^2, 3 for $R^2_{\hat{\mu}\hat{\nu}}$, and 2 for $R^2_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\delta}}$. These expressions are the same for any number n of flat dimensions, i.e. for any of the spaces $\mathscr{M}^n \times S^D$. This can be exploited for a simple check: since for $n+D=4$ and the Euler form coefficients $\alpha=1$, $\beta=-4$, $\gamma=1$ the 4th order lagrangian is a total derivative, its contribution for ($D=0$, $n=4$), ($D=1$, $n=3$), ($D=2$, $n=2$), ($D=3$, $n=1$) and ($D=4$, $n=0$) should drop out; moreover, for ($D=1$, $n=3$) the terms containing $\bar{y}$ and $\bar{y}^2$ have to vanish for all values of α, β and γ since for $D=1$ we have zero internal curvature. One can verify that the polynomials given in the appendix indeed have these properties. In writing them down we have not yet employed the "mass shell" condition (4.3), i.e. they can be used for the expansion around $\mathscr{M}^n \times S^D$ for an arbitrary radius $L=y^{-1/2}$ of the hypersphere.

5. Harmonic analysis and dimensional reduction

The fields appearing in (4.13) are functions of both the Minkowski space coordinates x^μ and the internal coordinates x^γ; to evaluate the action of $\mathring{D}^2$ on the latter, we now expand these functions in terms of the S^D tensor harmonics. Their $\mathring{D}^2$ eigenvalues and the corresponding degeneracies have been given by Rubin and Ordóñez [15] (cf. also ref. [16]). The results we will need in the following are summarized in table 1 : any vector field on S^D★, for instance, can be expanded in terms of the transverse vector harmonics T^α_{lm}, where l denotes a particular $\mathring{D}^2$

TABLE 1

Eigenvalues of $\mathring{D}^2$ acting on the tensor harmonics defined in the text

Eigenfunction		Eigenvalue	Degeneracy
symmetric tensors			
$T^{\alpha\beta}_{lm}$,	$l=2,3,\ldots$	$[l(l+D-1)-2]\bar{y}\equiv\eta_1\bar{y}$	$D_l(D,2)$*
$\underset{T}{L}{}^{\alpha\beta}_{lm}$,	$l=2,3,\ldots$	$[l(l+D-1)-(D+2)]\bar{y}\equiv\eta_2\bar{y}$	$D_l(D,1)$
$\underset{L}{L}{}^{\alpha\beta}_{lm}$,	$l=2,3,\ldots$	$[l(l+D-1)-2D]\bar{y}\equiv\eta_3\bar{y}$	$D_l(D,0)$
$\mathring{g}^{\alpha\beta}T_{lm}$,	$l=0,1,\ldots$	$l(l+D-1)\bar{y}\equiv\eta_4\bar{y}$	$D_l(D,0)$
vectors			
T^α_{lm},	$l=1,2,\ldots$	$[l(l+D-1)-1]\bar{y}\equiv\eta_5\bar{y}$	$D_l(D,1)$
L^α_{lm},	$l=1,2,\ldots$	$[l(l+D-1)-(D-1)]\bar{y}\equiv\eta_6\bar{y}$	$D_l(D,0)$
scalars			
T_{lm},	$l=0,1,\ldots$	$l(l+D-1)\bar{y}\equiv\eta_4\bar{y}$	$D_l(D,0)$

Explicit formulas for the degeneracies $D_l(D,0)$, etc. can be found in ref. [10].
*There is no transverse-traceless tensor for $D=2$.

★ From now on we always will assume $D \geqslant 2$.

eigenvalue and m is a degeneracy index, and the longitudinal vector harmonics

$$L^{\alpha}_{lm} \sim D^{\alpha} T_{lm} \tag{5.1}$$

formed from the scalar harmonics T_{lm}. Similarly, the expansion of a symmetric, rank-2 tensor field requires four types of harmonics: a transverse-traceless part $T^{\alpha\beta}_{lm}$, a longitudinal-transversal part

$$\underset{\mathrm{T}}{L}{}^{\alpha\beta}_{lm} \sim D^{\alpha} T^{\beta}_{lm} + D^{\beta} T^{\alpha}_{lm}, \tag{5.2}$$

a longitudinal-longitudinal part

$$\underset{\mathrm{L}}{L}{}^{\alpha\beta}_{lm} \sim D^{\alpha} L^{\beta}_{lm} + D^{\beta} L^{\alpha}_{lm} - \frac{2}{D} \mathring{g}^{\alpha\beta} D_{\gamma} L^{\gamma}_{lm} \tag{5.3}$$

and a trace part $\mathring{g}^{\alpha\beta} T_{lm}$. All harmonics are assumed to be normalized to unity. Their eigenvalues are shown in the table; for explicit expressions for the degeneracies $D_l(D,0)$, etc. we refer to Rubin and Ordóñez [15].

Now one can expand the fluctuations as*

$$h^{\alpha\beta}(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \mathrm{e}^{ik_\rho x^\rho} \left\{ \sum_{l=2}^{\infty} \sum_{m=1}^{D_l(D,2)} S_{1lm}(k) T^{\alpha\beta}_{lm}(x^\gamma) + \sum_{l=2}^{\infty} \sum_{m=1}^{D_l(D,1)} S_{2lm}(k) \underset{\mathrm{T}}{L}{}^{\alpha\beta}_{lm}(x^\gamma) + \sum_{l=2}^{\infty} \sum_{m=1}^{D_l(D,0)} S_{3lm}(k) \underset{\mathrm{L}}{L}{}^{\alpha\beta}_{lm}(x^\gamma) + \mathring{g}^{\alpha\beta}(x^\gamma) \sum_{l=0}^{\infty} \sum_{m=1}^{D_l(D,0)} S_{4lm}(k) T_{lm}(x^\gamma) \right\},$$

$$h^{\alpha\mu}(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \mathrm{e}^{ik_\rho x^\rho} \left\{ \sum_{l=1}^{\infty} \sum_{m=1}^{D_l(D,1)} V^{\mu}_{1lm}(k) T^{\alpha}_{lm}(x^\gamma) + \sum_{l=1}^{\infty} \sum_{m=1}^{D_l(D,0)} V^{\mu}_{2lm}(k) L^{\alpha}_{lm}(x^\gamma) \right\},$$

$$h^{\mu\nu}(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \mathrm{e}^{ik_\rho x^\rho} \sum_{l=0}^{\infty} \sum_{m=1}^{D_l(D,0)} H^{\mu\nu}_{lm}(k) T_{lm}(x^\gamma). \tag{5.4}$$

With respect to the Minkowski space coordinates we performed the usual Fourier

* To be slightly more general, we assume here and in the following n flat dimensions.

transform to momentum space. The fluctuations are now parametrized by an infinite set of scalar functions (in the sense of $\mathscr{M}^n$) S_{ilm}, $i=1,\dots,4$, vector functions V^{μ}_{ilm}, $i=1,2$, and tensor functions $H^{\mu\nu}_{lm}$. Taking the harmonics to be real, the reality of the fields implies $S^*_{ilm}(k)=S_{ilm}(-k)$, etc. In the same way we expand the external sources with coefficient functions Q_{ilm}, q^{μ}_{ilm} and $q^{\mu\nu}_{lm}$ replacing S_{ilm}, V^{μ}_{ilm} and $H^{\mu\nu}_{lm}$. Inserting (5.4) into (4.13) and performing the integration over the internal coordinates x^{α} by using the orthonormality relations of the tensor harmonics, one arrives at the following purely n-dimensional action for the various towers of fields:

$$
\begin{aligned}
S^{(2)} = -\frac{1}{2N}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\sum_{lm}\Bigg\{ & \sum_{i=1}^{4}\left[a^i_1 k^4 + a^i_2 k^2 + a^i_3\right]|S_i|^2 \\
& + \sum_{i=1}^{2}\left(-\left[F_1 k^4 + f^i_2 k^2 + f^i_3\right]|V^{\mu}_i|^2 + L V^{\mu *}_i k_{\mu} k_{\nu} V^{\nu}_i\right) \\
& + \left[G_1 k^4 + g_2 k^2 + g_3\right]|H^{\mu\nu}|^2 \\
& + \left[H_1 k^4 + h_2 k^2 + h_3\right]|H^{\mu}_{\mu}|^2 \\
& + S^*_4\left[C_1 D k^4 + c_2 k^2 + c_3\right]H^{\mu}_{\mu} \\
& + L\left(D S^*_4 k_{\mu} k_{\nu} H^{\mu\nu} + H^{\rho *}_{\rho} k_{\mu} k_{\nu} H^{\mu\nu} - H^{\rho *}_{\mu} k_{\rho} k_{\nu} H^{\mu\nu}\right) \\
& + \text{source terms}\Bigg\}. \qquad (5.5)
\end{aligned}
$$

(The indices l, m have been omitted for brevity.) The coefficients a^i_j, $f^i_j,\dots$ contain the quantities η_k, which are essentially the eigenvalues of $\mathring{D}^2$. They are defined in appendix C.

6. Irreducible decomposition and equations of motion

To obtain the linearized equations of motion in a simple form it is advantageous to decompose the fields and sources appearing in (5.5) into irreducible parts with respect to the Lorentz group. For the vector and tensor fields one writes

$$
\begin{aligned}
V^{\mu}_{ilm} &= \underset{\mathrm{T}}{V}{}^{\mu}_{ilm} + i k^{\mu}\phi_{ilm}, \qquad i=1,2, \\
H^{\mu\nu}_{lm} &= \underset{\mathrm{TT}}{H}{}^{\mu\nu}_{lm} + i\left(k^{\mu}\underset{\mathrm{T}}{H}{}^{\nu}_{lm} + k^{\nu}\underset{\mathrm{T}}{H}{}^{\mu}_{lm}\right) + \left(k^{\mu}k^{\nu} - \frac{1}{n}g^{\mu\nu}k^2\right)\eta_{lm} + \frac{1}{n}g^{\mu\nu}\sigma_{lm}, \qquad (6.1)
\end{aligned}
$$

where one requires

$$
k_{\mu}\underset{\mathrm{T}}{V}{}^{\mu}_{ilm} = k_{\mu}\underset{\mathrm{T}}{H}{}^{\mu}_{lm} = k_{\mu}\underset{\mathrm{TT}}{H}{}^{\mu\nu}_{lm} = \underset{\mathrm{TT}}{H}{}^{\mu}_{\mu lm} = 0. \qquad (6.2)
$$

Similar expansions are introduced for the sources. The functions corresponding to ϕ_{ilm}, $\underset{\mathrm{T}}{H}{}^{\mu}_{lm}$, η_{lm} and σ_{lm} are denoted by φ_{ilm}, W^{μ}_{lm}, w_{lm} and $q^{\mu}_{\mu lm}$. The continuity equation (4.10) now reads

$$k^2\varphi_{1,l=1,m}=0, \tag{6.3a}$$

$$k^2\varphi_{1lm}=\left(\tfrac{1}{2}\bar{y}\right)^{1/2}\left[l(l+D-1)-D\right]^{1/2}Q_{2lm}, \qquad l\geqslant 2, \tag{6.3b}$$

$$k^2\varphi_{2,l=1,m}=\left(\bar{y}D\right)^{1/2}Q_{4,l=1,m}, \tag{6.3c}$$

$$k^2\varphi_{2lm}=\bar{y}^{1/2}\left[(1-1/D)l(l+D-1)-(D-1)\right]^{1/2}Q_{3lm}$$
$$+\bar{y}^{1/2}\left[l(l+D-1)\right]^{1/2}Q_{4lm}, \qquad l\geqslant 2, \tag{6.3d}$$

$$k^2W^{\mu}_{l=0,m}=0, \tag{6.3e}$$

$$(n-1)k^2w_{l=0,m}+q^{\mu}_{\mu l=0,m}=0, \tag{6.3f}$$

$$k^2W^{\mu}_{lm}=\bar{y}^{1/2}\left[l(l+D-1)\right]^{1/2}\underset{\mathrm{T}}{q}{}^{\mu}_{2lm}, \qquad l\geqslant 1, \tag{6.3g}$$

$$\bar{y}^{1/2}\left[l(l+D-1)\right]^{1/2}\varphi_{2lm}=-\left(1-\frac{1}{n}\right)k^2w_{lm}-\frac{1}{n}q^{\mu}_{\mu lm}, \qquad l\geqslant 1. \tag{6.3h}$$

Correspondingly, the gauge condition (4.12), when expressed in terms of the fields appearing in the decomposition (6.1), leads to a similar set of equations. Because the gauge condition was used in deriving the action (5.5), the equations of motion for the various fields have to be determined by a constrained variation taking this condition into account. The fields $\underset{\mathrm{T}}{H}{}^{\mu\nu}$, $\underset{\mathrm{T}}{V}{}^{\mu}_{1}$ and S_1 do not couple to others and we can vary them independently. For the coupled sets $\{\underset{\mathrm{T}}{V}{}^{\mu}_{2},[\underset{\mathrm{T}}{H}{}^{\mu}];\, l\geqslant 1\}$, $\{S_2,[\phi_1];\, l\geqslant 2\}$, $\{S_4,\sigma,[\eta];\, l=0\}$, $\{S_4,\sigma,[\eta,\phi_2];\, l=1\}$ and $\{S_3,S_4,\sigma,[\eta,\phi_2];\, l\geqslant 2\}$ we eliminate the fields in the square brackets together with their sources using (4.12) and (6.3). We then perform an independent variation with respect to the others. The solution of the equations of motion has to be inserted into (4.11), which now reads

$$S^{(2)}_{\mathrm{phys}}=-\frac{1}{2N}\int\frac{\mathrm{d}^nk}{(2\pi)^n}\sum_{lm}\left\{\sum_{i=1}^{3}S_i^*Q_i+DS_4^*Q_4-2\sum_{i=1}^{2}\left(\underset{\mathrm{T}}{V}{}^{\mu *}_{i}\underset{\mathrm{T}}{q}{}_{i\mu}+k^2\phi_i^*\varphi_i\right)\right.$$
$$\left.+\underset{\mathrm{TT}}{H}{}^*_{\mu\nu}\underset{\mathrm{TT}}{q}{}^{\mu\nu}+2k^2\underset{\mathrm{T}}{H}{}^*_{\mu}W^{\mu}+\left(1-\frac{1}{n}\right)k^4\eta^*w+\frac{1}{n}\sigma^*q^{\mu}_{\mu}\right\}. \tag{6.4}$$

After a lengthy calculation one obtains the following action for the physical modes

$$S^{(2)}_{\mathrm{phys}}=\frac{1}{2N}\int\frac{\mathrm{d}^nk}{(2\pi)^n}\left[\sum_{lm}\left\{L_{\mathrm{T}}+L_{\mathrm{V1}}+L_{\mathrm{V2}}+L_{\mathrm{S1}}+L_{\mathrm{S2}}+L_{34q}\right\}+L^0_{4q}+L^1_{4q}\right], \tag{6.5}$$

where

$$L_{\mathrm{T}}=\left[G_1k^4+g_2k^2+g_3\right]^{-1}\left|\underset{\mathrm{TT}}{q}\,{}^{\mu\nu}_{lm}\right|^2,\qquad l\geqslant 0,\tag{6.6a}$$

$$L_{\mathrm{V1}}=-4\left[F_1k^4+f_2^1k^2+f_3^1\right]^{-1}\left|\underset{\mathrm{T}}{q}{}^{\mu}_{1lm}\right|^2,\qquad l\geqslant 1,\tag{6.6b}$$

$$L_{\mathrm{V2}}=\frac{4}{k^2}\left[k^2-l(l+D-1)\bar{y}\right]^2$$
$$\times\left(-k^2\left[F_1k^4+f_2^2k^2+f_3^2\right]+2l(l+D-1)\bar{y}\right.$$
$$\left.\times\left[G_1k^4+\left(g_2-\tfrac{1}{2}L\right)k^2+g_3\right]\right)^{-1}\left|\underset{\mathrm{T}}{q}{}^{\mu}_{2lm}\right|^2,\qquad l\geqslant 1,\tag{6.6c}$$

$$L_{\mathrm{S1}}=\left[a_1^1k^4+a_2^1k^2+a_3^1\right]^{-1}|Q_{1lm}|^2,\qquad l\geqslant 2,\tag{6.6d}$$

$$L_{\mathrm{S2}}=\frac{1}{k^2}\left[k^2-\{l(l+D-1)-D\}\bar{y}\right]^2$$
$$\times\left(k^2\left[a_1^2k^4+a_2^2k^2+a_3^2\right]-\tfrac{1}{2}\bar{y}\{l(l+D-1)-D\}\right.$$
$$\left.\times\left[F_1k^4+\left(f_2^1-L\right)k^2+f_3^1\right]\right)^{-1}|Q_{2lm}|^2,\qquad l\geqslant 2,\tag{6.6e}$$

$$L_{34q}=\tfrac{1}{2}\left(Q_3,Q_4,q_\mu^\mu\right)_{lm}\rho\mathbf{M}^{-1}\rho\left(Q_3,Q_4,q_\mu^\mu\right)^+_{lm},\qquad l\geqslant 2,\tag{6.6f}$$

$$L_{4q}^0=4\Xi^0\left\{D^2\left(\xi_1^0+\frac{1}{n(n-1)}\xi_2^0\right)|Q_4|^2+(n-1)^{-2}\xi_3^0|q_\mu^\mu|^2\right.$$
$$\left.\left.-D(n-1)^{-1}\xi_4^0Q_4^*q_\mu^\mu\right\}\right|_{l=0},\tag{6.6g}$$

$$L_{4q}^1=\Xi^1\left\{\left[(\omega_1)^2\xi_1^1+(\omega_2)^2\xi_2^1-\omega_1\omega_2\xi_3^1\right]|Q_4|^2\right.$$
$$+\left[(\omega_2)^2\xi_1^1+4(n-1)^{-2}\xi_2^1-2\omega_2(n-1)^{-1}\xi_3^1\right]|q_\mu^\mu|^2$$
$$+\left[2\omega_1\omega_2\xi_1^1+4\omega_2(n-1)^{-1}\xi_2^1\right.$$
$$\left.\left.\left.-\left((\omega_2)^2+2\omega_1(n-1)^{-1}\right)\xi_3^1\right]Q_4^*q_\mu^\mu\right\}\right|_{l=1}.\tag{6.6h}$$

This is our main result. It corresponds to the effective action (2.1) in sect. 2

expressed in terms of sources. The complicated k^2-dependent coefficients appearing in (6.6f)–(6.6h), in particular the matrices ρ and $\mathbb{M}$, are tabulated in appendix D. Eqs. (6.6a)–(6.6e) are already of the general form $(\text{source})^\star \cdot F^{-1}(k^2) \cdot (\text{source})$, so that the propagator $F^{-1}(k^2)$ for the various types of fields and hence the masses and residues we are interested in can be read off easily. For (6.6f)–(6.6h), however, a diagonalization with respect to the sources still has to be done. Fortunately, it turns out that for the stability discussion it is not necessary to explicitly perform this diagonalization (see below).

Looking at the Lorentz transformation properties of the sources, eq. (6.5) with (6.6) shows that the physical fluctuation (or particle) spectrum contains one tower of spin-2 particles, generated by $\underset{\mathrm{TT}}{q}^{\mu\nu}$, two towers of spin-1 particles associated with $\underset{\mathrm{T}}{q}_1^{\mu}$ and $\underset{\mathrm{T}}{q}_2^{\mu}$, as well as 5 towers of spinless excitations coupled to $Q_1, \dots, Q_4$ and q^{μ}_{μ}. Their stability properties will be discussed in the next section. Here we only add a brief comment on the apparent $k^2 = 0$ poles of propagators like that in (6.6c), which contains contributions of $\underset{\mathrm{T}}{q}_2^{\mu}$ and of the dependent source W^μ, which has been eliminated via (6.3g). Obviously, if $\underset{\mathrm{T}}{q}_2^{\mu}$ would be nonzero for $k^2 \to 0$, W^μ diverges in that limit. This should be interpreted that there is no massless excitation with the quantum numbers of W^μ or $\underset{\mathrm{T}}{q}_2^{\mu}$ *, but rather that for $k^2 \to 0$ the function $\underset{\mathrm{T}}{q}_2^{\mu}$ is an inadequate choice for a physically meaningful source. (We deal with a "coördinate singularity" in the space of source functions.) Since the definition of W^μ in terms of $\underset{\mathrm{T}}{q}_2^{\mu}$ breaks down for $k^2 \to 0$, we should use W^μ as the correct independent source in that region; this results in a propagator without a pole at $k^2 = 0$ and hence there is no massless particle associated with this spurious pole. It is clear from (6.6c) that the propagator for W^μ would not fall off as $1/k^4$ for $k^2 \to \infty$, as is naively expected since curvature effects are not assumed to play any role at sufficiently large momenta. For any choice, the physical content of the propagators, viz. the locations and the residues of the true poles, is the same.

We also checked that the residual gauge invariance, which is still present after imposing (4.12), is not sufficient to gauge away any of the massive states, as it should be. However, at the massless level of the S_{T} tower, say, it is possible to eliminate 3 of the 5 degrees of freedom contained in $\underset{\mathrm{TT}}{H}^{\mu\nu}$, so that one ends up with the two degrees of freedom of the graviton.

7. Masses and residues

In this section we investigate the pole structure of the propagators of (6.6) and its implications for the classical stability of our model**. Since the general form of the equations derived so far is quite difficult to survey, we now will restrict ourselves to

* Similar spurious massless states also appear in the procedure of Randjbar-Daemi, Salam and Strathdee [10].

** It is only at this point that we use (4.3) and (4.5).

two particular examples for the value of D, viz. $D=2$ and $D=9$, giving rise to a SU(2) and SO(10) Yang-Mills gauge group, respectively. Our aim is to find those domains in (α, β, γ) space, if any, where the stability criteria discussed in sects. 2 and 3 are fulfilled. It turns out that the masses and residues of the lowest lying poles depend on the ratio of α, β and γ only, but not on their absolute magnitude. Therefore it is convenient to introduce

$$\tilde{\alpha}=\alpha/2\zeta, \qquad \tilde{\beta}=\beta/2\zeta, \qquad \tilde{\gamma}=\gamma/2\zeta \tag{7.1}$$

as new parameters. Exploiting (4.4), we express $\tilde{\gamma}$ as

$$\tilde{\gamma}=\tfrac{1}{4}[1-2\tilde{\alpha}D(D-1)-2\tilde{\beta}(D-1)] \tag{7.2}$$

and use $(\tilde{\alpha}, \tilde{\beta}, \zeta)$ as the new set of independent parameters. We discuss the lowest excitations (of the expansion in k^2) for the various towers separately:

(a) The tensor tower T. The action is given by (6.6a). We first look at the term for $l=0$ which admits only a single m-value since $D_0(D,0)=1$ according to ref. [15]. Taking the polynomials from appendices B and C with $y=\bar{y}$ given by (4.3) and ε given by (4.5), one can show that $g_3(l=0)=0$ for all values of D and n, i.e.

$$L_{\mathrm{T}}(l=0)=\frac{1}{g_2(0)}\left[\frac{1}{k^2}-\frac{1}{k^2+g_2(0)/G_1}\right]\left|q_{00\,\mathrm{TT}}^{\mu\nu}\right|^2. \tag{7.3}$$

Here we recover the massless graviton pole together with a massive excitation with $(\text{mass})^2=-g_2(0)/G_1$. The first pole has the correct residue if one requires $g_2(0)<0$ since this leads to a positive euclidean action for the massless excitation and therefore to positive kinetic energy. From the appendices, one has

$$g_2(0)=-\tfrac{1}{2}[\beta(D-1)+2\gamma]\,\delta\zeta^{-1} \tag{7.4}$$

and since $\delta\zeta^{-1}$ is positive, it follows that one must require

$$\beta(D-1)+2\gamma>0. \tag{7.5}$$

In view of (4.6), this is precisely the condition which guarantees the positivity of M_{P}^2. Numerically, (7.5) means

$$\tilde{\alpha}<\tfrac{1}{4} \qquad \text{for } D=2,$$
$$\tilde{\alpha}<\tfrac{1}{144} \qquad \text{for } D=9. \tag{7.6}$$

For the massive excitations we only investigate the lowest pole in the small momentum approximation discussed in sect. 3. For $l\geqslant 1$ and to the first order in

k^2, the action reads

$$L_{\mathrm{T}}(l \geqslant 1) \approx \frac{1}{g_2(l)} \frac{1}{k^2 + g_3(l)/g_2(l)} \left| q^{\mu\nu}_{lm}{}_{\mathrm{TT}} \right|^2 ,$$

so that we must require $g_2(l) < 0$ and $g_3(l) > 0$ to ensure the correct residue and a positive (mass)^2 at the higher levels of the tower. Evaluating these rather complicated inequalities using appendices B and C with (4.3), eqs. (4.5) and (7.1) with (7.2) one finds several remarkably simple conditions.

(i) $\underline{D = 2}$. From $g_2(l = 1) < 0$ it follows that

$$\tilde{\alpha} < \tfrac{1}{4} - \tfrac{1}{5}\tilde{\beta} . \tag{7.7}$$

For higher values of l, the magnitude of the coefficient of $\tilde{\beta}$ monotonically increases to $\frac{1}{4}$:

$$\tilde{\alpha} < \tfrac{1}{4} - \tfrac{1}{4}\tilde{\beta} . \tag{7.8}$$

The second inequality, $g_3(l) > 0$, turns out to be independent of l; it leads to the constraint (7.8), too. Note that, as advertised, (7.7) or (7.8) is indeed independent of the absolute value of α, β and γ, i.e. independent of ζ.

(ii) $\underline{D = 9}$. In both inequalities the l-dependence is negligibly small and one obtains for any of them

$$\tilde{\alpha} < \tfrac{1}{144} - \tfrac{5}{48}\tilde{\beta} . \tag{7.9}$$

(b) The vector tower V1. The situation is similar to (a). One finds $f_3^1(l = 1) = 0$ for all D and n, so that there are again massless particles at the lowest level of the tower. Since their degeneracy is [15]

$$D_1(D,1) = \tfrac{1}{2}D(D+1) = \dim \mathrm{SO}(D+1) ,$$

and because we will discover no further massless excitations, these must be the gauge bosons. The Yang-Mills pole has the correct residue for

$$(D-1)\beta + (3D-2)\gamma > 0 . \tag{7.10}$$

At those points of the parameter space where in (7.10) the equality sign holds, the kinetic term of the gauge bosons vanishes or, equivalently, the gauge coupling becomes infinite. Similar to the graviton in (7.3), the gauge fields have massive companions with the wrong residue and (mass)$^2 = -f_2^1(1)/F_1$.

Considering the levels with $l \geqslant 2$ and neglecting the k^4 term, stability requires

$$f_2^1(l) < 0 , \qquad f_3^1(l) > 0 . \tag{7.11}$$

The numerical results for our examples are:

(i) $\underline{D=2}$. Condition (7.10) for the Yang-Mills residue again leads to (7.8); the same is true for the inequalities (7.11), which turn out to be l-independent.

(ii) $\underline{D=9}$. Condition (7.10) reads

$$\tilde{\alpha} < \tfrac{1}{144} - \tfrac{23}{225}\tilde{\beta}, \tag{7.12}$$

which numerically is almost the same as (7.9). The second inequality of (7.11) is exactly independent of l, the first to a very high degree of accuracy. They both lead to (7.9).

(c) The vector tower V2. The propagator (6.6c) has no true massless poles, i.e. there are no singularities if one uses W^{μ} as the independent source. Expanding to order k^2, one finds the stability criteria $g_3(l) > 0$ and

$$f_3^2 - 2l(l+D-1)\bar{y}\left(g_2 - \tfrac{1}{2}L\right) - g_3 > 0 .$$

The first condition already had appeared in (a), and after some lengthy calculations the second one is seen to imply the following constraints on the parameters:

(i) $\underline{D=2}$. For $l=1$ the inequality reads

$$\tilde{\alpha} < \tfrac{1}{4} - 0.350\tilde{\beta}. \tag{7.13}$$

For l becoming large, the coefficient of $\tilde{\beta}$ monotonically increases to -0.25, so that the bounds become less restrictive.

(ii) $\underline{D=9}$. For $l=1$ one has

$$\tilde{\alpha} < \tfrac{1}{144} - 0.172\tilde{\beta} \tag{7.14}$$

as the most restrictive bound; for large values of l, the inequality approaches (7.9).

(d) The scalar tower S1. There are no massless particles. Stability of the lowest massive excitations requires

$$a_2^1(l) < 0, \qquad a_3^1(l) > 0 . \tag{7.15}$$

The S_1 tower does not exist for $D=2$ (cf. table 1). For $D=9$ the first inequality is l-independent and leads to (7.9); the second one reads

$$\tilde{\alpha} < \tfrac{1}{144} - 0.106\tilde{\beta}$$

for $l=2$ and monotonically approaches (7.9) for l large. (In particular for $D=9$, "large" already means the second or third l-value.)

(e) The scalar tower S2. Proceeding as in (c), one finds no massless poles; the stability of the lowest massive ones requires $f_3^1(l) > 0$, which already appeared in (b)

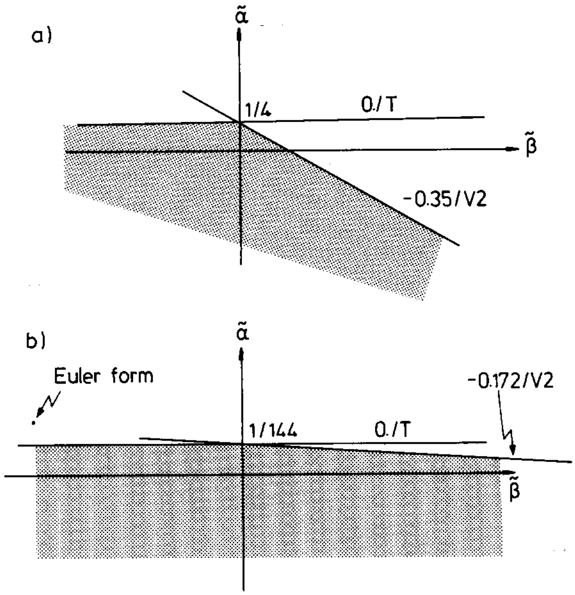

Fig. 1. In the shaded part of the $\tilde{\alpha}, \tilde{\beta}$-plane all stability requirements coming from the lowest poles of the T, V1, V2, S1 and S2 tower are fulfilled. (a) For $D = 2$, (b) for $D = 9$. The slope of the boundary lines and the respective towers as well as the location of the Euler form point ($\alpha = 1$, $\beta = -4$, $\gamma = 1$) for $D = 9$ are indicated.

for $l \geqslant 2$, as well as

$$a_3^2 - \tfrac{1}{2}\bar{y}\{l(l+D-1) - D\}(f_2^1 - L) - f_3^1 > 0. \tag{7.16}$$

This inequality is independent of l and is equivalent to (7.8) and (7.9) for $D = 2$ and 9, respectively.

All pairs $(\tilde{\alpha}, \tilde{\beta})$ fulfilling the stability criteria derived so far form an extended region in the $\tilde{\alpha}, \tilde{\beta}$ plane, which has been sketched in fig. 1. If one also would take the coupled scalar equations (6.6f, g, h) into account, its area could become smaller or even shrink to zero size*. However, already from the restrictions shown in the figure it can be seen that it is not possible to choose the parameters in a way so that the mass of the dangerous graviton-ghost at the $l = 0$ level of the tensor tower becomes much larger than the compactification scale $\bar{y}^{1/2} = L_0^{-1}$. The (mass)^2 of the second pole in the propagator of (7.3) is given by

$$M_c^2 = \frac{\beta(D-1) + 2\gamma}{(\beta + 4\gamma)[D(D-1)\alpha + (D-1)\beta + 2\gamma]}\delta. \tag{7.17}$$

* The stability analysis for the lowest excitation contained in L_{4q}^0 has been performed in the second paper of ref. [5] by different methods. For special values of the parameters we checked that the masses obtained there coincide with ours.

In terms of $\bar{y}$ this reads

$$M_c^2 = [1 - 4\tilde{\alpha} - \tilde{\beta}]^{-1}(1 - 4\tilde{\alpha})\bar{y} \qquad \text{for } D = 2,$$

$$M_c^2 = [1 - 144\tilde{\alpha} - 15\tilde{\beta}]^{-1}(1 - 144\tilde{\alpha})\bar{y} \qquad \text{for } D = 9. \tag{7.18}$$

Both expressions are positive in the allowed region, i.e. we have a ghost, but not a tachyon. For $D = 2$, the main obstruction against making M_c^2 large is (7.13); it can easily be seen that the highest value of M_c^2 compatible with this inequality, is $3.5\bar{y}$, a value which can not be considered "far beyond" the compactification scale. Similarly, for $D = 9$ the strongest restriction comes from (7.14); the maximum value of M_c^2 turns out to be $2.53\bar{y}$.

The conditions coming from the Yang-Mills ghost at the $l = 1$ level of the V1 tower are slightly less restrictive than those of the graviton. For $D = 2$ its $(\text{mass})^2$ is exactly, and for $D = 9$ approximately independent of $\tilde{\alpha}$ and $\tilde{\beta}$; the values are $4\bar{y}$ and $25\bar{y}$, respectively.

We conclude that the two requirements of low momentum stability and location of all higher poles far beyond L_0^{-2} are conflicting in our case. It seems not unlikely that low momentum stability is realized for a range of parameters. It is also well conceivable that the higher poles can be considered as artefacts of an insufficient expansion. However, the addition of terms with six and more derivatives, needed to remove all higher poles far beyond the inverse compactification radius, could easily modify drastically the quantitative low momentum stability analysis of this paper. Unfortunately, a reliable assessment of stability – or instability – is not possible in this context. This conclusion is independent of the number of flat dimensions and it is very likely that it remains unaltered for $D \neq 2, 9$. Better prospects will arise when the dimensionally continued Euler forms are not outside the parameter range of low momentum stability as it is the case in our example. (Cf. fig. 2b for $D = 9$. For $D = 2$, the Euler form does not admit solutions of the type $\mathscr{M}^n \times S^D$, since, according to (4.3) and (4.4), $\bar{y}$ would be divergent.)

We hope we have demonstrated that classical instability of effective actions with higher derivative terms is by no means obvious, but that a rather involved analysis is necessary for every given ground state solution. As a general rule, the location of the higher poles, which indicates the scale of breakdown of an approximation with few derivatives, will be far beyond the compactification scale whenever the higher derivative terms are "near" a generalized Euler form or when the two derivative approximation leads to classically stable compactification and the higher derivative terms are "sufficiently" small. (As an example, the $d = 6$ Einstein-Maxwell theory with monopole compactification on S^2 is expected to remain classically stable when sufficiently small higher derivative terms different from the generalized Euler forms are added.) A quantitative assessment of the meaning of "near" and "sufficiently small", however, will always require a detailed calculation.

One of the authors (C.W.) would like to thank S. Randjbar-Daemi for many fruitful discussions during early stages of this investigation.

Appendix A

Here we list the second variations needed in eq. (4.9) for an arbitrary background metric. We present the expressions without having performed any integration by parts. For the sake of simplicity, the hat over the indices has been omitted throughout.

$$
\begin{aligned}
(R)^{(2)} = {} & R_{\beta\mu}h^{\beta\gamma}h^{\mu}_{\gamma} - R_{\alpha\beta\gamma\mu}h^{\beta\gamma}h^{\alpha\mu} - 3h^{\beta\gamma}D_{\gamma}D_{\alpha}h^{\alpha}_{\beta} + 2h^{\mu\nu}D_{\mu}D_{\nu}h \\
& + 2h^{\mu\nu}D^2 h_{\mu\nu} - h^{\beta\gamma}D_{\alpha}D_{\beta}h^{\alpha}_{\gamma} - D_{\alpha}h^{\mu}_{\beta}D_{\mu}h^{\alpha\beta} + \tfrac{3}{2}D^{\mu}h_{\alpha\beta}D_{\mu}h^{\alpha\beta} \\
& - 2D^{\alpha}h^{\mu}_{\alpha}D_{\beta}h^{\beta}_{\mu} + 2D^{\alpha}h^{\mu}_{\alpha}D_{\mu}h - \tfrac{1}{2}D^{\mu}hD_{\mu}h\,,
\end{aligned}
$$

$$
\begin{aligned}
\left(\sqrt{|g|}\,R\right)^{(2)} = \sqrt{|g|}\,\Big\{ & \tfrac{1}{2}\left(\tfrac{1}{2}h^2 - h^{\mu\nu}h_{\mu\nu}\right)R \\
& - hR_{\mu\nu}h^{\mu\nu} + hD_{\mu}D_{\nu}h^{\mu\nu} - hD^2 h + (R)^{(2)}\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\left(\sqrt{|g|}\,R^2\right)^{(2)} = \sqrt{|g|}\,\Big\{ & \tfrac{1}{2}\left(\tfrac{1}{2}h^2 - h^{\mu\nu}h_{\mu\nu}\right)R^2 + 2hR\left[D_{\mu}D_{\nu}h^{\mu\nu} - R_{\mu\nu}h^{\mu\nu} - D^2 h\right] \\
& + 2\left(R_{\mu\nu}h^{\mu\nu}\right)^2 + 2\left(D_{\mu}D_{\nu}h^{\mu\nu}\right)^2 + 2\left(D^2 h\right)^2 \\
& - 4R_{\mu\nu}h^{\mu\nu}D_{\alpha}D_{\beta}h^{\alpha\beta} + 4R_{\mu\nu}h^{\mu\nu}D^2 h \\
& - 4D_{\mu}D_{\nu}h^{\mu\nu}D^2 h + 2R(R)^{(2)}\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\left(\sqrt{|g|}\,R_{\mu\nu}R^{\mu\nu}\right)^{(2)} = \sqrt{|g|}\,\Big\{ & \tfrac{1}{2}\left(\tfrac{1}{2}h^2 - h^{\mu\nu}h_{\mu\nu}\right)R_{\alpha\beta}R^{\alpha\beta} \\
& + 2hR^{\beta\gamma}R_{\alpha\beta\gamma\mu}h^{\alpha\mu} + 2R^{\beta\delta}R_{\alpha\beta\gamma\delta}h^{\alpha\mu}h^{\gamma}_{\mu} \\
& + R_{\beta\mu}R^{\beta\lambda}h^{\gamma}_{\lambda}h^{\mu}_{\gamma} - R^{\beta\lambda}R_{\gamma\mu}h^{\gamma}_{\lambda}h^{\mu}_{\beta} \\
& - 2R^{\beta\lambda}R_{\rho\beta\gamma\mu}h^{\gamma}_{\lambda}h^{\rho\mu} + 2R^{\alpha\beta\gamma\lambda}R_{\rho\beta\gamma\mu}h_{\alpha\lambda}h^{\rho\mu} \\
& - 2R^{\beta\lambda}h^{\gamma}_{\lambda}\Omega_{\beta\gamma} + 2R^{\alpha\beta\gamma\lambda}h_{\alpha\lambda}\Omega_{\beta\gamma} + hR^{\beta\gamma}\Omega_{\beta\gamma} \\
& + \tfrac{1}{2}\Omega_{\mu\nu}\Omega^{\mu\nu} + 2R^{\mu\nu}\Psi_{\mu\nu} \\
& + 2R^{\beta\delta}h^{\alpha\gamma}\left[D_{\alpha}D_{\gamma}h_{\beta\delta} - D_{\beta}D_{\gamma}h_{\alpha\delta}\right. \\
& \qquad \left. - D_{\alpha}D_{\delta}h_{\beta\gamma} + D_{\beta}D_{\delta}h_{\alpha\gamma}\right]\Big\}\,.
\end{aligned}
$$

In the last equation we introduced

$$\Omega_{\beta\gamma} \equiv D_\gamma D_\alpha h^\alpha_\beta - D_\beta D_\gamma h - D^2 h_{\beta\gamma} + D_\beta D^\alpha h_{\alpha\gamma}$$

and

$$\begin{aligned}\Psi_{\beta\delta} &\equiv g^{\alpha\gamma}\left(R_{\alpha\beta\gamma\delta}\right)^{(2)} \\ &= D_\mu h_{\alpha\delta} D^\mu h^\alpha_\beta - D_\alpha h^{\mu\alpha} D_\beta h_{\mu\delta} - D_\alpha h^{\mu\alpha} D_\delta h_{\mu\beta} + D_\alpha h^{\mu\alpha} D_\mu h_{\beta\delta} - D^\alpha h^\mu_\beta D_\mu h_{\alpha\delta} \\ &\quad + \tfrac{1}{2}\left\{ - D_\mu h D^\mu h_{\beta\delta} + D_\beta h^{\alpha\mu} D_\delta h_{\alpha\mu} + D_\beta h^\mu_\delta D_\mu h + D_\delta h^\mu_\beta D_\mu h \right\},\end{aligned}$$

$$\begin{aligned}&\left(\sqrt{|g|}\, R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}\right)^{(2)} \\ &= \sqrt{|g|}\, \Big\{ \tfrac{1}{2}\left(\tfrac{1}{2}h^2 - h^{\mu\nu}h_{\mu\nu}\right) R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \\ &\qquad - 2 h h^\alpha_\mu R^{\mu\beta\gamma\varepsilon} R_{\alpha\beta\gamma\varepsilon} + 7 R^{\alpha\beta\gamma\mu} R_{\alpha\beta\gamma\nu} h_{\mu\varepsilon} h^{\nu\varepsilon} \\ &\qquad - R^{\alpha\beta\gamma\mu} R_{\alpha\beta\nu\varepsilon} h^\varepsilon_\mu h^\nu_\gamma + 4 h R^{\alpha\beta\gamma\varepsilon} D_\beta D_\gamma h_{\alpha\varepsilon} \\ &\qquad - 4 R^{\alpha\beta\gamma\delta} h_{\delta\varepsilon}\left[D_\beta D_\gamma h^\varepsilon_\alpha + D_\alpha D^\varepsilon h_{\beta\gamma}\right] \\ &\qquad - 8 R^{\mu\beta\gamma\varepsilon} h^\alpha_\mu \left[D_\beta D_\gamma h_{\varepsilon\alpha} + D_\alpha D_\varepsilon h_{\beta\gamma}\right] \\ &\qquad + 2 R^{\alpha\beta\gamma\delta}\left[D_\mu h_{\alpha\delta} D^\mu h_{\beta\gamma} + 2 D_\beta h^\mu_\gamma D_\alpha h_{\mu\delta} \right. \\ &\qquad\qquad \left. + 2 D_\alpha h^\mu_\delta D_\gamma h_{\mu\beta} + 2 D_\alpha h^\mu_\gamma D_\mu h_{\beta\delta} + 2 D_\beta h^\mu_\delta D_\mu h_{\gamma\alpha}\right] \\ &\qquad + 2\left[D^\alpha D^\gamma h^{\varepsilon\beta} D_\alpha D_\gamma h_{\varepsilon\beta} - D^\alpha D^\gamma h^{\varepsilon\beta} D_\beta D_\gamma h_{\varepsilon\alpha} \right. \\ &\qquad\qquad \left. - D^\alpha D^\gamma h^{\varepsilon\beta} D_\alpha D_\varepsilon h_{\beta\gamma} + D^\alpha D^\gamma h^{\varepsilon\beta} D_\beta D_\varepsilon h_{\alpha\gamma}\right]\Big\}.\end{aligned}$$

Appendix B

In this appendix we define the coefficients appearing in eq. (4.13) for an arbitrary radius $L = y^{-1/2}$ of S^D. If the background is to fulfil the equations of motion, y has to be chosen equal to $\bar{y}$ of eq. (4.3) and ε must be adjusted according to (4.5).

$$A_1 = \tfrac{1}{2}\beta + 2\gamma,$$

$$A_2 = -D(D-1)y\alpha + 2y\beta + 2y(2D+1)\gamma + \tfrac{1}{2}\delta,$$

$$A_3 = -D(D-1)y\alpha + (-D+3)y\beta + 2Dy\gamma + \tfrac{1}{2}\delta,$$

$$A_4 = \left\{ -\tfrac{1}{2}D^4 + 3D^3 - \tfrac{13}{2}D^2 + 4D \right\} y^2\alpha + \left\{ -\tfrac{1}{2}D^3 + 3D^2 - \tfrac{13}{2}D + 6 \right\} y^2\beta$$
$$+ \{ -D^2 + 9D - 12 \} y^2\gamma + \left\{ \tfrac{1}{2}D^2 - \tfrac{3}{2}D + 2 \right\} y\delta - \tfrac{1}{2}\varepsilon ;$$

$$B_1 = 2\alpha + \tfrac{1}{2}\beta ,$$

$$B_2 = \{ D^2 - 5D + 4 \} y\alpha - 2y\beta - 6y\gamma - \tfrac{1}{2}\delta ,$$

$$B_3 = \{ D^2 - 5D + 4 \} y\alpha + \tfrac{1}{2}(D-5)y\beta - 2y\gamma - \tfrac{1}{2}\delta ,$$

$$B_4 = \left\{ \tfrac{1}{4}D^4 - \tfrac{5}{2}D^3 + \tfrac{33}{4}D^2 - 8D + 2 \right\} y^2\alpha + \left\{ \tfrac{1}{4}D^3 - \tfrac{5}{2}D^2 + \tfrac{33}{4}D - 8 \right\} y^2\beta$$
$$+ \left\{ \tfrac{1}{2}D^2 - \tfrac{9}{2}D + 8 \right\} y^2\gamma + \left\{ -\tfrac{1}{4}D^2 + \tfrac{5}{4}D - 2 \right\} y\delta + \tfrac{1}{4}\varepsilon ;$$

$$C_1 = 4\alpha + \beta ,$$

$$C_2 = \{ 2D^2 - 6D + 4 \} y\alpha - \delta ,$$

$$C_3 = \{ 2D^2 - 6D + 4 \} y\alpha + (D-1)y\beta + 4y\gamma - \delta ,$$

$$C_4 = \left(\tfrac{1}{2}D^2 - 2D \right)(D-1)^2 y^2\alpha + \left\{ \tfrac{1}{2}D^3 - 3D^2 + \tfrac{9}{2}D - 2 \right\} y^2\beta$$
$$+ \{ D^2 - 5D + 4 \} y^2\gamma + \left\{ -\tfrac{1}{2}D^2 + \tfrac{3}{2}D - 1 \right\} y\delta + \tfrac{1}{2}\varepsilon ;$$

$$F_1 = \beta + 4\gamma ,$$

$$F_2 = -2D(D-1)y\alpha + 4(D-1)y\gamma + \delta ,$$

$$F_3 = -2D(D-1)y\alpha - 2(D-1)y\beta - 4y\gamma + \delta ,$$

$$F_4 = (-D^2 + 2D)(D-1)^2 y^2\alpha + \{ -D^3 + 3D^2 - 3D + 1 \} y^2\beta$$
$$+ \{ -6D^2 + 14D - 8 \} y^2\gamma + \{ D^2 - 2D + 1 \} y\delta - \varepsilon ;$$

$$G_1 = \tfrac{1}{2}\beta + 2\gamma ,$$

$$G_2 = -D(D-1)y\alpha + \tfrac{1}{2}\delta ,$$

$$G_3 = -D(D-1)y\alpha - (D-1)y\beta - 2(D-1)y\gamma + \tfrac{1}{2}\delta ,$$

$$G_4 = -\tfrac{1}{2}D^2(D-1)^2 y^2\alpha - \tfrac{1}{2}D(D-1)^2 y^2\beta - D(D-1)y^2\gamma$$
$$+ \tfrac{1}{2}D(D-1)y\delta - \tfrac{1}{2}\varepsilon ;$$

$$H_1 = 2\alpha + \tfrac{1}{2}\beta ,$$

$$H_2 = D(D-1)y\alpha - \tfrac{1}{2}\delta ,$$

$$H_3 = D(D-1)y\alpha + \tfrac{1}{2}(D-1)y\beta - \tfrac{1}{2}\delta ,$$

$$H_4 = \tfrac{1}{4}D^2(D-1)^2 y^2\alpha + \tfrac{1}{4}D(D-1)^2 y^2\beta + \tfrac{1}{2}D(D-1)y^2\gamma$$
$$- \tfrac{1}{4}D(D-1)y\delta + \tfrac{1}{4}\varepsilon;$$
$$L = 4y\gamma.$$

Appendix C

The coefficients in the quadratic action (5.5) read as follows:

$$a_1^i = A_1, \qquad i = 1,2,3,$$
$$a_2^1 = -2A_1\eta_1\bar{y} - A_2,$$
$$a_3^1 = A_1\eta_1^2\bar{y}^2 + A_3\eta_1\bar{y} + A_4;$$
$$a_2^2 = -2A_1\eta_2\bar{y} - A_2,$$
$$a_3^2 = A_1\eta_2^2\bar{y}^2 + A_3\eta_2\bar{y} + A_4;$$
$$a_2^3 = -2A_1\eta_3\bar{y} - A_2,$$
$$a_3^3 = A_1\eta_3^2\bar{y}^2 + A_3\eta_3\bar{y} + A_4;$$
$$a_1^4 = DA_1 + D^2B_1,$$
$$a_2^4 = -2\left(DA_1 + D^2B_1\right)\eta_4\bar{y} - \left(DA_2 + D^2B_2\right),$$
$$a_3^4 = \left(DA_1 + D^2B_1\right)\eta_4^2\bar{y}^2 + \left(DA_3 + D^2B_3\right)\eta_4\bar{y} + \left(DA_4 + D^2B_4\right);$$
$$c_2 = -2C_1D\eta_4\bar{y} - DC_2,$$
$$c_3 = C_1D\eta_4^2\bar{y}^2 + DC_3\eta_4\bar{y} + DC_4;$$
$$f_2^1 = -2F_1\eta_5\bar{y} - F_2,$$
$$f_3^1 = F_1\eta_5^2\bar{y}^2 + F_3\eta_5\bar{y} + F_4;$$
$$f_2^2 = -2F_1\eta_6\bar{y} - F_2,$$
$$f_3^2 = F_1\eta_6^2\bar{y}^2 + F_3\eta_6\bar{y} + F_4;$$
$$g_2 = -2G_1\eta_4\bar{y} - G_2,$$
$$g_3 = G_1\eta_4^2\bar{y}^2 + G_3\eta_4\bar{y} + G_4;$$
$$h_2 = -2H_1\eta_4\bar{y} - H_2,$$
$$h_3 = H_1\eta_4^2\bar{y}^2 + H_3\eta_4\bar{y} + H_4.$$

Appendix D

Next we define the expressions appearing in the action of the coupled scalars (6.6f, g, h). These quantities depend on l as well as on k^2.

$$\xi_1^0=\left(H_1+\frac{1}{n}G_1\right)k^4+\left(h_2+\frac{1}{n}g_2\right)k^2+\left(h_3+\frac{1}{n}g_3\right),$$

$$\xi_2^0=G_1k^4+g_2k^2+g_3,$$

$$\xi_3^0=a_1^4k^4+a_2^4k^2+a_3^4,$$

$$\xi_4^0=C_1Dk^4+c_2k^2+c_3,$$

$$\xi_5^0=F_1k^4+f_2^2k^2+f_3^2,$$

$$\xi_6^0=a_1^3k^4+a_2^3k^2+a_3^3,$$

$$\xi_1^1=\xi_1^0+\frac{1}{n(n-1)}\xi_2^0,$$

$$\xi_2^1=\xi_3^0-\frac{\bar{y}D}{k^2}\xi_5^0+\frac{nD^2\bar{y}^2}{(n-1)k^4}\xi_2^0-LD(D-1)\bar{y}-\frac{LD^2\bar{y}^2}{k^2},$$

$$\xi_3^1=\xi_4^0+\frac{2D\bar{y}}{(n-1)k^2}\xi_3^0-DL\bar{y},$$

$$\Xi^0=\left[4\left(\xi_1^0+\frac{1}{n(n-1)}\xi_2^0\right)\xi_3^0-\left(\xi_4^0\right)^2\right]^{-1},$$

$$\Xi^1=\left[4\xi_1^1\xi_2^1-\left(\xi_3^1\right)^2\right]^{-1},$$

$$\omega_1=2D(1-2\bar{y}/k^2)+2n(n-1)^{-1}D^2\bar{y}^2/k^4,$$

$$\omega_2=2D(n-1)^{-1}\bar{y}/k^2,$$

$$\omega_3=\bar{y}^{1/2}\left[\left(1-\frac{1}{D}\right)l(l+D-1)-(D-1)\right]^{1/2}\Big/k^2,$$

$$\omega_4=\bar{y}^{1/2}\left[l(l+D-1)\right]^{1/2}/k^2,$$

$$\Omega_i=-n(n-1)^{-1}\left[l(l+D-1)\bar{y}\right]^{1/2}\omega_i/k^2,\qquad i=3,4,$$

$$\mathbb{M}_{11}=2\left[\xi_6^0-\omega_3^2k^2\left(\xi_5^0-Lk^2\right)+\frac{n-1}{n}\Omega_3^2k^4\xi_2^0-L\left(\frac{n-1}{n}\right)^2k^6\Omega_3^2\right],$$

$$\mathbb{M}_{22}=2\left[\xi_3^0-\omega_4^2k^2\left(\xi_5^0-Lk^2\right)+\frac{n-1}{n}\Omega_4^2k^4\xi_2^0+LD\frac{n-1}{n}\Omega_4k^4\right.$$
$$\left.-L\left(\frac{n-1}{n}\right)^2\Omega_4^2k^6\right],$$

$$\mathbb{M}_{33}=2\left[\xi_1^0+\frac{1}{n(n-1)}\xi_2^0\right],$$

$$\mathbb{M}_{12}=\mathbb{M}_{21}=-2\omega_3\omega_4k^2\left(\xi_5^0-Lk^2\right)+2\frac{n-1}{n}\Omega_3\Omega_4k^4\xi_2^0$$
$$+LD\frac{n-1}{n}\Omega_3k^4-2L\left(\frac{n-1}{n}\right)^2\Omega_3\Omega_4k^6,$$

$$\mathbb{M}_{13}=\mathbb{M}_{31}=n(n-1)^{-1}L\Omega_3k^4-2n^{-1}\Omega_3k^2\xi_2^0,$$

$$\mathbb{M}_{23}=\mathbb{M}_{32}=\xi_4^0-2n^{-1}\Omega_4k^2\xi_2^0+n(n-1)^{-1}L\Omega_4k^4,$$

$$\rho_{11}=2-4k^2\omega_3^2+2n^{-1}(n-1)k^4\Omega_3^2,$$

$$\rho_{22}=2D-4k^2\omega_4^2+2n^{-1}(n-1)k^4\Omega_4^2,$$

$$\rho_{33}=2(n-1)^{-1},$$

$$\rho_{12}=\rho_{21}=-4k^2\omega_3\omega_4+2n^{-1}(n-1)k^4\Omega_3\Omega_4,$$

$$\rho_{13}=\rho_{31}=-2n^{-1}k^2\Omega_3,$$

$$\rho_{23}=\rho_{32}=-2n^{-1}k^2\Omega_4.$$

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