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THE SU(N) LATTICE HIGGS MODEL AT STRONG GAUGE COUPLING

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A family of lattice Higgs models with matter fields carrying the fundamental representation of an SU(N) gauge group is analyzed at strong gauge coupling. Similarities with QCD with staggered fermions are emphasized. For large N the phase diagram is derived, and the scaling limit at the critical endpoint of the first-order Higgs-confinement transition line is investigated.

1. Introduction

From the moment that lattice versions of nonabelian gauge theories were written down, the strong coupling approximation has given important clues to the qualitative features of the nonperturbative dynamics. Such information is clearly desirable – if not indispensable – before a model is analyzed by purely numerical methods. The immediate interest in the pure gauge theory, for example, stemmed from the fact that confinement is apparent in the limit of strong coupling. Although it is remote from the weak coupling continuum limit, one expects qualitative features like a nonvanishing gap or string-tension to persist as long as one can move between the two points of the phase diagram without crossing a phase boundary. With lattice fermions coupled to the gauge fields, strong coupling again proved to be a valuable guide. It predicts the spontaneous breakdown of chiral symmetry [1–4] for staggered fermions and even gives reasonable rough estimates for the masses of mesons and baryons.

Recently the focus of interest of the lattice community has broadened from QCD to also include gauge-Higgs models [5]. One considers scalar matter coupled to lattice gauge fields in a way resembling the bosonic sector of the standard model. Such a theory has at least three bare coupling constants that have to be tuned simultaneously to a critical point in order to reach a continuum limit. So far the gaussian fixed-point, where all couplings are weak, has been investigated [6] and seems to lead to a noninteracting theory in the proper cutoff limit. Numerically it is indicated [7] that the system might exhibit critical behavior also on the boundary of a sheet of first-order transitions. The hyperplane $\beta = 0$ of the infinite gauge coupling

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is intersected by a first-order line with an endpoint. This is the submanifold of the phase diagram to be considered in this article. Judging from the data, it should not be an untypical representative of its neighborhood; the phase structure continues toward negative β values without drastic changes.

In sect. 2 we shall define a general N family of SU(N) gauge Higgs models and integrate out the gauge fields. The resulting scalar theory of the radial Higgs field is juxtaposed with the analogous theory of the chiral condensate of QCD. Both are still interacting theories with nontrivial effective action and can thus in general only be analyzed numerically. A dimer formulation [3] has been derived and simulated for QCD and is shown to emerge for the Higgs model, too. Here, however, we stay on the analytic track, and in sect. 3 we resort to taking the additional approximation of large N. Then a Landau description of the critical endpoint holds, and we have full control over the phase diagram and scaling behavior. In sect. 4 some conclusions are offered.

2. Effective theories of strongly gauge-coupled quark and Higgs fields

At strong coupling (no plaquette term) we wish to consider side-by-side lattice gauge fields coupled to fundamental Higgs fields

$$Z_{\mathrm{H}} = \int \mathrm{D}\varphi\, \mathrm{D}\varphi^+\, \mathrm{D}U \exp\left\{ - \sum_x \left[\frac{A}{2N} (\varphi^+ \varphi_x)^2 + B \varphi^+ \varphi_x \right] + \tfrac{1}{2} \sum_{x,\mu} \left[\varphi_x^+ U_{x\mu} \varphi_{x+\mu} + \varphi_{x+\mu}^+ U_{x\mu}^+ \varphi_x \right] \right\} \tag{1a}$$

and to staggered fermions

$$Z_{\mathrm{Q}} = \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi}\, \mathrm{D}U \exp\left\{ m \sum_x \bar{\psi}\psi_x + \tfrac{1}{2} \sum_{x,\mu} \Gamma_{x\mu} \left[\bar{\psi}_x U_{x\mu} \psi_{x+\mu} - \bar{\psi}_{x+\mu} U_{x\mu}^+ \psi_x \right] \right\}. \tag{1b}$$

Here x and μ denote the usual labels of sites and directions in a D-dimensional hypercubic lattice, and $\Gamma_{x\mu}$ are the standard sign factors. The gauge fields are integrated with the invariant group measure on every link, $\mathrm{D}U = \prod_{x,\mu} \mathrm{d}U_{x\mu}$. As we shall see, SU($N$) and U($N$) gauge groups are equivalent in the scalar theory. When coupled to fermions they differ by the propagation or absence of antisymmetric baryonic composites, and we shall restrict ourselves to the simpler U(N) case. For the phase structure and chiral symmetry breaking it has been found [8] that there is only a small quantitative difference between U(2) and SU(2). The measures of the complex scalars and anticommuting variables are

$$\mathrm{D}\varphi\,\mathrm{D}\varphi^{+} = \prod_{x}\prod_{n=1}^{N} \mathrm{d}\,\mathrm{Re}\,\varphi_x^n\,\mathrm{d}\,\mathrm{Im}\,\varphi_x^n, \tag{2a}$$

$$\mathrm{D}\psi\,\mathrm{D}\bar{\psi} = \prod_{x}\prod_{n=1}^{N} \mathrm{d}\psi_x^n\,\mathrm{d}\bar{\psi}_x^n, \tag{2b}$$

where n is the otherwise suppressed gauge group index. Finally m, B and $A > 0$ are real parameters. The parametrization in (1a) is connected to the standard lattice Higgs action

$$Z_{\mathrm{st}} = \int \mathrm{D}\varphi\,\mathrm{D}\varphi^{+}\,\mathrm{D}U \exp\Big\{-\sum_{x}\big[\lambda(\varphi^{+}\varphi_x - 1)^2 + \varphi^{+}\varphi_x\big] + k\sum_{x,\mu}\big[\varphi_x^{+}U_{x\mu}\varphi_{x+\mu} + \varphi_{x+\mu}^{+}U_{x\mu}^{+}\varphi_x\big]\Big\}$$

by

$$k = \frac{NB}{4A}\left(\left(1+\frac{4A}{NB^2}\right)^{1/2} - 1\right) = \frac{1}{2B}(1+\mathrm{o}(\lambda)) = \frac{1}{2B}(1+\mathrm{o}(1/N)),$$

$$\lambda = \frac{2A}{N}k^2, \tag{3}$$

and λ is small in the interesting sections of a typical phase diagram (U(1) or SU(2)) [7].

At strong coupling the gauge variables are simply random fields on the links, and they are integrated out link by link leaving behind a nearest neighbor interaction of gauge invariant composite fields. The relevant U(N) group integrals are found in ref. [3]:

$$\int_{(\mathrm{S})\mathrm{U}(N)} \mathrm{d}U \exp\{\varphi_x^{+}U\varphi_{x+\mu} + \varphi_{x+\mu}^{+}U^{+}\varphi_x\} = \sum_{k=0}^{\infty}\alpha(N,k)\big(\varphi^{+}\varphi_x\varphi^{+}\varphi_{x+\mu}\big)^k = H_N\big(\varphi^{+}\varphi_x\varphi^{+}\varphi_{x+\mu}\big) \tag{3a}$$

and

$$\int_{\mathrm{U}(N)} \mathrm{d}U \exp\{\bar{\psi}_x U\psi_{x+\mu} - \bar{\psi}_{x+\mu}U^{+}\psi_x\} = \sum_{k=0}^{N}(-)^k\alpha(-N,k)\big(\bar{\psi}\psi_x\bar{\psi}\psi_{x+\mu}\big)^k = H_{-N}\big(-\bar{\psi}\psi_x\bar{\psi}\psi_{x+\mu}\big), \tag{3b}$$

with

$$\alpha(N,k)=\frac{(N-1)!}{k!(N+k-1)!}\Rightarrow(-)^k\alpha(-N,k)=\frac{(N-k)!}{k!N!}. \tag{4}$$

We recognize that H_N is essentially a modified Bessel function

$$H_N(z^2/4)=(2/z)^{N-1}I_{N-1}(z). \tag{5}$$

Note that formally the result for fermions looks like the bosonic formula continued to negative N. The bosonic result is identical for SU(N) and for $U(N)$ since there are simply no SU(N) gauge invariant combinations of $\varphi, \varphi^\dagger$ to distinguish the different centers. Inserting now (3) into (1) we derive the effective theories

$$Z_{\mathrm{H}}=\int \mathrm{D}\varphi\,\mathrm{D}\varphi^\dagger \exp\left\{-\sum_x\left[\frac{A}{2N}(\varphi^+\varphi_x)^2+B\varphi^+\varphi_x\right]\right\}$$

$$\times\prod_{x,\mu}H_N\left(\tfrac{1}{4}\varphi^+\varphi_x\varphi^+\varphi_{x+\mu}\right), \tag{6a}$$

$$Z_{\mathrm{Q}}=\int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\exp\left\{m\sum_x\bar{\psi}\psi_x\right\}\prod_{x,\mu}H_{-N}\left(-\tfrac{1}{4}\bar{\psi}\psi_x\bar{\psi}\psi_{x+\mu}\right). \tag{6b}$$

Here Z_{Q} is still given as a Grassmann integral and thus defies a numerical treatment. However, the equivalent generalized monomer-dimer model [3] has been simulated very efficiently. It obtains by expanding H on every link, integrating exactly every term in the multiple expansion, and considering the sum over the various terms as a new statistical system. Then we have independent integer link variables $k_{x\mu}=0,1,\ldots,N$, and find

$$Z_{\mathrm{Q}}=\sum_{\{k_{x\mu}\}}\prod_{x,\mu}\gamma_{\mathrm{Q}}(k_{x\mu})\prod_x\rho_{\mathrm{Q}}(\sigma_x) \tag{7}$$

with the auxiliary field

$$\sigma_x=\sum_\mu\left(k_{x\mu}+k_{(x-\mu)\mu}\right) \tag{8}$$

counting the power of $\bar{\psi}\psi_x$ in the original theory and weights

$$\gamma_{\mathrm{Q}}(k)=\left(\frac{1}{4}\right)^k\frac{(N-k)!}{k!N!}, \tag{9a}$$

$$\rho_{\mathrm{Q}}(\sigma)=\frac{N!}{(N-\sigma)!}m^{N-\sigma}. \tag{10a}$$

The Higgs system Z_H could clearly be simulated as a one component scalar field theory by conventional methods, although a relatively extravagant action would have to be computed at every, say, Metropolis step. On the other hand it is clear that also this model can be written in dimer form (7), where now the links can be occupied by any number of dimers, $k_{x\mu} = 0, 1, \ldots,$ and the once again strictly positive weights are given as

$$\gamma_H(k) = \left(\frac{1}{4}\right)^k \frac{(N-1)!}{k!(N+k-1)!}, \tag{9b}$$

$$\rho_H(\sigma) = \int_0^\infty dR\, R^{N-1+\sigma} \exp\left\{-\frac{A}{2N}R^2 - BR\right\}. \tag{10b}$$

We are presently implementing a simulation of scalar models in dimer form to see if it is efficient enough to be of interest. The possible advantage would be that the computer has to deal only with integers, no functions have to be evaluated, and by changing the precomputed weights γ, ρ whole classes of models could be simulated.

Here, however, we carry on computing Z_Q, Z_H analytically in the large N limit.

3. The large N approximation

To bring out the leading large N behavior of the partition functions (6) we rescale all field by $\sqrt{N}$, put

$$H_N\left(\tfrac{1}{4}N^2 z^2\right) = \exp\{-NW(z^2)\}, \tag{11}$$

and end up with

$$Z_H = \int D\varphi\, D\varphi^+ \exp\left\{-N\left[\sum_x \tfrac{1}{2}A(\varphi^+\varphi_x)^2 + B\varphi^+\varphi_x + \sum_{x,\mu} W\left(\varphi^+\varphi_x\varphi^+\varphi_{x+\mu}\right)\right]\right\}, \tag{12a}$$

$$Z_Q = \int D\psi\, D\bar\psi \exp\left\{N\left[m\sum_x \bar\psi\psi_x + \sum_{x,\mu} W\left(-\bar\psi\psi_x\bar\psi\psi_{x+\mu}\right)\right]\right\}. \tag{12b}$$

Also, for large N, we use

$$W(z^2) = 1 - \sqrt{1+z^2} + \log\left(\tfrac{1}{2}\left(1+\sqrt{1+z^2}\right)\right) + o(1/N), \tag{13}$$

which follows from an asymptotic expansion of the Bessel function in (5) for large index or by the method outlined in ref. [3]. Thus $1/N$ is a loop counting parameter, and for $N \to \infty$ the saddle-point approximation becomes exact. Here it may be noted that in ref. [9] some results on the Higgs-gauge phase structure have been obtained from a leading order mean-field expansion. Since it can be rearranged as a $1/D$ expansion, it is valid in the limit of large euclidean dimension. The emerging phase diagram is of the same type as that which we shall find.

The Boltzmann weights in (6) and (12) depend only on the combinations $\varphi^+\varphi_x$ and $\bar{\psi}\psi_x$, which are locally gauge invariant – actually O(2N) and SP(2N,C) invariant – combinations of the fundamental matter fields. Physically, this is the result of the integrated gauge interaction which at infinite coupling has zero confinement radius. Clearly, one wants to eliminate the by now trivially redundant gauge degrees of freedom and go over to neutral fields. For the bosons this is simply achieved by integrating the angles in φ_x and remaining with the real positive field $R_x = \varphi^+\varphi_x$. For the fermions we use on every site the identity

$$\frac{1}{N!}\int \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\, f(\bar{\psi}\psi) = \oint \frac{\mathrm{d}\sigma}{2\pi i}\,\frac{1}{\sigma^{N+1}} f(\sigma)\,. \tag{14}$$

Both the Grassmann integral and the contour integral pick up the Nth order term in the expansion of f. The result for large N is

$$Z_{\mathrm{H}} = \prod_x \int_0^\infty \mathrm{d}R_x \exp\left\{ -N\left[\sum_x \left(\tfrac{1}{2}AR_x^2 + BR_x - \log R_x\right) + \sum_{x,\mu} W(R_x R_{x+\mu}) \right]\right\}, \tag{15a}$$

$$Z_{\mathrm{Q}} = \prod_x \oint \mathrm{d}\sigma_x \exp\left\{ N\left[\sum_x (m\sigma_x - \log \sigma_x) + \sum_{x,\mu} W(-\sigma_x\sigma_{x+\mu}) \right]\right\} \tag{15b}$$

The fermion-derived integral (15b) has the well known real constant saddle point [1–3]

$$\bar{\sigma} = \frac{D\sqrt{m^2+2D-1} - (D-1)m}{D^2+m^2} \underset{m\to 0^+}{\to} \frac{\sqrt{2D-1}}{D} \tag{16}$$

that exhibits chiral symmetry breaking. For the Higgs field we write

$$Z_{\mathrm{H}} = \exp\{-NVf(A,B)\}\,, \tag{17}$$

where V is the number of lattice sites, and have

$$f = \min_R U(R)\,, \tag{18}$$

$$U(R) = \tfrac{1}{2}AR^2 + BR - \log R + DW(R^2)\,. \tag{19}$$

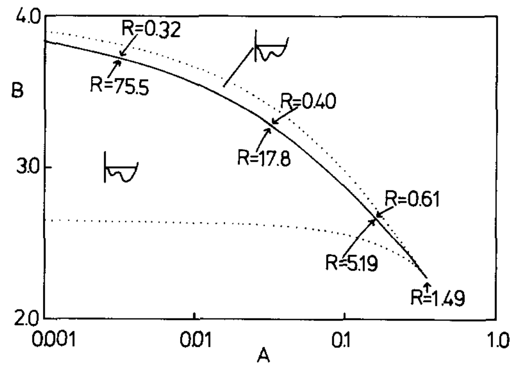

Fig. 1. Phase diagram of the gauge Higgs model at large N. The solid line is the first-order transition with critical endpoint, the dotted lines show the boundaries of metastable regions.

Here U is the effective (Landau) potential determining the phase structure of the large-N strongly gauge coupled Higgs gauge system.

Depending on the values of A and B the potential U has either one or two minima (phase coexistence, metastability) in the range $0 < R < \infty$. For $D = 4$ the corresponding phase diagram with the first-order line is displayed in the fig. 1. There is a unique critical endpoint with $U' = U'' = U''' = 0$, where two phases merge, and its coordinates A^*, B^* are summarized in table 1.

The scaling behavior close to a critical point is given by the mass of the quadratic fluctuations around the saddle point. In (15b) the stable fluctuations (steepest descent) around the real saddle point are imaginary, and, writing

$$\sigma_x = \bar{\sigma} + i(-)^{x_1 + x_2 + \cdots + x_D}\psi_x$$

we find a mass [3]

$$m_G^2 = 4\frac{\sqrt{1 - \bar{\sigma}^2}}{\bar{\sigma}}m. \tag{20}$$

This is the Goldstone mode of the staggered chiral symmetry, and the mean field critical exponent $\nu = \frac{1}{2}$ in (20) is called PCAC relation in this context. Carrying out

TABLE 1

Critical field, coupling constants, and quartic Landau potential coefficient in various dimensions

D	R^*	A^*	B^*	$C^* = U^{(4)}\vert_{R^*}$
2	2.70016	0.041889	1.64930	0.030315
3	1.81925	0.167659	2.01898	0.233418
4	1.48766	0.347248	2.28653	0.648432
$\to\infty$	$\left(\frac{8}{3D}\right)^{1/4}\left(1+\frac{5}{6}\left(\frac{2}{3D}\right)^{1/2}\right)$	$\frac{1}{2}D\left(1-\left(\frac{6}{D}\right)^{1/2}\right)$	$\frac{8}{3}\left(\frac{3}{8}D\right)^{1/4}$	$3D\left(1-5\left(\frac{2}{3D}\right)^{1/2}\right)$

(next term down by another power $D^{-1/2}$)

a similar expansion around a minimum of the action (15a) by writing

$$R_x = \bar{R} + \psi_x$$

a Higgs mass

$$m_H^2 = 4\sqrt{1+\bar{R}^2}\, U''(\bar{R}) \tag{21}$$

results. The extrema close to the critical endpoint are governed by the Landau potential

$$U(R^* + r) \cong U(R^*) - \varepsilon r + \tfrac{1}{2} a r^2 + \frac{C^*}{4!} r^4, \tag{22}$$

which arises from Taylor-expanding U around R^* for $A = A^* + a$, $B = B^* + b$ close to the critical point (a, b small). Here

$$C^* = U^{(4)}|_{R^*}, \qquad \varepsilon = -aR^* - b, \tag{23}$$

and the combination ε is chosen such that $\varepsilon = 0$ is tangent to the transition line at the endpoint. Numerical values for C^* are contained in the table. This implies close to the critical point

$$m_H^2 \cong 4\sqrt{1+R^{*2}}\left(a + \tfrac{1}{2} C^* \bar{r}^2\right), \tag{24}$$

with $\bar{r}$ determined to minimize (22). Therefore scaling laws

$$\bar{r} \cong |a|^{1/2} f_{\pm}\left(\frac{\varepsilon}{|a|^{3/2}}\right),$$

$$m_H^2 \cong 4\sqrt{1+R^{*2}}\,|a|\left(\tfrac{1}{2} C^* f_{\pm}^2\left(\frac{\varepsilon}{|a|^{3/2}}\right) \pm 1\right) \tag{26}$$

hold. The sign in (26) is the sign of a, and $f_{\pm}(x)$ minimizes the expression $-xf_{\pm} \pm \frac{1}{2} f_{\pm}^2 + C^* f_{\pm}^4/4!$. Special cases arise on the critical line $\varepsilon = 0$

$$m_H^2 = \begin{cases} 4\sqrt{1+R^{*2}}\, a & \text{for } a > 0 \\ -8\sqrt{1+R^{*2}}\, a & \text{for } a < 0, \end{cases} \tag{27}$$

and for $a = 0$

$$m_H^2 = 2\sqrt{1+R^{*2}}\, C^{*1/3} (6|b|)^{2/3}. \tag{28}$$

The scaling structure of this critical endpoint is the same as for the Z(2) Higgs-gauge system in the mean field analysis of Brézin and Drouffe [10], i.e. a φ^4 model.

For the chiral order parameter (16) the first $1/N$ correction has been computed [3] and led to no qualitative change in the picture. We would expect the same to hold true if one computed the first correction to the effective potential U in (19). In particular, such a correction would not change the Landau exponents. On the other hand the simulation of (6), presumably in dimer form (7), will be exact at $g^2=\infty$ for any N and thus reveal even effects that are nonanalytic in N, if present.

4. Conclusions

We analyzed U(N)-QCD with fermions and the fundamental (S)U(N) Higgs gauge model at infinite gauge coupling. In a remarkably parallel fashion both are converted to effective scalar theories of the chiral condensate $\bar{\psi}\psi$ and square radius of the Higgs field $\varphi^+\varphi$ describing the pion- and Higgs-boson excitations respectively. At finite N both theories may be looked upon as generalized monomer-dimer models. At diverging N a saddle-point expansion is applicable that incorporates real fluctuations around $\langle\varphi^+\varphi\rangle$ and staggered imaginary ones around $\langle\bar{\psi}\psi\rangle$ as the stable modes of steepest descent. In either case we find a diverging correlation length close to a critical point which the fermion system owes to the Goldstone phenomenon associated with the continuous chiral U(1). Criticality of the bosonic field theory is achieved by tuning two free parameters to the endpoint of a first-order line.

The effective theories (15) at $g^2=\infty$ possess rather unusual kinetic energy terms and integration measures. Nevertheless, by standard renormalization group and universality arguments, one would conclude that the infinitely many extra terms are irrelevant at $D=4$. The continuum pion and Higgs boson constructed at the strong coupling critical point are then the quanta of a one-component φ^4 theory and as such presumably noninteracting.

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