Z. Phys. C – Particles and Fields 34, 497–522 (1987)

Zeitschrift für Physik C Particles and Fields

Two-Jet Cross Section in e^+e^- Annihilation

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Received 17 November 1986

Abstract. We calculate the two-jet cross section in order α_s^2 in the framework of massless perturbative QCD, using for jet resolution the jet mass. We derive results for two different approaches. One approach is based on the singularity structure of the various contributions. The other one is adjusted to the methods used in three- and four-jet calculations so that the total cross section can be reconstructed.

1. Introduction

Much experimental [1–6] and theoretical work [7–18] has been carried out to analyse the $O(\alpha_s^2)$ corrections to differential three-jet cross sections in e^+e^- annihilation. Differential and integrated four-jet cross sections have been calculated in [19]. The integrated three-jet cross section is studied in [20].

In this paper we present the calculation of the two-jet cross section. This is an integrated cross section by definition, depending only on the resolution parameter that defines the jet. Although simpler in concept and although the $O(\alpha_s)$ two-jet cross section has been given as early as 1977 by Sterman and Weinberg [21] the $O(\alpha_s^2)$ corrections have not yet been calculated. To know the two-jet cross section up to $O(\alpha_s^2)$ is useful for several reasons. First it can be used to determine the coupling constant α_s or the scale parameter Λ by comparing the resolution dependence with experimental two-jet rates obtained from a cluster analysis of e^+e^- annihilation data [22]. Second it is important to know for consistency that the sums with the integrated three- and four-jet cross section yields the well known $O(\alpha_s^2)$ correction of σ_{tot}, which has been obtained independently via the optical theorem from the imaginary part of the vacuum polarization of the photon already some time ago [23].

One wants to calculate gluon corrections to Fig. 1. In this lowest order the two-jet cross section is identical to the total cross section

$$\sigma_0=\frac{4\pi\alpha^2}{3q^2}N_c\sum_f Q_f^2. \tag{1.1}$$

because only the two-parton diagram Fig. 1 contributes (α is the fine structure constant, N_c the number of colours, Q_f the flavour charges and the momenta are defined in Fig. 1).

In higher orders gluon radiation comes in and makes three- and four-jet events possible. However, qualitatively one expects these to be suppressed by powers of α_s compared to two-jet events.

The higher order contributions to the two-jet cross section have ultraviolet, infrared and collinear singularities. All these are regularized by going to $n=4-2\varepsilon$ dimensions. The ultraviolet singularities will be removed by renormalization in the $\overline{\text{MS}}$-scheme, while infrared and collinear singularities cancel in the sum of real and virtual contributions.

All calculations have been done in the Feynman gauge of massless QCD and in the one photon approximation. Also all correlations with the incoming beam have been integrated out. To define a jet one can use (ε, δ)-cuts, i.e. a cut for the energy and independently for the angle [21, 9]. We worked with an invariant mass cut defined as follows: Let $s_{ij}=(p_i$

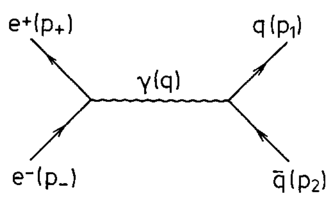

Fig. 1. Lowest order diagram for $e^+e^-\to q\bar{q}$

* This work has been funded by the German Federal Minister for Research and Technology (BMFT) under the contract number 054HH92P3

$+p_j)^2=2p_i p_j$ be the invariant mass of two partons i, j. Normalized to the energy, it is

$$y_{ij}=s_{ij}/q^2. \tag{1.2}$$

Then we say that i and j are in one jet, if y_{ij} is less than a given number y. One can easily see that this is both an energy and an angle cut.

The region of allowed y values is quite small:

$$0.02<y<0.06. \tag{1.3}$$

The lower limit comes from the fact that perturbation theory breaks down for very small values of y. One would have to take into account the radiation of an arbitrary number of gluons.

The upper limit is dictated by an additional approximation we are working with, namely we neglect terms of order y.

The outline of the paper is as follows. In Sect. 2 we will review briefly the derivation of the Sterman-Weinberg formula (two-jet cross section up to $O(\alpha_s)$).

In $O(\alpha_s^2)$ one has three classes of diagrams. There are two-parton diagrams which fully contribute to the two-jet case (Fig. 3). Here one has to do two virtual integrations. These diagrams are discussed in Sect. 3. Then one has three parton diagrams, where one virtual integration has to be done (Fig. 4). These diagrams only contribute to the two-jet cross section in case the outgoing gluon is soft or collinear with one of the quarks. This is discussed in Sect. 4. In Sect. 5 the renormalization is explained and the counterterms are derived.

Finally there are the four-parton diagrams which are tree level diagrams. They have to be integrated over those regions of phase space, where the four partons make up for two jets. In Sect. 6 and 8 we present two different approaches to handle them. In Sect. 7 and 9 we discuss the final results for the two approaches respectively.

To streamline the paper we have published many technical details in a separate paper [24] which we will often refer to.

2. Order α_s Two-Jet Cross Section

In order to elucidate the main steps in the higher order calculation we outline in this section the computation of the two-jet cross section in $O(\alpha_s)$ with invariant mass cut resolution. The relevant diagrams are shown in Fig. 2. The real diagrams yield the differential three-jet cross section to $O(\alpha_s)$ and contribute to the two-jet cross section only in case the gluon is collinear with one of the quarks or soft. (The region where the two quarks are collinear gives an $O(y)$ contribution, because there is no y_{12}-pole in the matrix-element (2.2).)

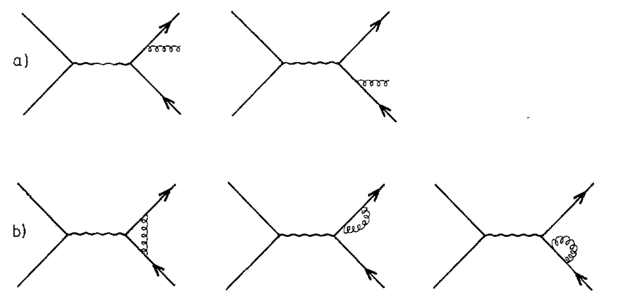

Fig. 2. a One gluon bremsstrahlung corrections to Fig. 1; **b** virtual one gluon corrections to Fig. 1

$$d^2\sigma_{q\bar{q}g}=\frac{\sigma^{(2)}}{\Gamma(1-\varepsilon)}\frac{\alpha_s(\mu^2)}{2\pi}C_F T(y_{13}, y_{23})\,dPS^{(3)}. \tag{2.1}$$

where

$$T(y_{13}, y_{23})=\left(\frac{y_{13}}{y_{23}}+\frac{y_{23}}{y_{13}}\right)(1-\varepsilon)+\frac{2y_{12}}{y_{13}y_{23}}-2\varepsilon. \tag{2.2}$$

Here $y_{12}=1-y_{13}-y_{23}$ from energy-momentum conservation and $\sigma^{(2)}=\sigma_0\left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon\frac{\Gamma(2-\varepsilon)}{\Gamma(2-2\varepsilon)}$ is the lowest order cross section in n dimensions. $C_F=4/3$ and μ is the up to now arbitrary mass parameter on which the coupling constant depends if one wants it to be dimensionless also in n dimensions.

$$dPS^{(3)}=\frac{q^2(4\pi/q^2)^{2\varepsilon}}{2^7\pi^3\Gamma(2-2\varepsilon)}\Theta(1-y_{13}-y_{23})\,y_{13}^{-\varepsilon}y_{23}^{-\varepsilon}\cdot(1-y_{13}-y_{23})^{-\varepsilon}dy_{13}\,dy_{23} \tag{2.3}$$

is the three-particle phase space element in n dimensions. To get (2.3) one has to integrate over the angles with respect to the e^+e^--beam direction.

In Fig. 6 we have marked the two-jet regions of this phase space. Integrating over the two strips one gets poles to up to second order in ε and powers of $\ln y$ up to second order. A typical integral is

$$\int_0^y dy_{13}\,y_{13}^{-\varepsilon}\int_0^y dy_{23}\,y_{23}^{-\varepsilon}\frac{1}{y_{13}y_{23}}=y^{-2\varepsilon}/\varepsilon^2 \tag{2.4}$$

The ε singularities are removed by adding the result of the virtual diagrams Fig. 2b and one can do the limit $\varepsilon\to 0$ and obtains [29]

$$\sigma_{2\text{-jet}}=\sigma_0\left[1+\frac{\alpha_s}{2\pi}C_F(-2\ln^2 y-3\ln y-1+\pi^2/3+O(y))\right]. \tag{2.5}$$

Table 1a–c. a $O(y)$ corrections to the integrated three-jet cross section in the region $\{y_{13}>y, y_{23}>y\}$ (see Fig. 6). **b** Integrated cross section in the region $\{y_{12}<y\}$ (see Fig. 7). **c** Leading terms of the three-jet cross section in order α_s (cf. (2.6)). The numbers in Table 1 are normalized to $\sigma_0 C_F \frac{\alpha_s}{2\pi}$. The error is less than 1 in the last decimal

y	**a** $O(y)_{3-\text{jet}}$	**b** $\sigma^{y_{12}<y}$	**c** $\sigma_{3-\text{jet}}$
0.05	0.640	0.212	8.172
0.02	0.331	0.186	18.082
0.01	0.194	0.119	27.807
0.005	0.111	0.043	39.454
0.002	0.052	0.021	57.809
0.001	0.029	0.012	73.921

Our aim is to extend this Sterman-Weinberg type formula to $O(\alpha_s^2)$. There one expects powers of fourth order in $\ln y$ and all powers of lower order including constants. Also one expects true nonabelian contributions ($\sim N_c$) and contributions from fermion loops which are proportional to the number of flavours $n_f=2T_R$. Before we continue to do this let us note that one can get the $O(y)$ correction to (2.5) easily by integrating over the three-jet region ($y_{13}>y$, $y_{23}>y$). Analytically one gets for the integrated three-jet cross section to order α_s [29]:

$$\sigma_{3\text{-jet}}=\sigma_0 \frac{\alpha_s}{2\pi} C_F\left(2 \ln^2 y+3 \ln y + \frac{5}{2}-\pi^2/3+O(y)\right). \tag{2.6}$$

There are no singularities in the three-jet region. So one can integrate numerically including terms of order y. Taking the difference with (2.6) one gets the numbers of Table 1a.

The *physical* two-jet cross region should also contain $y_{12}<y$. In this region the two quarks are in one jet and the other jet is a pure gluon jet. The contributions from this region are also order y, because there is no y_{12}-pole in (2.2). For convenience they are given in Table 1b.

In Fig. 9 we have drawn the Sterman-Weinberg formula including order y corrections for a wide range of y values (for $\alpha_s=0.12$). One sees that for $y\leqq 0.01$ the perturbative result is not useful, because the corrections to the tree level produce a change by more than a factor 2. For very small values of y one gets negative cross sections. For higher values of α_s the situation is even worse.

The higher order corrections in the "partial fractioned" and in the "physical" scheme will show a similar behaviour. They lead to negative cross sections already for $y\approx 0.01$. In the singular scheme (see Sect. 6) we shall find smaller corrections to the Sterman-Weinberg formula (see Fig. 10a and the discussion at the end of Sect. 8).

Note that by adding (2.5) and (2.6) one gets the total cross section to order α_s, which can be calculated independently from the imaginary part of the vacuum polarization of the photon via the optical theorem. All y dependence should drop out from the total cross section. So from Table 1 one can read off the $O(y)$ corrections to (2.5), too.

In the order α_s the structure of the $q\bar{q}g$ cross section (2.1) with (2.2) is such that all the terms proportional to $\ln y$ and the constant terms in (2.5) come from the integration of the pole terms in (2.2). The terms $O(y)$ which are neglected in (2.5) come from the strip $y_{12}<y$ and from taking into account exact kinematics. We see from Table 1 that the $O(y)$ terms are fairly small even for $y=0.05$ as compared to the dominant terms in $\sigma_{3\text{-jet}}$. They approach zero for $y\to 0$ whereas the dominant terms diverge.

In order α_s^2 we can proceed analogously. The total cross section σ_{tot} as calculated by the authors of [28] is obviously independent of y. In terms of jet cross sections we have

$$\sigma_{\text{tot}}=\sigma_{2\text{-jet}}+\sigma_{3\text{-jet}}+\sigma_{4\text{-jet}} \tag{2.7}$$

(since we go only up to $O(\alpha_s^2)$). Therefore it is an important check for these jet cross sections that the y-dependence in the sum cancels. So far this check could not be done since only three- and four-jet cross sections had been calculated. Actually to calculate $\sigma_{2\text{-jet}}$ including all $O(y)$ terms seems to be an unsurmountable task. Therefore we try to compute $\sigma_{2\text{-jet}}$ up to $O(1)$ terms. Then this $O(\alpha_s^2)$ $\sigma_{2\text{-jet}}$ is still a good check on the y-dependence of $\sigma_{3\text{-jet}}$ and $\sigma_{4\text{-jet}}$. The calculation of these cross sections will be presented in [20]. It turns out that they can be calculated with all the $O(y)$ correction terms although some parts only by numerical integration. Then after having checked that the $\ln^m y$ ($m=1, \ldots, 4$) terms cancel in the sum (2.7) and that for very small y the constant terms in the sum reproduce the known $O(\alpha_s^2)$ terms in σ_{tot} one can calculate the exact $\sigma_{2\text{-jet}}$ (with all $O(y)$ terms included) from the difference $\sigma_{2\text{-jet}}=\sigma_{\text{tot}}-\sigma_{3\text{-jet}}-\sigma_{4\text{-jet}}$ with $\sigma_{3\text{-jet}}$ and $\sigma_{4\text{-jet}}$ taken from [20].

3. The Two-Parton Diagrams

We want to calculate the contribution of the diagrams in Fig. 3. The techniques to calculate all the integrals needed have been presented in [24]. Also the results for all scalar diagrams are given there. So here it is enough to present the final results and to make some

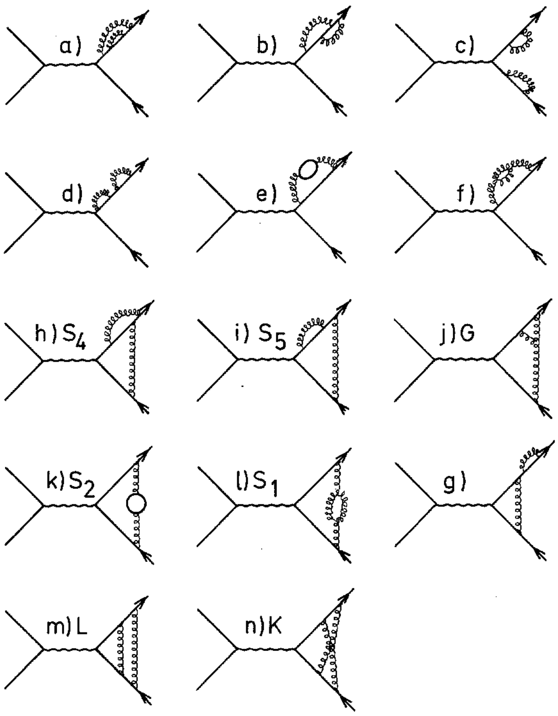

Fig. 3a–n. Virtual two gluon corrections to Fig. 1

general remarks: First the general form of the contribution of diagrams like Fig. 3 to the cross section is [29]

$$d\sigma_{e^+e^-\to q\bar{q}}=\frac{2\pi\alpha^2}{q^6}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{2\Gamma(2-2\varepsilon)}\cdot\left(-\frac{q^2}{3}g_{\mu\nu}H^{\mu\nu}_{e^+e^-\to q\bar{q}}\right) \tag{3.1}$$

where $H^{\mu\nu}$ is the hadronic tensor of those diagrams and $-\frac{q^2}{3}g_{\mu\nu}$ comes from averaging out initial state effects. Second the self energy insertions Fig. 3a–g can be consistently set to zero [9]⋆. This has already been done for the two self energy diagrams of Fig. 2b.

So we are left with the diagrams in Fig. 3h–n to be multiplied with Fig. 1. In addition there is the square of the vertex diagram in Fig. 2b, which will be called D. Including the lower orders we have

$$\sigma_{e^+e^-\to q\bar{q}}=\frac{\sigma^{(2)}}{1-\varepsilon}\left\{1-\varepsilon+C_F\frac{\alpha_s(\mu^2)}{2\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}A_1+C_F\left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2\left(\frac{4\pi\mu^2}{q^2}\right)^{2\varepsilon}A_2\right\}. \tag{3.2}$$

⋆ One can think of ultraviolet and infrared divergencies cancelling one another. One can calculate the ultraviolet divergence in principle by setting $p_i^2 \neq 0$

$$A_1=\frac{\Gamma^2(1-\varepsilon)\,\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left(-\frac{2}{\varepsilon^2}-\frac{1}{\varepsilon}+6\zeta_2-5+3\zeta_2\varepsilon-8\varepsilon\right) \tag{3.3}$$

is the lowest order result Fig. 2b (including terms of order ε which we shall need for renormalization later).

$$A_2=C_F D+N_c S_1+T_R S_2-2\left(C_F-\frac{N_c}{2}\right)S_4+2C_F S_5-N_c G+C_F L+\left(C_F-\frac{N_c}{2}\right)K. \tag{3.4}$$

$T_R=n_f/2$ comes from the diagram involving the fermion loops. Note that the q^2-dependence of all diagrams is universal (A_2 only depends on ε and the group parameters). The result for the single diagrams is

$$L=\mathrm{Re}\,(-1)^{-2\varepsilon}\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)/2\Gamma(1-3\varepsilon)\,\varepsilon^{-4}\cdot[1+\varepsilon+(\tfrac{13}{2}+2\zeta_2)\,\varepsilon^2+(\tfrac{67}{4}-2\zeta_2+26\zeta_3)\,\varepsilon^3+(\tfrac{429}{8}-9\zeta_2+100\zeta_3+45\zeta_4)\,\varepsilon^4]. \tag{3.5}$$

$\mathrm{Re}\,(-1)^{-2\varepsilon}$ comes from the fact that q^2 is timelike (see [24]).

$$K=\mathrm{Re}\,(-1)^{-2\varepsilon}\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)/2\Gamma(1-3\varepsilon)\,\varepsilon^{-4}\cdot[1+3\varepsilon+(12-6\zeta_2)\,\varepsilon^2+(42-6\zeta_2-30\zeta_3)\,\varepsilon^3+(146-30\zeta_2-54\zeta_3-67\zeta_4)\,\varepsilon^4]. \tag{3.6}$$

$$G=\mathrm{Re}\,(-1)^{-2\varepsilon}\frac{\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)}{2\varepsilon^4\Gamma(1-3\varepsilon)}\left[-\frac{1}{2}+\frac{\varepsilon}{2}+\frac{5}{2}\varepsilon^2+\left(\frac{63}{4}-\zeta_2\right)\varepsilon^3+\left(\frac{517}{8}-4\zeta_2-\zeta_3\right)\varepsilon^4\right]. \tag{3.7}$$

$$S_4=\mathrm{Re}\,(-1)^{-2\varepsilon}\frac{\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)}{2\varepsilon^3\Gamma(1-3\varepsilon)}[1+\left(\frac{9}{2}-2\zeta_2\right)\varepsilon+\left(\frac{87}{4}-7\zeta_2-2\zeta_3\right)\varepsilon^2+\left(\frac{693}{8}-31\zeta_2-7\zeta_3-9\zeta_4\right)\varepsilon^3]. \tag{3.8}$$

$$S_5=\mathrm{Re}\,(-1)^{-2\varepsilon}\frac{\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)}{2\varepsilon^3\Gamma(1-3\varepsilon)}\cdot\left[1+\frac{5}{2}\varepsilon+\frac{39}{4}\varepsilon^2+\frac{249}{8}\varepsilon^3\right]. \tag{3.9}$$

$$S_1=\mathrm{Re}(-1)^{-2\varepsilon}\frac{\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)}{2\varepsilon^3\Gamma(1-3\varepsilon)}\left[-\frac{5}{6}-\frac{61}{18}\varepsilon -\left(\frac{1530}{108}+\frac{5}{8}\zeta_2\right)\varepsilon^2-\left(\frac{31351}{648}+\frac{61}{9}\zeta_2+\frac{5}{3}\zeta_3\right)\varepsilon^3\right]. \tag{3.10}$$

$$S_2=\mathrm{Re}(-1)^{-2\varepsilon}\frac{\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)}{2\varepsilon^3\Gamma(1-3\varepsilon)}\cdot\left[\frac{2}{3}+\frac{22}{9}\varepsilon+\left(\frac{269}{27}+\frac{4}{3}\zeta_2\right)\varepsilon^2 +\left(\frac{5423}{162}+\frac{44}{9}\zeta_2+\frac{4}{3}\zeta_3\right)\varepsilon^3\right]. \tag{3.11}$$

$$D=\frac{\Gamma^4(1-\varepsilon)\,\Gamma^2(1+\varepsilon)}{4\varepsilon^4\Gamma^2(1-2\varepsilon)}\left[4+8\varepsilon+29\varepsilon^2+71\varepsilon^3+176\varepsilon^4\right]. \tag{3.12}$$

From this we find

$$A_2=\frac{\Gamma^3(1-\varepsilon)\,\Gamma(1+2\varepsilon)}{\Gamma(1-3\varepsilon)}\left\{C_F\left[\frac{2}{\varepsilon^4}+\frac{4}{\varepsilon^3}+\frac{29/2-12\zeta_2}{\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{281}{8}-21\zeta_2+6\zeta_3\right)+\frac{1413}{16}-\frac{151}{2}\zeta_2+42\zeta_3+67\zeta_4\right]\right.$$
$$+N_c\left[-\frac{11}{12\varepsilon^3}-\frac{1}{\varepsilon^2}\left(\frac{133}{36}-\frac{\zeta_2}{2}\right)-\frac{1}{\varepsilon}\left(\frac{3133}{216}-\frac{26}{3}\zeta_2-\frac{13}{2}\zeta_3\right)-\frac{4025}{81}+\frac{629}{18}\zeta_2+\frac{29}{3}\zeta_3-\frac{11}{4}\zeta_4\right]$$
$$\left.+T_R\left[\frac{1}{3\varepsilon^3}+\frac{11}{9\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{269}{54}-\frac{10}{3}\zeta_2\right)+\frac{5423}{324}-\frac{110}{9}\zeta_2+\frac{2}{3}\zeta_3\right]\right\}. \tag{3.13}$$

A few remarks concerning (3.13) are in order here. First one notes that the C_F-contribution carries the leading singularity $\sim\varepsilon^{-4}$. The leading singularities of the N_c and T_R-term are of order ε^{-3} and their coefficients are proportional to the zeroth order approximation of the β-function. This is not accidental. In fact the ε^{-4}, ε^{-3} and ε^{-2} contributions of our result can be generated using exponentiated expressions advocated by the authors of [25]. The coefficients of these poles are universal in the sense that knowing them for one process (e.g. ours) one can deduce them for other processes (e.g. deep inelastic scattering).

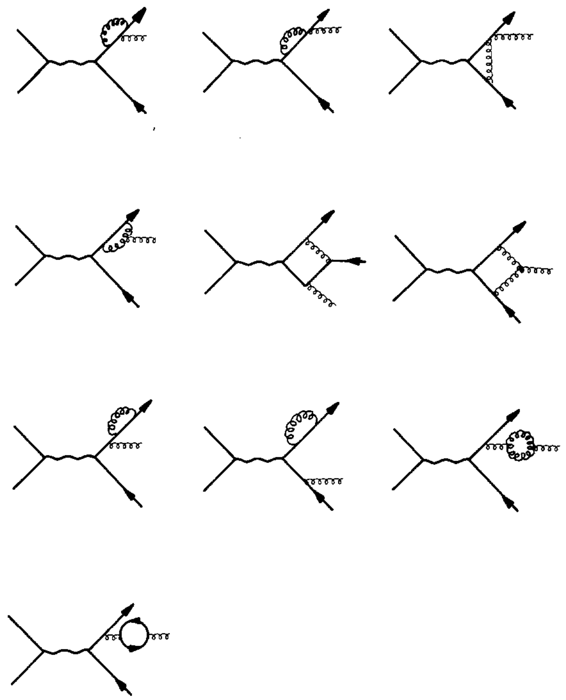

Fig. 4. Three parton contributions in order α_s^2

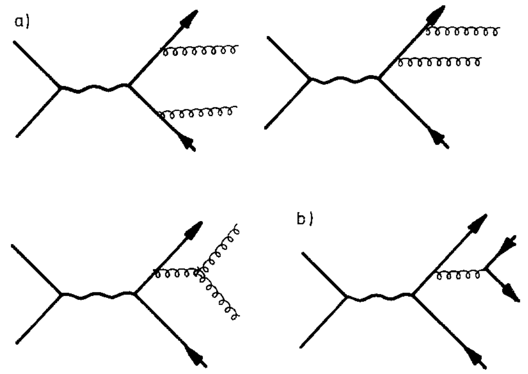

Fig. 5a, b. Four parton contributions in order α_s^2

Gonsalves [26] has published a result for A_2 which agrees with our (3.13) only in the leading singularity (i.e. ε^{-4} and ε^{-3} terms respectively). His result does not have the mentioned universality property [25]. In addition the cancellation of infrared singularities with the diagrams of Fig. 4 and 5 would not be possible with his result. With our result (3.13) we will be able to cancel all singularities to get a finite two-jet cross section.

We shall postpone the discussion of renormalization to Chapt. 5, after all virtual integrations have been done.

4. Three Parton Diagrams

Consider the one-loop corrections to the process

$$e^+(p_+)+e^-(p_-)\to q(p_1)+\bar{q}(p_2)+g(p_3) \qquad (4.1)$$

shown in Fig. 4. They have to be multiplied with the tree level contributions Fig. 2a*. This as well as the virtual loop integrations has been done by several groups [7–9, 16, 17], who have calculated differential three jet cross sections to $O(\alpha_s^2)$.

We are interested in integrated cross sections. So we have to do a phase space integration over the n-dimensional 3 particle phase space (2.3). This phase space is drawn in Fig. 6. We are not interested in the total cross section – we are interested in the separate two- and three-jet contributions of the three parton diagrams. So we have marked the two- and three-jet regions in Fig. 6. In fact Fig. 6 was already needed for the Sterman-Weinberg formula.

The result of the virtual integration of the diagrams in Fig. 4 is at most of order ε^{-2}. In the two-jet region the additional phase space integration can produce an additional ε^{-2} singularity. This means: we need the result of the virtual integrations including terms of order ε^2, whereas for calculating differential three-jet cross sections (thrust distributions etc.) terms up to $O(1)$ in ε were needed only. The exact basic loop integrals without ε expansion have been given in our earlier work [10], so that only the trace calculations had to be repeated. The following formulas give the result of the virtual integrations including terms of $O(\varepsilon^2)$

$$d\sigma_{e^+e^-\to q\bar{q}g}=\sigma^{(2)}\frac{\alpha_s(\mu^2)}{2\pi}\,C_F\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}dy_{13}\,dy_{23}$$
$$\cdot y_{13}^{-\varepsilon}y_{23}^{-\varepsilon}(1-y_{13}-y_{23})^{-\varepsilon}\Theta(1-y_{13}-y_{23})$$
$$\cdot T(y_{13},y_{23})/\Gamma(1-\varepsilon)$$
$$\cdot\left\{1+\frac{\Gamma^3(1-\varepsilon)\,\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}Re(-1)^{-\varepsilon}\right.$$
$$\left.\cdot\frac{\alpha_s(\mu^2)}{2\pi}\,C_F X\right\} \qquad (4.2)$$

where

$$X=\left(C_F-\frac{N_c}{2}\right)\left[\frac{-4+2A_1+2A_2}{\varepsilon^2}+\frac{1-2A_2-2A_1}{\varepsilon}\right.$$
$$\left.-5-8\varepsilon-16\varepsilon^2+2\frac{1-\varepsilon}{\varepsilon^2}(y_{12}^{-\varepsilon}+y_{13}^{-\varepsilon}+y_{23}^{-\varepsilon})\right]$$

* Once again diagrams with self energy insertions in outgoing parton legs can be consistently put to zero

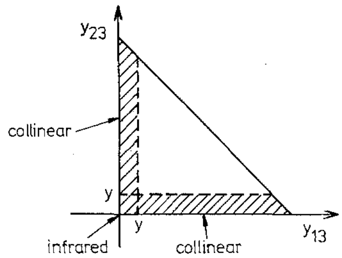

Fig. 6. Three particle phase space for $e^+e^-\to q\bar{q}g$. It is divided into a two-jet region which contains the singularities and a three-jet region free of singularities

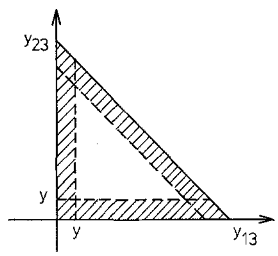

Fig. 7. Three particle phase space for $e^+e^-\to q\bar{q}g$. It is divided into the physical two-jet region and the physical three jet region

$$+\frac{N_c}{2}\left[\frac{2}{\varepsilon^2}(A_3-1)+\frac{1-2A_3}{\varepsilon}-5-8\varepsilon-16\varepsilon^2\right]$$
$$+T^{-1}(y_{13},y_{23})\left[\left(C_F-\frac{N_c}{2}\right)R_C+\frac{N_c}{2}R_N\right]. \qquad (4.3)$$

Here

$$A_1=\left(\frac{y_{12}\,y_{13}}{y_{12}\,y_{13}+y_{23}}\right)^{-\varepsilon}{}_2F_1\left(-\varepsilon,\,-\varepsilon,\,1-\varepsilon,\frac{y_{23}}{y_{12}\,y_{13}+y_{23}}\right)$$
$$-\left(\frac{y_{12}\,y_{13}}{y_{13}+y_{23}}\right)^{-\varepsilon}{}_2F_1\left(-\varepsilon,\,-\varepsilon,\,1-\varepsilon,\frac{y_{23}}{y_{13}+y_{23}}\right)$$
$$-\left(\frac{y_{12}\,y_{13}}{y_{12}+y_{23}}\right)^{-\varepsilon}{}_2F_1\left(-\varepsilon,\,-\varepsilon,\,1-\varepsilon,\frac{y_{23}}{y_{12}+y_{23}}\right)$$
$$A_2=A_1(1\leftrightarrow 2) \qquad (4.4)$$
$$A_3=A_1(1\to 3,\,3\to 2,\,2\to 1).$$

$$R_C=\frac{2}{\varepsilon^2}\left\{\left(2\frac{y_{12}}{y_{13}}+\frac{y_{23}}{y_{13}}\right)\right.$$
$$\left.\cdot\,(1-A_1-y_{12}^{-\varepsilon}-y_{13}^{-\varepsilon})+(1\leftrightarrow 2)\right\}$$
$$+\frac{1}{\varepsilon}\{8+N(-1,2,-4)+4(2+w)\,w(1-y_{12}^{-\varepsilon})\}$$

$$+4+2A_1+2A_2-4\frac{y_{13}}{y_{23}}(1-A_1)-4\frac{y_{23}}{y_{13}}(1-A_2)$$
$$+8w^2+12w+N(-1,5,-3)$$
$$+4y_{12}^{-\varepsilon}\left(1+\frac{y_{13}}{y_{23}}+\frac{y_{23}}{y_{13}}-2w^2-2w\right)$$
$$+y_{13}^{-\varepsilon}\left(-1-\frac{y_{12}}{y_{13}}-\frac{y_{23}}{y_{13}}+4\frac{y_{13}}{y_{23}}-\frac{y_{23}}{y_{12}+y_{23}}\right)$$
$$+y_{23}^{-\varepsilon}\left(-1-\frac{y_{12}}{y_{23}}-\frac{y_{13}}{y_{23}}+4\frac{y_{23}}{y_{13}}-\frac{y_{13}}{y_{12}+y_{13}}\right)$$
$$+\varepsilon\left\{4+2A_1+2A_2-2\frac{y_{13}}{y_{23}}(1-A_1)\right.$$
$$-2\frac{y_{23}}{y_{13}}(1-A_2)+N(-2,5,-5)+16w^2+24w$$
$$+2y_{12}^{-\varepsilon}\left(\frac{y_{13}}{y_{23}}+\frac{y_{23}}{y_{13}}+2\right)$$
$$+y_{13}^{-\varepsilon}\left(2\frac{y_{13}}{y_{23}}-\frac{y_{12}}{y_{13}}-\frac{y_{23}}{y_{12}+y_{23}}\right)$$
$$\left.+y_{23}^{-\varepsilon}\left(2\frac{y_{23}}{y_{13}}-\frac{y_{12}}{y_{23}}-\frac{y_{13}}{y_{12}+y_{13}}\right)\right\}$$
$$+\varepsilon^2\left\{16+N(-4,8,-12)\right.$$
$$+32w^2(1-y_{12}^{-\varepsilon})+(48-32y_{12}^{-\varepsilon})w$$
$$+y_{13}^{-\varepsilon}\left(-5-2\frac{y_{12}}{y_{13}}-\frac{y_{23}}{y_{13}}-2\frac{y_{23}}{y_{12}+y_{23}}\right)$$
$$\left.+y_{23}^{-\varepsilon}\left(-5-2\frac{y_{12}}{y_{23}}-\frac{y_{13}}{y_{23}}-2\frac{y_{13}}{y_{12}+y_{13}}\right)\right\}. \tag{4.5}$$

$$R_N=\frac{1}{\varepsilon}\{6A_3+2(1+y_{13}^{-\varepsilon}+y_{23}^{-\varepsilon})+N(-1,4,-4)\}$$
$$+2+2A_3+y_{13}^{-\varepsilon}\left(1+\frac{y_{12}}{y_{13}}+\frac{y_{23}}{y_{13}}-\frac{y_{23}}{y_{12}+y_{23}}\right)$$
$$+N(-1,5,-3)+y_{23}^{-\varepsilon}\left(1+\frac{y_{12}}{y_{23}}+\frac{y_{13}}{y_{23}}\right.$$
$$\left.-\frac{y_{13}}{y_{12}+y_{13}}\right)+\varepsilon\left\{4+N(-2,7,-5)\right.$$
$$+y_{13}^{-\varepsilon}\left(-2+\frac{y_{12}}{y_{13}}-\frac{y_{23}}{y_{12}+y_{23}}\right)$$
$$\left.+y_{23}^{-\varepsilon}\left(-2+\frac{y_{12}}{y_{23}}-\frac{y_{13}}{y_{12}+y_{13}}\right)\right\}$$
$$+\varepsilon^2\left\{8+N(-4,12,-12)\right.$$
$$+y_{13}^{-\varepsilon}\left(-3+2\frac{y_{12}}{y_{13}}-2\frac{y_{23}}{y_{12}+y_{23}}\right)$$
$$\left.+y_{23}^{-\varepsilon}\left(-3+2\frac{y_{12}}{y_{23}}-2\frac{y_{13}}{y_{12}+y_{13}}\right)\right\}. \tag{4.6}$$

$$w:=\frac{y_{12}}{y_{13}+y_{23}}. \tag{4.7}$$

$$N(a,b,c)=(1-y_{13})^{-\varepsilon}\left[\frac{a\,y_{13}\,y_{23}}{(y_{12}+y_{23})^2}+\frac{b\,y_{13}}{y_{12}+y_{23}}\right.$$
$$\left.+\frac{c\,y_{23}}{y_{12}+y_{23}}\right]+(1\leftrightarrow 2). \tag{4.8}$$

First we want to integrate this over the two jet region which can be written as

$$R_{2\text{-jet}}=\int_0^y \mathrm{d}y_{13}\left\{2\int_0^1 \mathrm{d}y_{23}-\int_0^y \mathrm{d}y_{23}\right\} \tag{4.9}$$

because of symmetry properties. (4.9) is valid only to order 1 in y. Working in this approximation one has to look for poles in y_{13} in (4.2–8).

Most of the integrals are elementary. The terms with denominator $(y_{13}+y_{23})^{-1}$ can be integrated by splitting the y_{23}-integration into

$$\int_0^y \mathrm{d}y_{23}=\int_0^{y_{13}} \mathrm{d}y_{23}+\int_{y_{13}}^y \mathrm{d}y_{23}. \tag{4.10}$$

The only difficulty lies in integrating the hypergeometric functions in (4.5). They originate from the box diagrams in Fig. 4. In Sect. 3 of [24] we have shown how to treat such hypergeometric functions by integrating the scalar box diagram. The box diagrams of Fig. 4 can be reduced to scalar box integrals and certain vertex integrals already at the virtual level [7]. This is why the hypergeometric functions in (4.2) show up only in the combination (4.4) which is typical for the scalar box diagram [24].

What is needed of (4.4) for our integration neglecting terms $O(y)$ is collected in Table 2. The integrations

Table 2. Approximate formulas for certain hypergeometric functions as explained in the text

$y_{13}<y,\ y_{23}>y$
$A_1\approx -y_{13}^{-\varepsilon}(1-y_{23})^{-\varepsilon}{}_2F_1(-\varepsilon,-\varepsilon,1-\varepsilon,y_{23})$
$A_2\approx 1-(1-y_{23})^{-\varepsilon}-y_{23}^{-\varepsilon}$
$A_3\approx -y_{13}^{-\varepsilon}y_{23}^{-\varepsilon}{}_2F_1(-\varepsilon,-\varepsilon,1-\varepsilon,1-y_{23})$
$y_{13}<y,\ y_{23}<y$
A_1, A_3 as above
$A_2\approx -y_{23}^{-\varepsilon}$

can then be done straightforwardly after expanding the hypergeometric function ${}_2F_1(-\varepsilon, -\varepsilon, 1-\varepsilon, \cdot)$. The final result for the two-jet contribution of the three-parton diagrams is:

$$\sigma_{2\text{-jet}}(q\bar{q}g)=\frac{\sigma^{(2)}}{1-\varepsilon} C_F \frac{\alpha_s(\mu^2)}{2\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon \cdot\left\{B_1+\frac{\alpha_s(\mu^2)}{2\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon B_2\right\}. \tag{4.11}$$

$$B_1=\frac{\Gamma^2(1-\varepsilon)\,\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left\{\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}+4-4\zeta_2 -3\ln y-2\ln^2 y+\varepsilon\left(7-8\zeta_3+\zeta_2 +(4\zeta_2-4)\ln y+\frac{7}{2}\ln^2 y+2\ln^3 y\right)\right\}. \tag{4.12}$$

B_1 is the order α_s contribution which we have included for convenience.

$$\begin{aligned}B_2=\frac{\Gamma(1-\varepsilon)\,\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot &\left\{C_F\left[-\frac{4}{\varepsilon^4}-\frac{8}{\varepsilon^3}-\frac{1}{\varepsilon^2}\left(\frac{59}{2}-24\zeta_2\right)\right.\right.\\
&-\frac{1}{\varepsilon}\left(\frac{147}{2}-30\zeta_2-36\zeta_3\right)-\frac{598}{3}+128\zeta_2-35\zeta_4\\
&+\frac{6}{\varepsilon^2}\ln y+\frac{1}{\varepsilon}(22-16\zeta_2)\ln y+\frac{4}{\varepsilon^2}\ln^2 y\\
&-\frac{1}{\varepsilon}\ln^2 y-\frac{4}{\varepsilon}\ln^3 y+(65-20\zeta_2-56\zeta_3)\ln y\\
&\left.-\frac{7}{2}\ln^2 y-\ln^3 y+\frac{7}{3}\ln^4 y\right]\\
&+N_c\left[-\frac{1}{2\varepsilon^4}-\frac{1}{\varepsilon^3}-\frac{1}{\varepsilon^2}\left(\frac{5}{2}-\zeta_2\right)\right.\\
&-\frac{1}{\varepsilon}(6-5\zeta_2+8\zeta_3)-\frac{55}{12}+10\zeta_2+11\zeta_3-21\zeta_4\\
&+\frac{3}{\varepsilon^2}\ln y+\frac{5}{\varepsilon}\ln y+\frac{2}{\varepsilon^2}\ln^2 y-\frac{5}{\varepsilon}\ln^2 y\\
&-\frac{4}{\varepsilon}\ln^3 y+(12-12\zeta_2+16\zeta_3)\ln y\\
&\left.\left.-(5+4\zeta_2)\ln^2 y+6\ln^3 y+\frac{14}{3}\ln^4 y\right]\right\}.\end{aligned} \tag{4.13}$$

Let us also discuss the contribution of the three-parton diagrams to the integrated three-jet cross section. For one this can give us a check on all the logarithms in (4.11), since in the sum of the integrated two- and three-jet contributions all y dependence should cancel. This way we get also the contribution of the three-parton diagrams to the total cross section which is perhaps interesting by itself.

In the three-jet region of Fig. 6 all the phase space integrals are finite. This means that we can take the result from [7, 9] without the terms $\sim\varepsilon$ instead of (4.2) as integrand. Just as at the end of Sect. 2 one can do the integrations both analytically (neglecting $O(y)$ contributions) and numerically. For the analytical integration we rewrite the result in [7, 9] as

$$\begin{aligned}S_{\text{virtual}}=\int_y^1 dy_{13}\int_y^{1-y_{13}} dy_{23}&\left\{T(y_{13},y_{23})\,(y_{13}\,y_{23}\,y_{12})^{-\varepsilon}\right.\\
&\cdot\left[-2C_F\,y_{12}^{-\varepsilon}/\varepsilon^2-N_c(y_{12}/y_{13}\,y_{23})^\varepsilon/\varepsilon^2-3C_F/\varepsilon\right]\\
&+T(y_{13},y_{23})\left(4\zeta_2\left(C_F+\frac{N_c}{2}\right)-8C_F+N_c(\ln^2 y_{12}\right.\\
&\left.-\ln y_{12}\ln(y_{13}\,y_{23})+\ln y_{13}\ln y_{23})\right)\\
&\left.+F(y_{13},y_{23})\right\}.\end{aligned} \tag{4.14}$$

The function F which is independent of ε is defined in [7, 9].

For the terms in (4.14) containing ε-poles one has to keep $\varepsilon\neq 0$. Despite this they lead to elementary integrals. The other terms which can be integrated in 4 dimensions, can all be taken from standard tables [27]. The final result is

$$\begin{aligned}\sigma_{3\text{-jet}}(q\bar{q}g)=&\frac{\sigma^{(2)}}{1-\varepsilon}\frac{\alpha_s(\mu^2)}{2\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon C_F\\
&\cdot\left\{2\ln^2 y+3\ln y+\frac{5}{2}-2\zeta_2+\frac{\Gamma^2(1-\varepsilon)\,\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\right.\\
&\left.\cdot\left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon Re(-1)^{-\varepsilon}\frac{\alpha_s(\mu^2)}{2\pi}\Delta^{(3)}\right\}.\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}\Delta^{(3)}=&-\left(\frac{14}{3}N_c+\frac{7}{3}C_F\right)\ln^4 y+\left(C_F-6N_c\right.\\
&\left.+\frac{4}{\varepsilon}(C_F+N_c)\right)\ln^3 y+\left[\left(\frac{7}{2}-12\zeta_2\right)C_F+(5-2\zeta_2)N_c\right.\\
&\left.+\frac{5N_c+C_F}{\varepsilon}-\frac{4C_F+2N_c}{\varepsilon^2}\right]\ln^2 y\\
&+\left[(2\zeta_2+56\zeta_3-65)C_F+(3\zeta_2-16\zeta_3-12)N_c\right.\end{aligned}$$

$$-\frac{1}{\varepsilon}(5N_c+(22-16\zeta_2)C_F)-\frac{1}{\varepsilon^2}(3N_c+6C_F)\Big]\ln y$$
$$+\frac{1}{\varepsilon^2}\left[\left(2\zeta_2-\frac{5}{2}\right)N_c+(4\zeta_2-5)C_F\right]$$
$$+\frac{1}{\varepsilon}\left[\left(4\zeta_2+8\zeta_3-\frac{29}{2}\right)N_c\right.$$
$$\left.+(20\zeta_2+24\zeta_3-38)C_F\right]$$
$$+N_c\left(\frac{51}{2}\zeta_2+4\zeta_3+19\zeta_4-75\right)$$
$$+C_F(81\zeta_2+90\zeta_3+31\zeta_4-195). \tag{4.16}$$

One can see easily that in the sum (4.11)+(4.15) all logarithms drop out and one is left with the contribution of the three-parton diagrams to the total cross section:

$$\sigma_{\rm tot}(q\bar{q}g)=\frac{\sigma^{(2)}}{1-\varepsilon}\frac{\alpha_s(\mu^2)}{2\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}C_F$$
$$\cdot\left\{\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon^2}+\frac{13}{2}-4\zeta_2\right)\right.$$
$$+\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}Re(-1)^{-\varepsilon}\frac{\alpha_s(\mu^2)}{2\pi}$$
$$\cdot\left[-\frac{1}{\varepsilon^4}\left(4C_F+\frac{N_c}{2}\right)-\frac{1}{\varepsilon^3}(8C_F+N_c)\right.$$
$$-\frac{1}{\varepsilon^2}\left(\left(\frac{69}{2}-16\zeta_2\right)C_F+\left(5-\frac{3}{2}\zeta_2\right)N_c\right)$$
$$+\frac{1}{\varepsilon}\left(\left(26\zeta_2+60\zeta_3-\frac{223}{2}\right)C_F+\left(6\zeta_2-\frac{41}{2}\right)N_c\right)$$
$$+\left(\frac{241}{2}\zeta_2+90\zeta_3+101\zeta_4-\frac{1183}{3}\right)C_F$$
$$\left.\left.+\left(28\zeta_2+15\zeta_3-\frac{31}{8}\zeta_4-\frac{955}{12}\right)N_c\right]\right\}. \tag{4.17}$$

Now we come to the numerical analysis of (4.14) in the three-jet region. We report only the $O(y)$ corrections to the various terms in (4.16) which we write as

$$\Delta^{(3)}(\text{order }y)=\left(-\frac{2}{\varepsilon^2}Q+Q_1^C/\varepsilon+f^C\right)C_F$$
$$+(-Q/\varepsilon^2+Q_1^N/\varepsilon+f^N)N_c \tag{4.18}$$

Q can be obtained from Table 1 since it is the result of the integral over $T_v=T|_{\varepsilon=0}$. The other numerical results are found in Table 3.

Table 3. $O(y)$ corrections to the three-jet contributions of the diagrams in Fig. 4
Order y corrections for the three parton diagrams in the three-jet region = −order y corrections for the three parton diagrams in the two jet region

y	Q_1^C	Q_1^N	f^C	f^N
0.05	−6.1886	−6.4066	−13.260	−40.843
0.02	−3.891	−4.202	− 9.728	−34.101
0.01	−2.612	−2.854	− 7.537	−27.65
0.005	−1.695	−1.681	− 6.05	−21.60
0.002	−0.92	−1.01	− 4.52	−15.17
0.001	−0.59	−0.62	− 3.40	−11.5

The errors are one unit in the last digit.
All the contributions go to 0 for $y\to 0$

Now we meet a surprise. In contrast to $O(\alpha_s)$ the $O(y)$ corrections are rather large here, especially for the N_c-term (~30% at $y=0.05$). Therefore we should be prepared that the $O(\alpha_s^2)$ contributions to the Sterman-Weinberg formula (2.5) which we are going to derive will have non-negligible corrections for larger y's, $y=0.05$ say, caused by $O(y)$ terms.

5. Renormalization

We have now calculated all virtual corrections of $e^+e^-\to$ QCD-quanta to order α_s^2. So it is reasonable to do the renormalization.

Renormalization commutes with all the phase space integrations. So it is similar for differential and for integrated cross sections. (We shall concentrate on the two-jet case.)

Because of chiral invariance our massless theory gets no mass renormalization. So our final expression (3.2)+(4.11) is renormalized by using the renormalized coupling g_R instead of the naked coupling g_b:

$$g_R=\sqrt{Z_3}\,Z_2/Z_1\,g_b. \tag{5.1}$$

In (5.1) Z_1 is the renormalization constant for the quark-antiquark-gluon vertex, Z_2 for the quark field and Z_3 for the gluon field. We need these quantities only to order α_s, because the $O(\alpha_s)$ Sterman-Weinberg formula gets no contribution from renormalization.

Many schemes have been discussed on what to absorb into the coupling constant (5.1) [26]. We worked in the $\overline{\rm MS}$ scheme which is most standard, technically the most convenient and which also seems to lead to the smallest higher order corrections in most cases*. It is defined as follows:

* For a discussion see [29]

One absorbs only terms proportional to $\varepsilon^{-1}+\ln 4\pi-\gamma$ into the coupling constant, i.e. terms which are proportional to the pure ultraviolet singularity plus certain constants which are an artefact of dimensional regularisation. These constants compensate for the factor $(4\pi)^\varepsilon$ and certain Γ-functions in (3.2) and (4.11). The ultraviolet singularity of (3.2)+(4.11) is mixed up with the infrared singularities in our procedure. However, from the structure of the renormalization constants one can deduce that it is only a weak singularity ($\sim\varepsilon^{-1}$) compared to the infrared singularities. This is the reason why no $\ln^2(4\pi)$ etc. terms enter the $\overline{\text{MS}}$ prescription.

In fact one can produce the counterterm to (3.2)+(4.11) by inserting

$$\alpha_{sb}=\alpha_{sR}\left\{1-\left(\frac{11}{6}N_c-\frac{2}{3}T_R\right)\frac{\alpha_{sR}}{2\pi}\left(\frac{1}{\varepsilon}+\ln 4\pi-\gamma\right)\right\} \tag{5.2}$$

into the Sterman-Weinberg formula (2.5) keeping the terms of order α_s^2. Here $\frac{11}{6}N_c-\frac{2}{3}T_R$ is the first order approximation to the β-function. As (2.5) is finite one sees at once that the counterterm is of order ε^{-1}.

One should note that because of the singularity $\sim\varepsilon^{-1}$ one needs the Sterman-Weinberg formula including terms of order ε. These can be calculated by the same methods by which one arrives at (2.5). We have given these order ε corrections to A_1 and B_1 already in (4.12) and (3.3).

Using (5.2) one gets for the counterterm

$$\begin{aligned}\sigma_{2\text{-jet}}^{CT}=&-\frac{\sigma^{(2)}}{1-\varepsilon}C_F\left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2\left(\frac{4\pi\mu^2}{q^2}\right)^{2\varepsilon}\\&\cdot\left(\frac{11}{6}N_c-\frac{2}{3}T_R\right)\left(\frac{1}{\varepsilon}-\gamma\right)[2\varepsilon\ln^3 y\\&+\left(\frac{7}{2}\varepsilon-2\right)\ln^2 y+(4\varepsilon(\zeta_2-1)-3)\ln y\\&+(4\zeta_2-4\zeta_3-1)\varepsilon+2\zeta_2-1].\end{aligned} \tag{5.3}$$

If one adds the counterterm (5.3) to the result (3.2) +(4.11) obtained so far one knows that the sum is free of ultraviolet singularities. All remaining singularities must be infrared and collinear ones and they must be cancelled by the two-jet contributions of the diagrams in Fig. 5, to which we turn our attention now.

6. Four Parton Diagrams: The Singular Approach

In this section and the following sections we describe the integration of the four-parton diagrams (Fig. 5) over such regions of phase space corresponding to unresolved two- and three-parton configurations. Based on our experience with $e^+e^-\to 3$ jets [9, 12, 13, 17] we approximate the four-parton matrix elements by their singular contributions which are responsible for the infrared/collinear singular terms $\sim\varepsilon^{-n}$ ($n=1-4$). The motivation for proceeding in this way is that the four-parton matrix elements involve so many terms of very complicated structure that we want to prove the cancellation of all singularities $\sim\varepsilon^{-n}$ first before we try to evaluate the exact matrix elements. We also might hope that the result obtained with the singular terms already gives us a reasonable good approximation to the final result. However, this is not the case, cf. Sect. 9. The diagrams of Fig. 5 stand for the processes

$$e^+(p_+)+e^-(p_-)\to q(p_1)+\bar{q}(p_2)+g(p_3)+g(p_4). \tag{6.1}$$

and

$$e^+(p_+)+e^-(p_-)\to q(p_1)+\bar{q}(p_2)+q(p_3)+\bar{q}(p_4). \tag{6.2}$$

The 4-dimensional matrix elements for these processes can be found in several papers [19, 7]. They are finite in the four jet region (all $y_{ij}>y$), so for four jet cross sections the 4 dimensional expressions are sufficient. For two- and three-jet cross sections one has to extend some of the terms to n dimensions, namely those which yield singularities when integrated over the two- and three-jet regions. In the three-jet case these n-dimensional corrections are very simple. There the singular terms factorize into a tree level type amplitude (Fig. 2a) and an Altarelli-Parisi kernel, the generalization of both factors to n dimensions being very well known [7]. In the two-jet limit there are other n-dimensional corrections which we shall present in the following.

The integration of the four-parton diagrams over the two-jet region is rather involved. One has two topologically different kinematical configurations depending on whether three partons are collinear (e.g. $y_{134}<y$), the second jet then consists just of one parton, or whether both jets are made of two partons (e.g. $y_{13}<y$ and $y_{24}<y$). Neglecting order y terms the second configuration can be neglected for most of the diagrams (e.g. for the $q\bar{q}q\bar{q}$-diagrams).

The techniques of how to do the phase space integrations are discussed in our paper [24]. However, the difficulty comes in not only through the phase space integration. In addition one has to be careful to integrate over any region of four-particle phase space at most once. For example if one has to integrate over the two regions $y_{134}<y$ and $y_{234}<y$ one cannot simply add the two results but has to subtract the overlap contribution. This problem already ap-

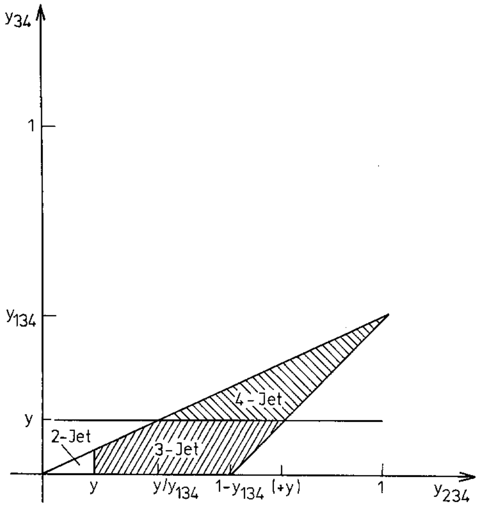

Fig. 8. Four particle phase space in a 34-system for fixed y_{134}. It is divided into the regions $\{y_{134}<y\}$, $\{y_{34}<y, y_{134}>y\}$ and $\{y_{34}>y\}$

pears at order α_s (Fig. 6) where it is easily manageable because only one region of overlap exists. In our case however there are four two-jet configurations* and lots of overlaps exist. In this section we shall take care of all these overlap regions. However, we will only calculate the singular terms of the two-jet contributions of the four-parton diagrams proving the infrared cancellations and including only those finite logarithms and constants that arise out of these singularities. Thus, for example, $-2\ln y$ arises from $y^{-2\varepsilon}/\varepsilon=-2\int_0^y dy_{134}\, y_{134}^{-1-2\varepsilon}$, where the singularity y_{134}^{-1} comes from the matrix elements. We are not sure whether there are constants or even logarithms in the nonsingular parts of the matrix elements. To examine this question would be rather cumbersome because of the appearance of terms of order y/ε. These make it impossible to integrate the nonsingular parts of the matrix elements numerically.

To avoid overlap and $O(y/\varepsilon)$-terms in the three-jet case the authors of [7] have used partial fractioning of the matrix elements. With this method they have calculated inclusive cross sections where all 3- and 4-parton contributions are summed. In [20] we have carried through this approach also for the integrated three-jet cross section. In order to check all these calculations it would be good to have an independent calculation of the two-jet cross section in this approach to see whether the total cross section can be reconstructed. Such a calculation will be done in Sect. 8.

But let us return to the singular approach. As long as one is interested in the case that (at least) parton 1 and 3 are going in one jet, the following formula for the n-dimensional four-particle phase is most convenient [7]

$$\begin{aligned} dPS^{(4)} &= g_0\, dy_{123}\, dy_{134}\, dy_{13} (y_{123}\, y_{134}-y_{13})^{-\varepsilon} y_{13}^{-\varepsilon} \\ &\cdot (y_{13}+1-y_{123}-y_{134})^{-\varepsilon}\, \Theta(y_{13})\, \Theta(y_{123}\, y_{134}-y_{13}) \\ &\cdot \Theta(y_{13}+1-y_{123}-y_{134})\, dv\, v^{-\varepsilon}(1-v)^{-\varepsilon} d\theta' \\ &\cdot \sin^{-2\varepsilon}\theta' \end{aligned} \tag{6.3a}$$

$$g_0=\left(\frac{4\pi}{q^2}\right)^{3\varepsilon} \frac{S q^4 \Gamma(1-\varepsilon)}{2^{11+2\varepsilon}\pi^6 \Gamma(1-2\varepsilon)\Gamma(2-2\varepsilon)}. \tag{6.3b}$$

Here S is a statistical factor, $S=2$ for (6.1), $S=4$, for (6.2). $\mathbf{p}_2$ has been put along the z-axis here and $\mathbf{p}_1+\mathbf{p}_3=0$. The phase space can almost fully be described by invariants $(y_{ijk}:=y_{ij}+y_{jk}+y_{ik})$ with the exception of the azimuthal angle θ' of p_1 and the variable v which contains its polar angle θ with the z-axis, $v:=\frac{1}{2}(1-\cos\theta)$ $(0<\theta'<\pi,\ 0<\theta<\pi)$.

Exchanging the roles of the partons one can derive other phase space formulas from (6.3) which are useful if other jet configurations are considered.

The combination $y_{13}+1-y_{123}-y_{134}$ which appears in (6.3) is just y_{24}, as momentum conservation shows. This means that (6.3) is not only suited to implement the two-jet condition $(y_{134}<y)$ but also the condition $(y_{24}<y,\ y_{13}<y)$.

We remind the reader how one treats (6.3) in the three-jet limit ($y_{13}<y$, all other $y_{ij}>y$). To get differential distributions one can interpret y_{123} and y_{134} as effective three-particle variables of an effective three-particle phase space (I=1+3, II=2, III=4). Then one only has to integrate over v, θ' and y_{13}. If one neglects order y contributions one can restrict oneself to the poles $\sim y_{13}^{-1}$ in the transition probabilities. Then one has to consider only a few terms, because all the invariants reduce very much in this limit, e.g.

$$\begin{aligned} y_{14} &= (y_{134}-y_{13})\,(v(1-\gamma)+\gamma(1-v) \\ &\quad -2\cos\theta' \sqrt{v(1-v)\,\gamma(1-\gamma)}), \end{aligned} \tag{6.4}$$

where $\gamma:=y_{13}\, y_{24}/(y_{134}-y_{13})\,(y_{123}-y_{13})$, reduces to

$$y_{14}\cong y_{\mathrm{I\,III}}\cdot v. \tag{6.5}$$

We find a strong reduction of the matrix elements also in the two-jet case $(y_{13}<y,\ y_{24}<y)$ where only terms $\sim y_{13}^{-1} y_{24}^{-1}$ are important, if one neglects order y contributions. Such terms only come from the plan-

* $\{y_{134}<y\}$
$\cup\{y_{234}<y\}$
$\cup\{y_{13}<y,\ y_{24}<y\}\cup\{y_{23}<y,\ y_{14}<y\}$

This can be read off directly from the singularities of the propagators of the 4-parton diagrams ar and nonplanar QED type contributions* and only in the C_F^2 contributions and are of the form

$$M(y_{13}\to 0, y_{24}\to 0)=\frac{C_F}{y_{13}\,y_{24}}\,P^n_{qq}(v)\,P^n_{qq}(1-y_{134}), \quad (6.6)$$

where P^n_{qq} is the generalization of the Altarelli-Parisi function to n dimensions

$$P^n_{qq}(v)=2\frac{v}{1-v}+(1-\varepsilon)(1-v). \quad (6.6a)$$

In (6.6) and in the following we leave out the factor $\left(\frac{\alpha_s}{2\pi}\right)^2 C_F$.

(6.6) has to be integrated over that part of (6.3) where $y_{13}<y$, $y_{24}<y$, $y_{134}>y$. (We will consider $y_{134}<y$ later.) For $y_{13}\to 0$ and $y_{24}\to 0$ (6.3) reduces very much so that the integration is easy. One has

$$PS^{(4)}(y_{13}<y, y_{24}<y, y_{134}>y)=g_0\int_0^y dy_{13}\,y_{13}^{-\varepsilon}\cdot\int_0^y dy_{24}\,y_{24}^{-\varepsilon}\int_y^1 dy_{134}\,y_{134}^{-\varepsilon}(1-y_{134})^{-\varepsilon}\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}. \quad (6.7)$$

In contrast to this and to the three-jet case configurations with three-partons in one jet are more involved. One reason is that a formula like (6.4) is still quite complicated in the region $y_{134}<y$. (In the region $y_{234}<y$ it is appropriate to choose another coordinate system.) In this region one has to look for double poles ($\sim y_{134}^{-2}$) in the transition probabilities. One can work with the approximation

$$\lim_{y_{134}\to 0}\frac{y_{14}}{y_{134}}=v(1-z)+z\,y_{24}(1-v)-2\cos\theta'\sqrt{v(1-v)\,z(1-z)\,y_{24}} \quad (6.8)$$

In our technical paper [24] we have described how to integrate a term with y_{14} in the denominator, because this is a rather characteristic case. This case occurs for four types of contributions

(A) The planar QED type contributions of Fig. 5a ($\sim C_F$)

(B) the nonplanar QED type contributions of Fig. 5a $\left(\sim C_F-\frac{N_c}{2}\right)$

(C) the interferences of QED diagrams with diagrams involving the three gluon coupling in Fig. 5a ($\sim N_c/2$)

(D) the nonplanar contributions of Fig. 5b.

* We use the same classification of transition probabilities as in [7]

All other contributions contain at most poles in y_{234} and y_{34} besides y_{134}-poles. Instead of using the "13-system" which leads to (6.3) in this case it is appropriate to use a "34-system" for them, which one can get from the "13-system" by interchanging partons 1 and 4. (This strategy is useless for the y_{14}-poles, because they appear in combination with y_{13}, y_{23}, y_{24} or y_{34}.)

Let us begin with these simpler contributions: First there are planar contributions of Fig. 5b which pick up a factor $T_R=n_f/2$ because of the additional fermion loop involved. In the limit $y_{134}\to 0$ they reduce to

$$M_T(y_{134}\to 0)=\frac{2}{3}\,T_R\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)\,y_{134}^2}\cdot\left\{\frac{1}{z}\left(1-\frac{7}{3}\varepsilon+\frac{13}{9}\varepsilon^2\right)-1+\frac{7}{3}\varepsilon+\frac{2-\frac{8}{3}\varepsilon+\frac{2}{9}\varepsilon^2+\frac{4}{27}\varepsilon^3}{y_{234}^2\,z}+\frac{-2+\frac{8}{3}\varepsilon-\frac{2}{9}\varepsilon^2}{y_{234}\,z}\right\}. \quad (6.9)$$

(6.9) has been written down in a 34-system. The θ'-integration, which is trivial here, and the v-integration have already been carried out. A variable $z=y_{34}/y_{134}\,y_{234}$ instead of y_{34} can be introduced as integration variable, which has the advantage that it is integrated between 0 and 1.

(6.9) should be integrated with

$$R_{34}=\int_0^y dy_{134}\left\{2\int_0^1 dy_{234}-\int_0^y dy_{234}\right\}\cdot y_{134}^{1-2\varepsilon}y_{234}^{1-2\varepsilon}(1-y_{234})^{-\varepsilon}\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}. \quad (6.10)$$

In (6.10) the factor 2 is the contribution from the region ($y_{234}<y$) and the contribution from the overlap region ($y_{134}<y$, $y_{234}<y$) is subtracted in order that it be not counted twice.

(6.9) can be integrated with the measure (6.10) quite easily. So it is not necessary to give the result here. It will be included as the T_R-term in (6.25). We only note in passing that it contains poles of at most third order in the dimensional parameter ε. (There is one ε^{-1} for $y_{34}\to 0$, and one for $y_{134}\to 0$ and $y_{234}\to 0$, respectively.) This gives just what one needs to cancel the T_R-part of (3.4) and (5.3).

The second class of simple contributions comes from the "pure QCD" transition probabilities in Fig. 5a. Topologically they have the same structure as the T_R contributions. Analytically they lead to an expression similar to (6.9).

$$M_P(y_{134}\to 0)=\frac{5}{6}N_c\frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)\,y_{134}^2}\cdot\left\{-1+\frac{14}{15}\varepsilon+\frac{1}{z}\left(1-\frac{14}{15}\varepsilon-\frac{\varepsilon^2}{45}\right)+\frac{1}{y_{234}z}\left(-2-\frac{2}{15}\varepsilon-\frac{4}{45}\varepsilon^2\right)+\frac{1}{y_{234}^2 z}\left(2+\frac{2}{15}\varepsilon+\frac{4}{45}\varepsilon^2+\frac{8}{135}\varepsilon^3\right)\right\}. \tag{6.11}$$

This is integrated just like (6.9) and gives part of the N_c-contribution in the total result (6.25). Let us now come to the contribution of type E, i.e. those nonplanar transition probabilities of Fig. 5b which survive Furry's theorem. In a sense they are still simpler than (6.9) and (6.11). The reason is that in their sum all y_{ij}-poles drop out. Therefore they give order y contributions in the three-jet regions.

However, they have singularities in one two-jet variable y_{ijk}, say y_{134}. So from the two-jet region $y_{134}<y$ one expects a $1/\varepsilon$ singularity after integration. In fact in the limit $y_{134}\to 0$ the matrix elements squared read

$$\begin{aligned}M_E(y_{134}\to 0)=&\frac{C_F-N_c/2}{y_{134}^2}\left\{\frac{1}{z}\left(1-\frac{v}{y_{14}/y_{134}}\right)\right.\\&\cdot\left[\frac{y_{234}}{y_{124}}((v-\varepsilon)\,y_{234}+1-2v)\right.\\&\left.+1-\varepsilon+(v-v\varepsilon+\varepsilon^2)\,y_{234}\right]\\&+\frac{y_{234}}{y_{124}\,y_{14}/y_{134}}(y_{234}^2(v+v\varepsilon-v^2-\varepsilon)\\&+y_{234}(1-2v\varepsilon+\varepsilon)-1)\\&-2\frac{y_{14}/y_{134}-v}{z}(1-\varepsilon)+2y_{234}(\varepsilon+\varepsilon^2-2)\\&+\frac{1}{y_{14}/y_{134}}[1-\varepsilon+y_{234}(1-2v\varepsilon+\varepsilon-\varepsilon^2)\\&+y_{234}^2(v(1+\varepsilon-\varepsilon^2)\\&\left.+v^2(1-\varepsilon)-2\varepsilon+\varepsilon^2-2z(1-\varepsilon))]\right\}.\end{aligned} \tag{6.12}$$

The expression has been written in such a way that the absence of a pole in z is explizit. Just as in (6.9) and (6.11) we have worked in a 34-system here. This means that y_{14}/y_{134} should be approximated by (6.8) with y_{24} replaced by $y_{12}\approx 1-y_{234}$. (v and z of course change their meaning.) y_{124} can be approximated by $y_{124}\approx 1-y_{234}(1-v)$.

The details of the integrations in (6.12) have been described in our technical paper [24]. There we have devoted some care to describe how terms with y_{14} in the denominator are treated. Such terms will also be important in the following where they occur in connection with stronger poles in ε.

Concerning (6.12) one should note that there is no overlap problem because there is no such term as $y_{134}^{-2}y_{234}^{-2}$ or $y_{134}^{-2}y_{124}^{-2}$. So one can integrate with

$$\begin{aligned}R'_{34}=&2\int_0^y dy_{134}\,y_{134}^{1-2\varepsilon}\int_0^1 dy_{234}\,y_{234}^{1-2\varepsilon}(1-y_{234})^{-\varepsilon}\\&\cdot\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\\&\cdot\int_0^\pi d\theta'\sin^{-2\varepsilon}\theta'/N_{\theta'}.\end{aligned} \tag{6.13}$$

The result is

$$\begin{aligned}Z_E=N\cdot\frac{C_F-N_c/2}{\Gamma^2(1-\varepsilon)}&\left[\frac{1}{\varepsilon}\left(-\frac{13}{2}+6\zeta_2-4\zeta_3\right)+\frac{113}{4}-12\zeta_2\right.\\&\left.+50\zeta_3-75\zeta_4+(13-12\zeta_2+8\zeta_3)\ln y\right],\end{aligned} \tag{6.14}$$

where $N:=\sigma^{(2)}(\alpha_s/2\pi)^2(4\pi\mu^2/q^2)^{2\varepsilon}C_F$.

Now we come to the remaining contributions of Fig. 5a in the limit $y_{134}\to 0$. In a 13-system ($z=y_{13}/y_{123}\,y_{134}$) they are

$$\begin{aligned}M_{ABC}(y_{134}\to 0)=&\frac{1}{y_{134}^2}\left\{C_F\left[\frac{1}{z}(1-\varepsilon)(1-v)\,P_{qq}^n(y_{123})\right.\right.\\&+(1-\varepsilon)^2 v\,y_{123}+\frac{2v\,y_{123}^2}{z\,y_{24}\,y_{234}}-\frac{2v\,y_{123}^2(1+y_{24})}{y_{234}\,y_{24}\,y_{14}/y_{134}}\\&+\frac{2v\,y_{123}}{(1-v)\,y_{24}\,y_{14}/y_{134}}+\frac{1-\varepsilon}{y_{14}/y_{134}}\\&\left.\cdot\left(2\frac{y_{24}}{1-v}+(1-\varepsilon)\,y_{123}^2\,v\right)\right]\\&+\left(C_F-\frac{N_c}{2}\right)\left[2\frac{v}{(1-v)\,z}P_{qq}^n(y_{123})\right.\\&-\frac{2v\,y_{123}^2}{z\,y_{24}\,y_{234}}+2\varepsilon(1-\varepsilon)\,y_{123}+\frac{2v\,y_{123}^2(1+y_{24})}{y_{234}\,y_{24}\,y_{14}/y_{134}}\\&+\left(\frac{y_{123}\,v^3}{y_{24}(1-v)}-\varepsilon\frac{y_{123}}{y_{24}}v(1-v)+(1-\varepsilon)(-\varepsilon\,y_{24}\,y_{123}\right.\\&\left.+(1+\varepsilon)\,v\,y_{24}\,y_{123}-1+y_{123}\,v^2)\right)\frac{2}{y_{14}/y_{134}}\\&-\frac{2t}{z\,y_{14}/y_{134}}\left(y_{123}(1-\varepsilon)\left(\frac{v}{1-v}-v-\varepsilon\right)+\frac{v}{1-v}(1+\varepsilon)\right.\\&\left.\left.-v(1-\varepsilon)-\frac{1}{y_{24}}P_{qq}^n(v)\right)\right]\end{aligned}$$

$$+\frac{N_c}{2}\left[\frac{2v}{z(1-v)}P^n_{qq}(y_{123})-\frac{2vy^2_{123}}{zy_{234}y_{24}}\right.$$

$$+\frac{2t}{zy_{34}/y_{134}}\left(\frac{y_{123}}{y_{234}}P^n_{qq}(y_{123})-2+\frac{4}{1-v}-2(1-\varepsilon)v\right.$$

$$\left.+(1-\varepsilon)y_{24}\left(1+v-\frac{2}{1-v}\right)\right.$$

$$\left.+\frac{1}{y_{24}}\left(1+\varepsilon+(1-\varepsilon)v-\frac{4}{1-v}\right)\right)+\frac{1}{y_{34}/y_{134}}$$

$$\cdot\left(2(1-\varepsilon)\frac{y_{24}}{y_{234}}+\frac{1}{y_{24}}(2(1-\varepsilon)+2v^2(1-\varepsilon)+4v\varepsilon)\right.$$

$$+4\varepsilon+v(4-8\varepsilon)-4v^2(1-\varepsilon)-\frac{4}{1-v}$$

$$+y_{24}\left(-4\varepsilon-8v(1-\varepsilon)+2v^2(1-\varepsilon)+\frac{4}{1-v}\right)$$

$$\left.\left.+y^2_{24}(1-\varepsilon)\left(4v-\frac{2}{1-v}\right)\right)\right]\Bigg\}. \tag{6.15}$$

Here the term $\sim C_F$ corresponds to the planar QED type contributions A, the term $\sim C_F-\frac{N_c}{2}$ to the nonplanar QED type contributions B and the term $\sim N_c$ to the interferences C. In all three cases the most singular terms containing a pole in z have been written down first. All the other terms can be arranged in such a way that poles in z drop out ($t:=2\cos\theta'\sqrt{v(1-v)z(1-z)y_{24}}$). This means that poles in z never occur in conjunction with poles in y_{14} or y_{34}.

In (6.15) $\lim_{y_{134}\to 0} y_{34}/y_{134}$ can be determined from (6.8) by remembering $y_{34}=y_{134}-y_{13}-y_{14}$. For y_{123} one can use the approximation $y_{123}=1-y_{24}$. There are no poles in y_{123}, but in y_{24}. In this sense the situation is asymmetrical compared to (6.10). For y_{234} one can use the approximation $y_{234}=1-vy_{123}$. There are many cancellations in the terms $\sim y_{234}^{-1}$ of (6.15), as will be demonstrated in our three-jet work [20]. This does not mean however, that there is no contribution from the region $y_{234}<y$. There is an expression symmetrical to (6.15) with 1 and 2 interchanged. One can also see from (6.15) that there must be an overlap contribution from terms like $1/(y^2_{134}z(1-v)y_{24})$, if one remembers that $y_{23}=(y_{123}-y_{13})(1-v)$.

For the contribution $\sim N_c$ it would be equally reasonable to use a 34-system instead of a 13-system, because the typical denominator contains $y_{34}y_{13}y_{24}$. Then either y_{13} or y_{24} would be a complicated variable of the type of (6.8).

In Sect. 8 we shall pursue a different strategy (at least for the N_c-term of (6.15)). There we shall use a different phase space system for each single expression to be integrated. For instance a term with denominator $y_{34}y_{13}y_{24}$ can be treated in a 13-system, where the role of 2 and 4 is interchanged compared to the 13-system we are using here. In this way one can avoid a complicated θ'-integration for all but a few terms. But let us come back to the integration of (6.15). Most of the technical details of the integrals have been presented in our paper [24]. Therefore we concentrate here on the treatment of the overlap regions.

As can be seen from (6.6) the N_c-term has no contributions from the overlap region ($y_{13}<y$, $y_{24}<y$, $y_{134}<y$) and ($y_{23}<y$, $y_{14}<y$, $y_{134}<y$). Thus we could use (6.10) if we were working in a 34-system. In a 13-system the condition $y_{234}<y$ (which is part of the overlap condition) is nontrivial, because y_{234} is not an integration variable, $y_{234}=1-vy_{123}+O(y_{134})$. But for the integration measure one can write

$$2\int_0^y dy_{134}\,y_{134}^{1-2\varepsilon}\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}\int_0^1 dy_{24}\,y_{24}^{-\varepsilon}(1-y_{24})^{-\varepsilon}$$

$$\cdot\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\int_0^\pi\frac{d\theta'}{N_\theta'}\sin^{-2\varepsilon}\theta'$$

$$-\int_0^y dy_{134}\,y_{134}^{1-2\varepsilon}\int_0^y dy_{234}\,y_{234}^{1-2\varepsilon}\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}$$

$$\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\int_0^\pi\frac{d\theta'}{N_{\theta'}}\sin^{-2\varepsilon}\theta' \tag{6.16}$$

where it is assumed that the second term acts on the N_c-part of (6.15) given in a 34-system. $N_{\theta'}$ is defined after (B.7).

For the second term one needs only the infrared singular contributions of the N_c-term (gluons 3, 4 infrared). These are

M_{BC}(IR gluons)

$$=\frac{N_c}{2}\frac{1}{y^2_{134}}\left[\frac{1}{(1-v)y_{13}/y_{134}}+\frac{1}{v\cdot y_{14}/y_{134}}\right.$$

$$+\frac{1}{v(1-v)y_{13}/y_{134}\,y_{14}/y_{134}}-\frac{1}{z}\left(1-\frac{1}{v}-\frac{1}{1-v}\right)$$

$$\left.+\frac{1}{zy_{14}/y_{134}}\left(1+\frac{1}{1-v}\right)+\frac{1}{zy_{13}/y_{134}}\left(1+\frac{1}{v}\right)\right]. \tag{6.17}$$

The terms without poles in z come from the diagrams of class B, the terms with poles in z come from diagrams of class C. y_{13} and y_{14} can be approximated by

$$y_{13}/y_{134}=v(1-z)+z(1-v) -2\cos\theta'\sqrt{v(1-v)\,z(1-z)}$$
$$y_{14}/y_{134}=(\text{the same with } v\leftrightarrow 1-v,\ \cos\theta'\leftrightarrow -\cos\theta'). \tag{6.18}$$

Therefore (6.17) can be reduced very much by symmetry arguments:

$$M_{BC}(\text{IR gluons})=\frac{N_c}{2}\frac{1}{y_{134}^2}\left[\frac{8}{z\,y_{14}/y_{134}}+\frac{2}{z(1-v)\,y_{14}/y_{134}}-\frac{1}{z}\left(1-\frac{2}{1-v}\right)\right]. \tag{6.19}$$

The θ'-integration can now be done using [24]

$$\int_0^\pi \frac{d\theta'\sin^{-2\varepsilon}\theta'}{N_{\theta'}\,y_{14}/y_{134}} =(s_+/2)^{2\varepsilon}r_-^{-1-2\varepsilon}{}_2F_1\left(-\varepsilon,\,-2\varepsilon,\,1-\varepsilon,\,\frac{s_-}{s_+}\right), \tag{6.20}$$

where $s_\pm=r_+\pm r_-$ and $r_\pm=|v(1-z)\pm z(1-v)|$.

The examples described in [24] all refer to the first region of (6.16). In contrast to them here the hypergeometric function may not be approximated by 1. Instead one should use the full series representation

$$_2F_1(-\varepsilon,\,-2\varepsilon,\,1-\varepsilon,\,x)=1+\frac{2\varepsilon^2}{\Gamma(1-2\varepsilon)}\cdot\sum_1\frac{\Gamma(k-2\varepsilon)}{k!(k-\varepsilon)}x^k. \tag{6.21}$$

The series in (6.21) is nonleading, so it can be integrated term by term using suitable approximations. After the integration one ends up with a series which can be summed by standard methods.

The integrals with the first term in (6.21) can all be integrated by standard techniques. One of them will be needed in Sect. 8, so we include its result for convenience here:

$$\int_0^1 dz\,z^{-1-\varepsilon}(1-z)^{-\varepsilon}\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-1-\varepsilon}\cdot\{\Theta(z-v)\,z^{2\varepsilon}(1-v)^{2\varepsilon}(z-v)^{-1-2\varepsilon}+\Theta(v-z)\,v^{2\varepsilon}(1-z)^{2\varepsilon}(v-z)^{-1-2\varepsilon}\}$$
$$=\frac{3}{\varepsilon^2}-10\zeta_2-26\zeta_3\,\varepsilon-11.5\zeta_4\,\varepsilon^2. \tag{6.22}$$

Note that there is a factor $y^{-4\varepsilon}/4\varepsilon^2$ in front coming from the y_{134}- and y_{234}-integration which fully decouple from all other integrations. (6.17) then leads to

$$Z_{\text{class }B,C}(\text{IR gluons})=N\frac{N_c}{2}\frac{y^{-4\varepsilon}}{4}\left[-\frac{1}{2\varepsilon^4}+\frac{\gamma}{\varepsilon^3}+\frac{1}{\varepsilon^2}\left(2\zeta_2-\gamma^2-\frac{1}{2}\right)+\frac{1}{\varepsilon}\left(\frac{29}{6}\zeta_3+\gamma+\frac{2}{3}\gamma^3-4\zeta_2\gamma-1-\frac{1}{2}\zeta_2\right)\right.$$
$$\left.-2+\frac{37}{4}\zeta_4+\frac{3}{2}\zeta_2+\zeta_2\gamma+4\zeta_2\gamma^2-\frac{5}{2}\zeta_3-\frac{29}{3}\zeta_3\gamma+2\gamma-\gamma^2+\frac{1}{3}\gamma^4\right]. \tag{6.23}$$

For the C_F-terms of (6.15) the overlap structure is more complicated than (6.16), because one also has to prevent the regions ($y_{13}<y$, $y_{24}<y$, $y_{134}<y$) and ($y_{14}<y$, $y_{23}<y$, $y_{134}<y$) being counted twice. Using the symmetries of the matrix elements one can derive the following formula (see Appendix A):

$$2\int_0^y dy_{134}\,y_{134}^{1-2\varepsilon}\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}\int_y^1 dy_{24}\,y_{24}^{-\varepsilon}(1-y_{24})^{-\varepsilon}\cdot\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\int_0^\pi\frac{d\theta'}{N_{\theta'}}\sin^{-2\varepsilon}\theta'$$
$$+\int_0^y dy_{134}\,y_{134}^{1-2\varepsilon}\int_0^y dy_{234}\,y_{234}^{1-2\varepsilon}\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}\cdot\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\int_0^\pi\frac{d\theta'}{N_{\theta'}}\sin^{-2\varepsilon}\theta' \tag{6.24}$$

where, just as in (6.16), the first term has to be read in the usual 13-system ($z=y_{13}/y_{134}\,y_{123}$, $v=y_{12}/(y_{123}-y_{13})\ldots$) and the second term in the 34-system ($z=y_{34}/y_{134}\,y_{234},\ldots$).

All the integrals can be calculated with the methods described in [24]. Adding all contributions gives

$$\sigma_{2\text{-jet}}(q\bar{q}gg,\,q\bar{q}q\bar{q})=N/((1-\varepsilon)\,\Gamma^2(1-\varepsilon))\,\{C_F\,F_C+N_c\,F_N+T_R\,F_T\} \tag{6.25}$$

$$F_C=2/\varepsilon^4+4/\varepsilon^3+(15-12\zeta_2)/\varepsilon^2+(307/8-9\zeta_2-30\zeta_3)/\varepsilon+5335/48-261/4\zeta_2-7\zeta_3-10\zeta_4$$
$$+\ln y(-6/\varepsilon^2-(22-16\zeta_2)/\varepsilon-251/4+17\zeta_2+44\zeta_3)+\ln^2 y(-4/\varepsilon^2+1/\varepsilon+10-6\zeta_2)$$
$$+\ln^3 y(4/\varepsilon+7)-\tfrac{1}{3}\ln^4 y \tag{6.26}$$

$$F_N = 1/(2\varepsilon^4) + 23/(12\varepsilon^3) + (223/36 - \tfrac{3}{2}\zeta_2)/\varepsilon^2 + (4033/216 - 10\zeta_2 + \zeta_3/2)/\varepsilon + 434561/7776 - 27\zeta_2 - 625/12\zeta_3 + 781/16\zeta_4 + \ln y(-3/\varepsilon^2 - 21/(2\varepsilon) + 58/3\zeta_2 - 10\zeta_3 - 403/12) + \ln^2 y(-2/\varepsilon^2 + 4/(3\varepsilon) + 121/8 + 6\zeta_2) + \ln^3 y(4/\varepsilon + 4/3) - 14/3 \ln^4 y. \quad (6.27)$$

$$F_T = -1/(3\varepsilon^3) - 11/(9\varepsilon^2) - (233/54 - 2\zeta_2)/\varepsilon - 18092/1296 + 16/3\zeta_2 + 16/3\zeta_3 + (23/3 - 8/3\zeta_2 + 2/\varepsilon)\ln y + (4/(3\varepsilon) - 10/9)\ln^2 y - 8/3\ln^3 y. \quad (6.28)$$

7. Result and Discussion of the Singular Approach

The sum (3.2)+(4.11)+(5.3)+(6.25) is finite for $\varepsilon \to 0$.

All the terms in the sum have been derived by expanding expressions of the form $y^{-k\varepsilon}/\varepsilon^n$, $k, n = 0, 1, 2, 3, 4$. (Some finite logarithms may still be missing in (7.2), see the discussion at the beginning of Sect. 8.) The cancellation of the singularities can best be understood in terms of these quantities. For instance the leading singularity is cancelled in the following way

$$\lim_{\varepsilon \to 0} C_F \varepsilon^{-4}\{2 + 4y^{-2\varepsilon} - 8y^{-\varepsilon} + 8y^{-2\varepsilon} - 8y^{-3\varepsilon} + 2y^{-4\varepsilon}\} = 2C_F \ln^4 y. \quad (7.1)$$

The first term in (7.1) comes from the two-parton diagrams, the next two terms come from the three-parton diagrams and the rest from the tree diagrams.

The right hand side of (7.1) will be the leading logarithm of our two-jet cross section (7.2). Thus we see how the leading singularity generates the leading logarithm. Analogously the nonleading logarithms are generated by the nonleading singularities. The result is

$$\sigma_{2\text{-jet}}/\sigma_0 = 1 + \frac{\alpha_s(q^2)}{2\pi} C_F(-2\ln^2 y - 3\ln y + 2\zeta_2 - 1) + \left(\frac{\alpha_s(q^2)}{2\pi}\right)^2 C_F(C_F Z_C^s + N_c Z_N^s + T_R Z_T^s) \quad (7.2)$$

$$Z_C^s = 2\ln^4 y + 6\ln^3 y + \left(\frac{13}{2} - 6\zeta_2\right)\ln^2 y + \left(\frac{9}{4} - 3\zeta_2 - 12\zeta_3\right)\ln y + \frac{1}{8} - \frac{51}{4}\zeta_2 + 11\zeta_3 + 4\zeta_4 \quad (7.3)$$

$$Z_N^s = \frac{11}{3}\ln^3 y + \left(2\zeta_2 - \frac{169}{36}\right)\ln^2 y + \left(6\zeta_3 - \frac{57}{4}\right)\ln y + \frac{31}{9} + \frac{32}{3}\zeta_2 - 13\zeta_3 + \frac{53}{2}\zeta_4. \quad (7.4)$$

$$Z_T^s = -\frac{4}{3}\ln^3 y + \frac{11}{9}\ln^2 y + 5\ln y + \frac{19}{9} - \frac{38}{9}\zeta_2. \quad (7.5)$$

In (7.2) we have included the Sterman-Weinberg contributions $O(\alpha_s)$.

We have identified the arbitrary parameter μ^2 with the energy of the virtual photon. In order α_s^2 this can be justified by renormalization group considerations [29].

The coefficients of the cubic term in (7.4) and (7.5) are such that they can be absorbed into the running coupling constant by going from $\alpha_s(q^2)$ to

$$\alpha_s(yq^2) = \alpha_s(q^2)\left(1 - \frac{\alpha_s(q^2)}{2\pi} \cdot \left(\frac{11}{6}N_c - \frac{2}{3}T_R\right)\ln y + O(\alpha_s^2)\right). \quad (7.6)$$

This is a necessary condition for the exponentiation properties of our result which we will discuss now. There is a conjecture by Smilga [30] that the leading and even the next to leading logarithms of the two-jet cross section exponentiate to any order in α_s. This means they can be generated by writing the leading and next to leading logarithm of the Sterman-Weinberg result into an exponential $\exp\left(\frac{\alpha_s}{2\pi} C_F(-2\ln^2 y - 3\ln y)\right)$. Note that it is important here to choose $\alpha_s = \alpha_s(yq^2)$, because otherwise one could never generate a N_c or T_R-contribution.

We can prove Smilga's conjecture to order α_s^2, which means we can generate all quartic and cubic terms in $\ln y$ in our result (7.2)–(7.5)*. It should be stressed however that the nonleading logarithms do not exponentiate. One has to add a correction factor to take them into account

$$\sigma_{2\text{-jet}}/\sigma_0 = \exp\left[\frac{\alpha_s(yq^2)}{2\pi} C_F(-2\ln^2 y - 3\ln y)\right] \cdot \left\{1 + \frac{\alpha_s(yq^2)}{2\pi}(2\zeta_2 - 1)C_F + \left(\frac{\alpha_s}{2\pi}\right)^2 C_F(C_F\tilde{Z}_C + N_c\tilde{Z}_N + T_R\tilde{Z}_T)\right\} \quad (7.7)$$

$$\tilde{Z}_T = \frac{29}{9}\ln^2 y + \left(\frac{17}{3} - \frac{4}{3}\zeta_2\right)\ln y + \frac{19}{9} - \frac{38}{9}\zeta_2 \quad (7.8)$$

$$\tilde{Z}_C = -2\zeta_2\ln^2 y - \left(\frac{3}{4} - 3\zeta_2 + 12\zeta_3\right)\ln y + \frac{1}{8} - \frac{51}{4}\zeta_2 + 11\zeta_3 + 4\zeta_4 \quad (7.9)$$

* This is in agreement with earlier leading logarithm calculations for the off shell quark form factor [30] and the two-jet cross section for massive quarks [32]

$$\tilde{Z}_N=\left(2\zeta_2-\frac{367}{36}\right)\ln^2 y+\left(\frac{11}{3}\zeta_2+6\zeta_3-\frac{193}{12}\right)\ln y$$
$$+\frac{31}{9}+\frac{32}{3}\zeta_2-13\zeta_3+\frac{53}{2}\zeta_4. \quad (7.10)$$

Formulas (7.2) and (7.7) are identical, if one neglects terms of order α_s^3.

The corrections (7.8)–(7.10) typically contain squares of logarithms.

In Fig. 9 we have drawn the Sterman-Weinberg result and our two formulas (7.2) and (7.7) for a wide range of y values and for $\alpha_s(q^2)=0.12$. In this plot also the order $\alpha_s\, y$ contributions of table 1 are taken into account. The region of physical y-values is on the right half of the diagram.
In that region the α_s^2 contribution is of the order of 10% which is reasonable for a perturbative result. In the physical region (7.2) and (7.7) practically give the same result. Only for small y the exponentiated version improves the result. One can see this from the fact that for $y\to 0$ the Sterman-Weinberg formula and our naked result (7.2) become negative. This is an indication for the breakdown of perturbation theory. The exponentiated curve remains positive. The exponentiation sums up multigluon radiation at leading logarithmic level.

Instead of calculating $\sigma_{2-\text{jet}}/\sigma_0$ one could have given the two-jet multiplicity $\sigma_{2-\text{jet}}/\sigma_{\text{tot}}$, where [23]

$$\sigma_{\text{tot}}=\sigma_0\left\{1+\frac{3}{2}C_F\frac{\alpha_s(q^2)}{2\pi}+\left(\frac{\alpha_s(q^2)}{2\pi}\right)^2 C_F\right.$$
$$\left.\cdot\left[-\frac{3}{8}C_F+\left(4\zeta_3-\frac{11}{2}\right)T_R+\left(\frac{123}{8}-11\zeta_3\right)N_c\right]\right\}. \quad (7.11)$$

This will be done in Sect. 9, where we shall compare (7.2) with the result of the partial fractioning approach, which will contain all finite logarithms and constants and all order y corrections.

8. Four Parton Diagrams: The Partial Fractioning Approach

It is the purpose of this section to go beyond the approximations applied in Sect. 6, where we took into account only the singular terms in the four-parton cross section, which are responsible for the infrared/collinear singularities. This means we must integrate the complete four-parton matrix elements over the two-jet region. This task seems impossible due to the fact that the four-parton terms are too complicated to allow an exact integration over several variables in n-dimensions in analytic form. In addition we want to define the two-jet region in such a way that the integration over its complement in the four-parton phase space yields the three- and four-jet cross section. In our complementary work on integrated three- and four-jet cross sections [20] we describe how through partial fractioning of the four-parton matrix elements we separate the singular contribution in such a form that it is integrable analytically in n-dimensions and the remainder is finite and integrable in $n=4$ dimensions which can be done partly numerically. These singular terms are also suitable for integration over the two-jet region. But the terms which are non-singular in the three-jet region have still singularities in the two-jet region. Nevertheless we accomplish to integrate them analytically although only approximately, neglecting contributions $O(y)$. For an exact integration with all terms $O(y)$ included further partial fractioning would be necessary. In the partial fractioned expressions we have presented [20] one avoids y/ε terms only in the three-jet region. In the two-jet region $O(y/\varepsilon)$ terms prohibiting numerical integration are still present.

Our result which is valid up to terms $O(y)$ can be added to the integrated three- and four-jet cross section [20]. This sum should yield the $O(\alpha_s^2)$ terms in σ_{tot} at least for very small values of y ($y<10^{-3}$). We shall find rather big $O(y)$ contributions for the N_c-term (just as in Table 3). Once the total cross section is reconstructed one can be quite sure that the differential and integrated three- and four-jet cross sections are correct (even including terms of order y). Then the exact two-jet cross section can be calculated via the sum rule

$$\sigma_{2-\text{jet}}=\sigma_{\text{tot}}-\sigma_{3-\text{jet}}-\sigma_{4-\text{jet}}.$$

In the partial fractioning approach (PF approach) the contributions (6.9), (6.11) and (6.12) remain unchanged. The reason is that there is not more than one pole in any y_{ij} at a time, so no partial fractioning has to be done. (The variables y_{ijk} are not subject to partial fractioning.) In the PF approach only the expression (6.15) is changed. The matrix elements squared contributing to (6.15) are rewritten as [20]

$$M=C_F\left[\frac{R}{y_{13}}+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\, 3\leftrightarrow 4)\right]$$
$$+N_c\left[\frac{S}{y_{34}}+\frac{T}{y_{13}}+(1\leftrightarrow 2)+(3\leftrightarrow 4)\right.$$
$$\left.+(1\leftrightarrow 2,\, 3\leftrightarrow 4)\right] \quad (8.1)$$

where R, S and T have the property that they remain finite for any one y_{ij} going to zero. This property can be achieved by partial fractioning. Consider for example the equation

$$\frac{1}{y_{13}\,y_{23}}=\frac{1}{y_{13}(y_{13}+y_{23})}+(1\leftrightarrow 2). \tag{8.2}$$

The first term on the right hand side has a singularity only for $y_{13}\to 0$.

The partial fractioned expressions in four dimensions are given in our three-jet paper [20]. Here they have to be supplemented by the n-dimensional corrections that are singular when integrated over the two-jet regions.

The partial fractioning can be chosen in such a way that the "13-terms" R/y_{13} and T/y_{13} give finite or order y contributions when integrated over the two-jet regions $(y_{234}<y, y_{134}>y, y_{13}>y)$ and $(y_{14}<y, y_{23}<y, y_{134}>y, y_{13}>y)$. Analogously the 34-term S/y_{34} gives finite or order y contributions in the regions $(y_{13}<y, y_{24}<y, y_{134}>y, y_{234}>y)$ and $(y_{14}<y, y_{23}<y, y_{134}>y, y_{234}>y)$. These regions have been integrated over numerically and for the time being treated together with the finite three- and four-jet contributions in [20]. So for R and T we define an "unphysical" two-jet cross section with the help of the region $(y_{134}<y)\cup(y_{13}<y, y_{24}<y)$ and for S with the help of the region $(y_{134}<y)\cup(y_{13}<y, y_{24}<y)$ and for S with the help of the region $(y_{134}<y)\cup(y_{234}<y)$. This is sufficient for the reconstruction of the total cross section. To get a physical two-jet cross section one should disentangle the finite two-jet contributions from the three- and four-jet cross section. This is done in [20] and the result is summarized and discussed in Sect. 9.

In fact the singularity structure (poles in ε) of this "unphysical" two-jet cross section in the PF approach will turn out to be the same as the singularity structure of (6.25). The difference is in the finite terms ($\ln^2 y$, $\ln y$ and constants).

To understand the core of the matter let us reconsider the example (8.2). If one wants to integrate $1/y_{13}(y_{13}+y_{23})$ over the two-jet region of Fig. 6 one can first integrate over the stripe $(y_{13}<y)$ thereby getting all ε-singularities. The (finite) contribution from the region $(y_{13}>y, y_{23}<y)$, which is $\sim\zeta_2$, can be corrected for afterwards.

Now why can we restrict our attention to the 13- and 34-terms of (8.1) and (8.2)? The reason is the following: Because of symmetry

$$\text{(full phase space)}\left[C_F\frac{R}{y_{13}}+N_c\left(\frac{S}{y_{34}}+\frac{T}{y_{13}}\right)\right]$$
$$=\frac{1}{4}\ \text{(full phase space)}\ M. \tag{8.3}$$

For the y_{13}-terms (full phase space) can be divided into four disjoint regions: the here for convenience called four-jet region $(y_{13}>y)$, the three-jet region $(y_{13}<y, y_{24}>y, y_{134}>y)$ and the two-jet regions $(y_{134}<y)$ and $(y_{13}<y, y_{24}<y, y_{134}>y)$. (All are unphysical in the sense described above.) For the y_{34}-term the division is as follows: the four-jet region is $(y_{34}>y)$, the three-jet region is $(y_{34}<y, y_{134}>y, y_{234}>y)$ and the two-jet regions are $(y_{134}<y)$ and $(y_{234}<y, y_{134}>y)$.

Let us start now with the two-jet contributions to the C_F-term. We begin with the region $(y_{13}<y, y_{24}<y, y_{134}>y)$*, for which we had a very simple approximation of the matrix elements in Sect. 6. Here we have to integrate

$$M_{\mathrm{PF}}(y_{13}\to 0, y_{24}\to 0)=C_F\frac{P^n_{qq}(1-y_{134})}{y_{13}(y_{13}+y_{24})}$$
$$\cdot\left(2\frac{y_{123}\,v}{y_{13}+y_{23}}+(1-\varepsilon)(1-v)\right)+\Delta K_T+I_N. \tag{8.4}$$

ΔK_T and I_N will be defined later. We use the 13-system of Sect. 6 here. In (8.4) the first term corresponds to the old approximation (6.6). However, integrating it with the methods described in our technical paper [24] gives a very different result as compared to the result of integrating (6.6). The difference is already in the leading contribution (which is ε^{-3} here, because y_{134} is not allowed to approach zero). The appearance of this difference is not astonishing. As a consequence of the partial fractioning the various contributions are distributed in another way. Only in the final result the singularity structure should be the same. To be definite let us shortly describe how one integrates the term $1/(y_{13}(y_{13}+y_{24})(y_{13}+y_{23})\,y_{134})$. In the 13-system $y_{23}=(y_{123}-y_{13})(1-v)$. Then the v-integration yields

$$\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}(y_{13}+y_{23})^{-1}$$
$$=\frac{1}{y_{123}}\left[-\frac{\Gamma^2(1-\varepsilon)}{\varepsilon\Gamma(1-2\varepsilon)}+\frac{1}{\varepsilon}\Gamma(1-\varepsilon)\,\Gamma(1+\varepsilon)\,y_{13}^{-\varepsilon}y_{123}^{\varepsilon}\right]$$
$$+O(y_{13}). \tag{8.5}$$

It is a characteristic feature of the PF approach that the approximation $(y_{13}+y_{23})^{-1}\approx y_{23}^{-1}$ cannot be made here [7, 24].

With (8.5) the y_{134}-integration becomes simple. Then the y_{13}- and y_{24}-integrations can be done by

* We should have noted already in Sect. 6 that the proper upper limit for the y_{13}-integration is $y_{13}<\min(y, y_{134}\,y_{123})$. However, here we have $y_{123}\approx 1-y_{134}$, so $y>y_{134}\,y_{123}$ only for $y_{134}<y$, which is excluded, or for $y_{123}<y$, which gives an order y contribution, because there is no pole in y_{123}

expanding $(y_{13}+y_{24})^{-1}$ around $y_{13}=0$ for $y_{13}<y_{24}$ and around $y_{24}=0$ for $y_{13}>y_{24}$.

Now we come to the correction terms K_T and I_N in (8.4).

$$I_N=4y_{12}\,y_{123}\,y_{124}/((y_{13}+y_{24})(y_{13}+y_{14})\cdot(y_{13}+y_{23})(y_{14}+y_{24}))$$

gives a finite though nonnegligible contribution here*. It is the only integral for which we found no analytical expression. However, it can be integrated numerically because it is finite. The numerical result can be fitted by $4(0.28-1.47\ln y)$**. It will be included in the final results (8.19) and (8.20) for the C_F-term.

$$\Delta K_T=\frac{2}{y_{13}}\left(y_{134}(1-\varepsilon)+\frac{y_{123}}{y_{134}}\right)\cdot\left[\frac{1}{y_{234}}\left(\frac{v}{y_{13}+y_{24}}+\frac{y_{123}}{y_{13}+y_{23}}\right)-\frac{y_{123}\,v}{(y_{13}+y_{24})(y_{13}+y_{23})}\right] \tag{8.6}$$

gives a leading contribution $\sim\ln y/\varepsilon$. It contains those y_{13}-y_{234}-pole terms that are not absorbed into the terms singular for $y_{13}\to 0$ since their sum is finite for $y_{13}\to 0$. Without approximation ΔK_T reads

$$\begin{aligned}\Delta K_f=\frac{2}{y_{13}}\Bigg\{&\frac{y_{14}(1-y_{24})}{y_{234}(y_{13}+y_{24})}(1-\varepsilon)\\&+\frac{y_{12}}{y_{134}\,y_{234}(y_{13}+y_{24})}+\frac{y_{14}}{y_{134}}(1-\varepsilon)\\&-\frac{y_{12}(1+y_{134})}{y_{134}\,y_{234}}+\frac{y_{12}\,y_{123}(1+y_{34})}{y_{134}\,y_{234}(y_{13}+y_{23})}\\&+\frac{y_{12}\,y_{14}(1-\varepsilon)}{y_{234}(y_{13}+y_{23})}-\frac{y_{12}}{(y_{13}+y_{24})(y_{13}+y_{23})}\\&\cdot\left(y_{134}(1-\varepsilon)+\frac{y_{123}-2y_{13}}{y_{134}}\right)\Bigg\}.\end{aligned} \tag{8.7}$$

In (8.6) y_{234} can be approximated by $1-v+y_{24}$. With this ΔK_T can be integrated analytically. In fact the v-integration is very similar to (8.5). For the terms containing $(y_{13}+y_{23})\,y_{234}$ in the denominator one should write

$$\frac{1}{\left(1-v+\frac{y_{13}}{y_{123}}\right)(1-v+y_{24})}=\frac{1}{y_{24}-\frac{y_{13}}{y_{123}}}\cdot\left(\frac{1}{1-v+\frac{y_{13}}{y_{123}}}-\frac{1}{1-v+y_{24}}\right). \tag{8.8}$$

y_{13}- and y_{24}-integration can then be done by expanding $\left(y_{24}-\frac{y_{13}}{y_{123}}\right)^{-1}$.

Now we come to the contribution of R/y_{13} in the region $y_{134}<y$. First we have the terms singular for $y_{13}\to 0$ which in the limit $y_{134}\to 0$ read

$$M_{\rm PF}^{\rm sing}=\frac{C_F}{y_{134}^2\,z(1-y_{24})}\left[2\frac{1-y_{24}}{y_{13}+y_{24}}+(1-\varepsilon)\,y_{24}\right]\cdot\left[2\frac{1-y_{24}}{y_{13}+y_{23}}-2+(1-\varepsilon)(1-v)\right]. \tag{8.9}$$

Once again the expression is written in the usual 13-system and again one must not use $(y_{13}+y_{24})^{-1}\approx y_{24}^{-1}$ and $(y_{13}+y_{23})^{-1}\approx y_{23}^{-1}$ here. However, the y_{24}- and v-integration can be done with the help of a formula similar to (8.5). The result is

$$I_{\rm PF}^{\rm sing}=\frac{C_F}{y_{134}^2\,z}\left(c+\frac{2}{\varepsilon}\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\,z^{-\varepsilon}y_{134}^{\varepsilon}\right)\cdot\left(d+\frac{2}{\varepsilon}\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\,z^{-\varepsilon}y_{134}^{\varepsilon}\right), \tag{8.10}$$

$$c=2\frac{\Gamma(2-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(2-2\varepsilon)}+(1-\varepsilon)\frac{\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(3-2\varepsilon)}, \tag{8.11}$$

$$d=2\frac{\Gamma(2-2\varepsilon)\Gamma(-\varepsilon)}{\Gamma(2-3\varepsilon)}+(1-\varepsilon)\frac{\Gamma(2-\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(3-3\varepsilon)}. \tag{8.12}$$

The y_{134}- and the z-integration are then straightforward.

From the terms nonsingular for $y_{13}\to 0$ ("three-jet-finite") only a few survive in the region $y_{134}<y$. In the usual 13-system they are:

$$\begin{aligned}M_{\rm PF}^{\rm fin}=(1-\varepsilon)&\left[\frac{v-(1-v)\,y_{24}}{y_{134}^2(1-y_{24})}-\frac{y_{23}\,\varepsilon_2}{y_{13}\,y_{14}\,y_{24}\,y_{134}}+\varepsilon\left(\frac{y_{12}}{y_{134}^2}+2\frac{y_{24}}{y_{134}^2}\right)\right]+\frac{\varepsilon}{2}(1+\varepsilon)\frac{\varepsilon_2}{y_{13}\,y_{14}\,y_{134}}\\&+I_N+K_{41}^f+\Delta K_f.\end{aligned} \tag{8.13}$$

Here $\varepsilon_2=y_{14}\,y_{23}+y_{13}\,y_{24}-y_{12}\,y_{34}$ and

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* I_N will also give contributions in the regions $y_{134}<y$ which are even singular. For these we shall be able to offer analytical results

** One gets this fit from the numerical integration of

$$4\int_0^1 dx\int_0^1 dz\frac{\ln(xy)\ln(1-x+z)}{(x+z)(x-z)},$$

which is the relevant approximation of I_N in the region $(y_{13}<y,\,y_{24}<y,\,y_{134}>y)$

$$K_{41}^f = \frac{2}{y_{134}^2} \frac{2vy_{24} - v(1-v) + vy_{24}^2(1-v) - y_{24} + \frac{2t}{z}(v + y_{24} - vy_{24})}{(y_{24}+y_{13})\left(\frac{y_{14}}{y_{134}} + zy_{123}\right)(1-v+zy_{134})}. \quad (8.14)$$

Most of the terms in (8.13) give $O(\varepsilon^{-1})$, only I_N and ΔK_f contribute $O(\varepsilon^{-2})$. The integration of I_N over the region $(y_{134}<y)$ has been described in our technical paper [24].

To integrate the terms in (8.13) containing ε_2 one should use a 13-system, where the role of partons 2 and 4 is interchanged. In this system

$$\varepsilon_2 = 2y_{123}y_{134}[zy_{24}v + \cos\theta' \sqrt{v(1-v)z(1-z)y_{24}}] + O(y_{134}^2), \quad (8.15)$$

$$y_{14} = y_{134}(1-zy_{123})v, \quad (8.16)$$

so that the θ'-integration becomes trivial.

In the same system one should integrate part of K_{41}^f. (For that purpose (8.14) has to be transformed.) We will not go into the details of this calculation. We only note that in intermediate steps poles of second order in ε appear. Also one has to subtract and add many pole terms before one can apply such formulas as (8.5). The final result is

$$(y_{134}<y)\,K_{41}^f = \frac{y^{-2\varepsilon}}{\varepsilon}\left(2\zeta_3 - \frac{7}{4}\right) + \frac{2}{3}\frac{y^{-3\varepsilon}}{\varepsilon}(1-\zeta_3) + 5.957. \quad (8.17)$$

The bracket $(y_{134}<y)$ means that part of the 4 particle phase space, where $y_{134}<y$.

To calculate ΔK_f we return to our old 13-system. In the limit of small y_{134} (8.7) reduces to

$$\Delta K_f = \frac{4}{y_{134}y_{234}(y_{13}+y_{23})(y_{13}+y_{24})}. \quad (8.18)$$

The approximation $y_{234} \approx 1 - vy_{123}$ can again be justified. So one has a denominator of the form $(1-y_{123}(1-zy_{134}))\ (1-v(1-zy_{134}))(1-y_{123}v)$. Before one can apply (8.5) to the v-integration one has to do a partial fractioning. The y_{123}-integration can then be handled similarly. The result one finally obtains has the same singularity structure as the C_F-term F_C in (6.25). Only the finite contributions differ.

$$F_C^{\mathrm{PF}} = \frac{2}{\varepsilon^4} + \frac{4}{\varepsilon^3} + \frac{1}{\varepsilon^2}(15 - 12\zeta_2) + \frac{1}{\varepsilon}\left(\frac{307}{8} - 9\zeta_2 - 30\zeta_3\right) - 6.920$$
$$+ \ln y \cdot \left(-\frac{6}{\varepsilon^2} - \frac{1}{\varepsilon}(22 - 16\zeta_2) + 33.118\right) + \ln^2 y \cdot \left(-\frac{4}{\varepsilon^2} + \frac{1}{\varepsilon} + 10 - 8\zeta_2\right) + \left(\frac{4}{\varepsilon} + 7\right)\ln^3 y - \frac{1}{3}\ln^4 y. \quad (8.19)$$

This changes Z_C of (7.3) to

$$Z_C^{\mathrm{PF}} = 2\ln^4 y + 6\ln^3 y + \left(\tfrac{13}{2} - 8\zeta_2\right)\ln^2 y - 2.094\ln y + 5.218. \quad (8.20)$$

The finite contribution $4\cdot(0.28 - 1.47\ln y)$ from I_N is included here. Therefore the finite simple logarithms and constants have numerical coefficients.

For small y the most prominent change of (8.20) as compared to (7.3) is the term $-2\zeta_2\ln^2 y$. One should stress that with this new result the term $\sim\zeta_2\ln^2 y\,C_F$ still does not exponentiate (cf. (7.9)).

For physical values of y the $\ln y$-term in (8.20) is just as important. It has changed strongly as compared to (7.3).

Now we come to the N_c-term. What is left is to calculate the contributions of class C. All other contributions can be either taken from Sect. 6 or they are hidden in (8.4), (8.9) and (8.13) where the class B terms must be multiplied with $C_F - \frac{N_c}{2}$ instead of C_F.

In total the N_c-contributions of (6.15) are to be replaced by

$$(U_1 + U_2 + U_T - \tfrac{1}{2}W_1 - \tfrac{1}{2}W_T)N_c. \quad (8.21)$$

Here U_1, U_2 and U_T come from the class C contributions and will be defined below. W_1 are the contributions from class B in the region $(y_{134}<y)$:

$$W_1 = (y_{134}<y)\left\{K_{41}^f - (1-\varepsilon)\frac{y_{23}\varepsilon_2}{y_{13}y_{14}y_{24}y_{134}} + \varepsilon(1-\varepsilon)\frac{y_{12}+2y_{24}}{y_{134}^2} + \frac{2y_{12}}{y_{13}(y_{13}+y_{23})} \cdot \left[\left(\frac{y_{134}}{y_{13}+y_{24}} + \frac{y_{24}}{y_{134}}\right)(1-\varepsilon) + \frac{y_{123}}{y_{134}(y_{13}+y_{24})}\right] + \frac{\varepsilon}{2}(1+\varepsilon)\frac{\varepsilon_2}{y_{13}y_{14}y_{134}} + \frac{2y_{12}y_{123}}{y_{13}(y_{13}+y_{23})y_{134}y_{234}}\right\} \quad (8.22)$$

W_T are the contributions from class B in the region $(y_{13}<y, y_{24}<y, y_{134}>y)$. Remember that for the singular approach of Sect. 6 these contributions cancelled against those of class C in the same region (compare (6.6)). This is no longer true here.

$$W_T=(y_{13}<y, y_{24}<y, y_{134}>y) \cdot \left\{\frac{2y_{12}y_{123}}{y_{13}y_{134}(y_{13}+y_{23})(y_{13}+y_{24})} + \frac{2y_{12}}{y_{13}y_{234}(y_{13}+y_{23})} P^n_{qq}(1-y_{134})\right\}. \quad (8.23)$$

The bracket $(y_{13}<y, y_{24}<y, y_{134}>y)$ stands for integration over four particle phase space with the additional restrictions $y_{13}<y$, $y_{24}<y$ and $y_{134}>y$. The finite contribution $4(0.28-1.47\ln y)$ from I_N is not included here, but will be included later.

The integrals in W_1 and W_T have already been done in connection with the C_F-term.

For the C_F-term we have used the explicit decomposition of R/y_{13} into terms singular for $y_{13}\to 0$ and into three-jet-finite terms. For the N_c-contributions of class C we will not use such a decomposition but instead will make contact with the strategies of Sect. 6.

We begin with the terms to be treated as 34-pole term (i.e. two-jet region$=(y_{134}<y)\cup(y_{234}<y, y_{134}>y)$). Again we include the n-dimensional corrections, wherever necessary. We leave out those terms which give order y contributions after integration. In the region $(y_{234}<y, y_{134}<y)$

$$U_2=(y_{134}<y, y_{234}<y)\frac{1}{y_{34}} \cdot \left\{\frac{y_{12}}{2(y_{34}+y_{13})(y_{34}+y_{24})} - \frac{y_{12}}{2y_{234}} + \frac{y_{12}y_{123}}{(y_{13}+y_{34})y_{234}} + \frac{y_{14}y_{134}(1-\varepsilon)}{(y_{13}+y_{34})y_{234}} - \frac{y_{12}}{y_{134}y_{234}}\right\}. \quad (8.24)$$

are the relevant terms.

Both the 34- and the 13-terms have to be integrated over $(y_{134}<y)$. Therefore they can be treated together. This means some of the partial fractioning can be undone, so that the integrations are less involved. Finally the following expression has to be integrated over the region $y_{134}<y$:

$$U_1=(y_{134}<y)\left\{\frac{y_{12}}{2y_{34}(y_{13}+y_{34})(y_{24}+y_{34})} + \frac{y_{12}}{y_{13}(y_{13}+y_{24})(y_{13}+y_{34})} + \frac{y_{12}}{(y_{13}+y_{24})(y_{13}+y_{34})(y_{24}+y_{34})} + \frac{y_{12}y_{123}}{y_{34}y_{13}y_{234}} + \frac{1-\varepsilon}{y_{34}} \left[\frac{y_{14}y_{134}}{y_{13}y_{234}} + \frac{y_{14}y_{24}}{y_{13}y_{134}} + \frac{y_{13}}{y_{134}^2}\right] + \frac{y_{12}}{y_{34}y_{13}} - \frac{y_{12}}{y_{34}y_{134}y_{234}} - \frac{y_{12}y_{123}}{y_{13}y_{134}y_{234}} - \frac{3y_{12}}{2y_{34}y_{134}} + \frac{1-\varepsilon}{2y_{34}y_{134}}(5y_{24}+y_{23}-1) + \frac{\varepsilon_1}{y_{13}y_{34}y_{134}}\left[\frac{5-3\varepsilon}{4} + \frac{(1-\varepsilon)y_{23}}{2y_{24}} + \frac{y_{23}(3-\varepsilon)+y_{24}(1+\varepsilon)}{4y_{234}}\right]\right\}. \quad (8.25)$$

Here $\varepsilon_1=y_{14}y_{23}-y_{12}y_{34}-y_{13}y_{24}$ annihilates the singularities for $y_{13}\to 0$, $y_{34}\to 0$ and $y_{24}\to 0$. (This is the reason why no partial fractioning is necessary for the terms proportional to ε_1).

The contribution of the 13-terms of class C in the region $(y_{13}<y, y_{24}<y, y_{134}>y)$ will be called U_T. Only a few 13-terms are relevant, namely

$$U_T=(y_{13}<y, y_{24}<y, y_{134}>y) \cdot \left\{\frac{y_{12}}{y_{13}(y_{13}+y_{24})(y_{13}+y_{34})} + \frac{y_{12}}{(y_{13}+y_{24})(y_{13}+y_{34})(y_{24}+y_{34})} + \frac{y_{12}+y_{14}y_{134}(1-\varepsilon)}{y_{13}(y_{13}+y_{34})y_{234}}\right\}. \quad (8.26)$$

It is important that the decomposition into 13- and 34-terms presented here is the same as in our three-jet calculations [20]. Otherwise the parts of phase space over which the terms are integrated would not sum up to the total phase space. One would not be able to reconstruct the total cross section.

Of course it is still possible to take advantage of the symmetries of the integration regions. For instance the region $(y_{134}<y)$ is symmetrical under $3\leftrightarrow 4$ exchange. So the term $y_{12}/y_{13}y_{134}$ will yield the same as $y_{12}/y_{14}y_{134}$ in this region etc. Note however that the region $(y_{134}<y)$ has no $(1\leftrightarrow 2, 3\leftrightarrow 4)$-symmetry. This is the reason, why the first three terms of (8.25) cannot be interpreted as $y_{12}/2y_{13}y_{24}y_{34}$, although they have been constructed out of this by partial fractioning:

$$\frac{y_{12}}{2y_{13}\,y_{34}\,y_{24}}+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\, 3\leftrightarrow 4)$$

$$=\frac{y_{12}}{2y_{34}(y_{13}+y_{34})(y_{24}+y_{34})}$$

$$+\frac{y_{12}}{y_{13}(y_{13}+y_{34})(y_{24}+y_{13})}$$

$$+\frac{y_{12}}{(y_{13}+y_{24})(y_{13}+y_{34})(y_{24}+y_{34})}$$

$$+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\, 3\leftrightarrow 4). \qquad (8.27)$$

Most of the integrals in (8.24), (8.25) and (8.26) are standard by now. Because of the appearance of y_{34} in the denominator one should work either in a 34-system or in a 13-system, where $y_{34}=(y_{134}-y_{13})(1-v)$ is simple. In this way the θ'-integral can be made trivial in all cases. The integrals are then typically of the form

$$I(k, l, m, n, a, b)=\int_0^1 dy_{123}\, y_{123}^{k-2\varepsilon}(1-y_{123})^{l-\varepsilon}$$

$$\cdot\int_0^1 dz\, z^{m-\varepsilon}(1-z)^{n-\varepsilon}$$

$$\cdot\int_0^1 dv\, v^{a-\varepsilon}(1-v)^{b-\varepsilon}$$

$$(1-v(1-z\,y_{123}))^{-1}$$

$$k, n, a=0, 1, 2, \ldots, l, m, b=-1, 0, 1, \ldots. \qquad (8.28)$$

Ways to integrate (8.28) have been described in [24]. So instead of giving further details let us present the final result for the N_c-term, the analogue of equations (8.19) and (8.20):

$$F_N^{\mathrm{PF}}=\frac{1}{2\varepsilon^4}+\frac{23}{12\varepsilon^3}+\left(\frac{223}{36}-\frac{3}{2}\zeta_2\right)/\varepsilon^2$$

$$+\left(\frac{4033}{216}-10\zeta_2+\frac{1}{2}\zeta_3\right)/\varepsilon+11.56$$

$$+\ln y\left(-\frac{3}{\varepsilon^2}-\frac{21}{2\varepsilon}-17.16\right)$$

$$+\ln^2 y\left(-\frac{2}{\varepsilon^2}+\frac{4}{3\varepsilon}+\frac{121}{18}+4\zeta_2\right)$$

$$+\ln^3 y\left(\frac{4}{\varepsilon}+\frac{4}{3}\right)-\frac{19}{4}\ln^4 y \qquad (8.29)$$

$$Z_N^{\mathrm{PF}}=-\tfrac{1}{12}\ln^4 y+\tfrac{11}{3}\ln^3 y-\tfrac{169}{36}\ln^2 y$$

$$-10.4\ln y+51.29. \qquad (8.30)$$

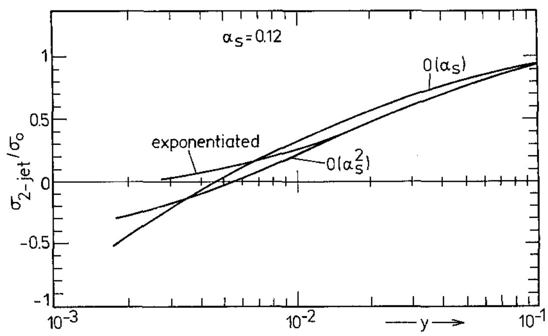

Fig. 9. Two jet cross section of the singular approach as a function of y in units of σ_0 for $a_s=0.12$. $O(\alpha_s^2)$ is according to equations (7.2–5). $O(\alpha_s)$ is the exact lowest order result including all non-leading terms in y and "exponentiated" is according to (7.7–10)

The contribution $-4(0.28-1.47\ln y)$ is also included in (8.29) and (8.30).

The most prominent change in Z_N^{PF} as compared to Z_N^{S} is the appearance of a $\ln^4 y$-term although with a rather small coefficient. This makes it impossible to exponentiate the partial fractioned result. We can see from Fig. 9 that exponentiation is important for $y<0.01$. ($\alpha_s=0.12$). Therefore for physical applications our final result is useful only in the region $0.02\leqq y\leqq 0.05$. This conclusion is supported by the fact that for $y<0.01$ our physical two-jet cross section is smaller than $\frac{1}{10}$ of the $O(\alpha_s)$ Sterman-Weinberg result (see Sect. 9). Perturbation theory breaks down in that region. For higher values of α_s perturbation theory breaks down already for higher values of y.

The origin of the additional $\ln^4 y$-term lies in the different treatment of the term (8.27) in the two approaches. So let us compare them: In the full singular approach the left hand side of (8.27) was integrated over the region

$$(y_{134}<y)+(y_{134}>y,\, y_{234}<y)$$

$$+(y_{13}<y,\, y_{24}<y,\, y_{134}>y,\, y_{234}>y)$$

$$+(y_{14}<y,\, y_{23}<y,\, y_{134}>y,\, y_{234}>y,$$

$$y_{13}>y \quad \text{or} \quad y_{24}>y).$$

This is just the sum of the phase space regions (6.24) and (6.7) written differently (see appendix A). It was shown that the last two regions in this expression do not contribute to the N_c-term. However, for the PF approach the region $(y_{13}<y,\, y_{24}<y,\, y_{134}>y)$ is essential (at least for the 13-terms). It even contributes to the singularities. In the singular approach, however, the N_c-terms coming from the contribution of type B) and C), as defined in Sect. 6, compensate each other in the region $(y_{13}<y,\, y_{24}<y,\, y_{134}>y)$. The re-

Table 4. $O(\alpha_s^2)$ corrections to the two-jet cross section in various schemes. The normalization of these corrections is as in (7.2). Also given are the two-jet multiplicities in $O(\alpha_s)$ and in $O(\alpha_s^2)$ for the various schemes for $\alpha_s=0.12$

y	$Z_T^S=Z_T^{PF}$	Z_T^{phys}	Z_C^S	Z_C^{PF}	Z_C^{phys}	Z_N^S	Z_N^{PF}	Z_N^{phys}	$m_2(\alpha_s)$	m_2^S	m_2^{PF}	m_2^{phys}
0.05	27.00	30.03	17.49	−48.50	−10.91	− 56.06	− 64.97	−104.12	0.789	0.762	0.708	0.670
0.04	36.20	38.87	31.47	−42.44	− 1.76	− 80.14	− 95.10	−133.88	0.739	0.696	0.629	0.594
0.02	74.(14)	75.85	121.27	20.70	65.22	−179.44	−218.90	−255.14	0.551	0.466	0.348	0.322
0.01	128.28	129.39	317.57	187.16	240.46	−321.44	−395.95	−430.33	0.315	0.214	0.0285	0.012

gion ($y_{134}>y$, $y_{234}<y$, $y_{13}>y$) (for the y_{13}-term) which gives finite contributions is left out here. It is included in our calculation of the 3- and 4-jet cross section and appears there as a finite contribution to $\sigma_{2-\text{jet}}$ [20].

In summary the singular terms proportional to ε^{-n} in the two approaches are quite distinct in intermediate steps, but agree in the sums (8.19) and (8.29). The finite terms are different however.

9. Results and Conclusions

With (8.20) and (8.30) we have obtained the two-jet cross section in the PF-scheme (cf. (7.2) and remember that the T_R-term is the same for both methods). However, this is not a physical cross section. Some finite two-jet contributions have not been treated here (e.g. ($y_{12}<y$, $y_{34}<y$). Instead they have been called three- and four-jet temporarily in our paper [20]). This is correct for the reconstruction of the total cross section, if only every contribution is counted exactly once. To get the physical two-jet cross section one should disentangle those finite contributions from the three- and four-jet numbers. This way one gets the exact physical three- and four-jet cross sections. Subtracting them from the total cross section (7.11) one is led to the exact physical two-jet cross section. This all has been done numerically in [20]. Here we only quote the numbers for the physical values of Z_T, Z_C, Z_N (cf. (7.2)) and compare them to the corresponding values obtained with the singular approach and PF approach respectively (Table 4). We give these numbers only in the region* $0.01<y<0.05$. From Table 4 one concludes that the $O(\alpha_s^2)$ corrections differ in the two schemes.

In Table 4 m_2^S, m_2^{PF} and m_2^{phys} are the two-jet multiplicities in the singular, partial fractioned and physical scheme. One gets them by dividing by the total cross section of (7.11). For $m_2(\alpha_s)$ we have divided by $\sigma_0(1+\alpha_s/\pi)$. In the physical two-jet multiplicities m_2^{phys} all order y corrections are included, especially corrections of order $y\cdot\alpha_s$ (see Table 1). In the PF and singular multiplicities m_2^{PF} and m_2^S they are not included. We worked with $\alpha_s=0.12$. The numbers are drawn in Fig. 10a.

* cf. the discussion at the end of Sect. 8

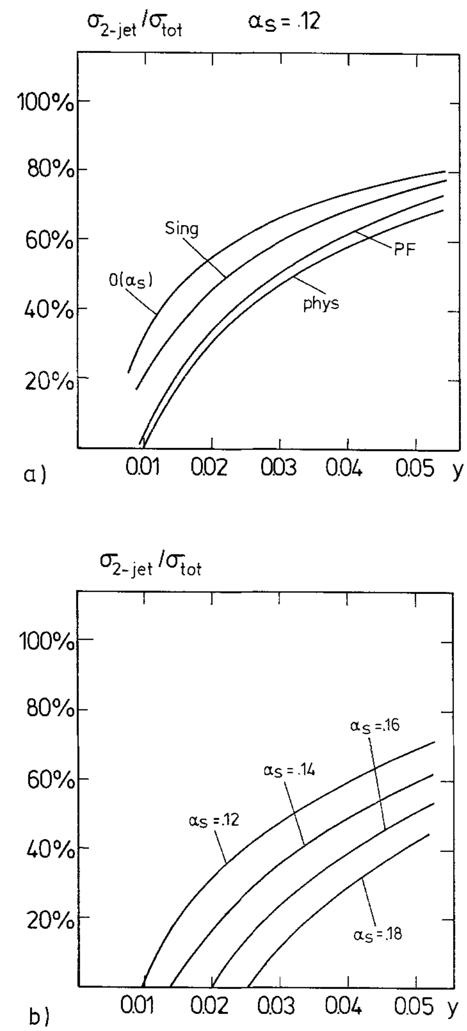

Fig. 10. a Comparison of the two jet multiplicities in the various approaches for $\alpha_s=0.12$. $O(\alpha_s)$ is the lowest order result according to (2.5). "Sing" is the result of the singular approach (7.2–5), "PF" is the result of the partial fractioned approach equations (7.2), (7.4), (8.20) and (8.30) and "phys" is the final physical result of [20]; **b.** physical two jet multiplicities for various couplings $\alpha_s=0.12$, 0.14, 0.16 and 0.18

Table 5. Physical values for the two-, three- and four-jet multiplicities according to [20] for $\alpha_s=0.12$. For comparison we have also included the numbers one receives in $O(\alpha_s)$ including the $O(y)$ corrections of Tables 1a and b

y	$m_2(\alpha_s)$	$m_3(\alpha_s)$	m_2	m_3	m_4
0.05	0.789	0.211	0.670	0.326	0.005
0.04	0.739	0.261	0.594	0.397	0.009
0.02	0.551	0.449	0.322	0.637	0.041
0.01	0.315	0.685	0.012	0.871	0.117

From Fig. 10a we conclude that the PF and physical results give a larger correction to the Sterman-Weinberg formula than does the singular result. This feature already appeared on the level of differential three-jet cross sections and gave rise to some discussion [7, 9] there. For the three-jet case we have resolved it in [20].

As a sort of summary in Table 5 we have given the physical results for the two-, three- and four-jet multiplicities as given in [20]. As noted in [20], an abelian theory ($N_c^{\text{abel}}=0$, $T_R^{\text{abel}}=3\cdot n_f$, $C_F^{\text{abel}}=1$ [29]) would lead to quite distinct results, because the N_c-contributions are large in the case of QCD. For the abelian version of the theory in the $\overline{\text{MS}}$-scheme and with the q^2-scale in α_s no value of α_s exists, such that $m_3>5\%$ at $y=0.05$, which is a strong contradiction to experimental results (see [29] for a discussion).

Let us repeat that with the PF values of Table 4 the total cross section can be reconstructed, if one adds the results of the PF three- and four-jet cross sections [20] and goes to the limit of small y ($y \leqq 10^{-3}$). In this sense the numbers in Table 4 are cross checked. In Fig. 10b also the α_s dependence of our physical result is shown. The coupling constants used are 0.12, 0.14, 0.16 and 0.18, which corresponds to $\Lambda_{\overline{\text{MS}}}=86$, 215, 420 and 710 MeV at $q^2=34$ GeV, respectively. We see that a measurement of the two-jet rate with an error less than 10% would determine $\Lambda_{\overline{\text{MS}}}$ quite accurately.

Thus the two-jet cross section is a possibility to test the structure of higher order QCD matrix elements. After having done a cluster analysis of the hadronic final states it should be possible to obtain two-jet multiplicities in a range of cuts between $y=0.01$ and $y=0.1$ and so check via the y-dependence our higher order QCD calculations and to determine $\Lambda_{\overline{\text{MS}}}$.

Appendix A

Two Jet Phase Space for Symmetrical Matrix Elements

We want to derive the equations (6.7) and (6.24) of the full dressing approach.

In the following we denote by ( ) subsets of four particle phase space which are thought to operate on the matrix elements (with the appropriate n-dimensional integration measure).

In the singular approach the integrand is symmetrical in $(1\leftrightarrow 2)$, $(3\leftrightarrow 4)$ and $(1\leftrightarrow 2, 3\leftrightarrow 4)$.

Originally the two-jet region is

$$\begin{aligned}R^s_{2\text{-jet}} &= (y_{134}<y)+(y_{234}<y, y_{134}>y)\\ &+(y_{13}<y, y_{24}<y, y_{134}>y, y_{234}>y)\\ &+(y_{14}<y, y_{23}<y, y_{134}>y, y_{234}>y)\\ &-(y_{14}<y, y_{23}<y, y_{13}<y, y_{24}<y, y_{134}>y, y_{234}>y).\end{aligned} \tag{A.1}$$

Here the last region gives an order y contribution for every term in the matrix element. So we leave it out. Using the $(1\leftrightarrow 2)$-symmetry we find

$$\begin{aligned}R^s_{2\text{-jet}} &= 2(y_{134}<y)-(y_{134}<y, y_{234}<y)\\ &\quad+2(y_{13}<y, y_{24}<y, y_{134}>y, y_{234}>y).\end{aligned} \tag{A.2}$$

Now one has

$$\begin{aligned}&(y_{13}<y, y_{24}<y, y_{134}>y, y_{234}>y)\\ &=(y_{13}<y, y_{24}<y, y_{134}>y)\\ &\quad-(y_{13}<y, y_{134}>y, y_{234}<y)\end{aligned} \tag{A.3}$$

because $y_{234}<y$ implies $y_{24}<y$.

Also

$$\begin{aligned}&(y_{13}<y, y_{134}>y, y_{234}<y)\\ &=(y_{134}<y, y_{24}<y)-(y_{134}<y, y_{234}<y)\end{aligned} \tag{A.4}$$

because of $(1\leftrightarrow 2, 3\leftrightarrow 4)$ symmetry and because $y_{234}<y$ implies $y_{24}<y$. Inserting (A.3) and (A.4) into (A.2) one gets

$$\begin{aligned}R^s_{2\text{-jet}} &= 2(y_{13}<y, y_{24}<y, y_{134}>y)\\ &\quad+2(y_{134}<y)-2(y_{134}<y, y_{24}<y)\\ &\quad+(y_{134}<y, y_{234}<y)\end{aligned} \tag{A.5}$$

which is the content of (6.7) and (6.24).

Appendix B

Phase Space Formulas

The phase space for j massless state particles in n dimensions is

$$PS^{(j)} = (2\pi)^{j+n(1-j)} \int \Pi d^n p_i\, \delta^+(p_i^2)\, \delta^n\left(q-\sum_{i=1}^{j} p_i\right). \tag{B.1}$$

For $j=3$ and q^2-channel processes it can be fully expressed by the invariants y_{13}, y_{23}:

$$PS^{(3)}=\frac{q^2(4\pi/q^2)^{2\varepsilon}}{128\pi^3\Gamma(2-2\varepsilon)}\int_0^1 dy_{13}\,y_{13}^{-\varepsilon}\cdot\int_0^{1-y_{13}} dy_{23}\,y_{23}^{-\varepsilon}(1-y_{13}-y_{23})^{-\varepsilon}. \tag{B.2}$$

For $j=4$ two angle variables θ, θ' are needed. They are defined as follows. One chooses a system, where $\mathbf{p}_1+\mathbf{p}_3=0$ and where $\mathbf{p}_2 \parallel \mathbf{e}_z$ [7]

$$p_1=\tfrac{1}{2}\sqrt{s_{13}}(1, \ldots, \sin\theta\cos\theta', \cos\theta) \tag{B.3}$$

$$p_2=\frac{s_{123}-s_{13}}{2\sqrt{s_{13}}}(1, \ldots, 0, 1), \tag{B.4}$$

$$p_3=\tfrac{1}{2}\sqrt{s_{13}}(1, \ldots, -\sin\theta\cos\theta', -\cos\theta), \tag{B.5}$$

$$p_4=\frac{s_{134}}{2\sqrt{s_{13}}}(1, \ldots, \sin\beta, \cos\beta). \tag{B.6}$$

Setting $v=\frac{1}{2}(1-\cos\theta)$ one gets

$$\begin{aligned}PS^{(4)}=&\frac{q^4(4\pi/q^2)^{3\varepsilon}}{2048\pi^5\Gamma(2-2\varepsilon)\Gamma(1-\varepsilon)}\\&\cdot\int dy_{123}\,dy_{134}\,dy_{13}(y_{134}y_{123}-y_{13})^{-\varepsilon}\\&\cdot(y_{13}+1-y_{123}-y_{134})^{-\varepsilon}y_{13}^{-\varepsilon}\Theta(y_{13})\\&\cdot\Theta(y_{134}y_{123}-y_{13})\,\Theta(y_{13}+1-y_{134}-y_{123})\\&\cdot\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\int_0^\pi\frac{d\theta'}{N_{\theta'}}\sin^{-2\varepsilon}\theta'.\end{aligned} \tag{B.7}$$

$N_{\theta'}=2^{2\varepsilon}\pi\Gamma(1-2\varepsilon)/\Gamma^2(1-\varepsilon)$ is the normalization of the θ'-integration. For integrations over full of phase space a representation of $PS^{(4)}$ is useful, where all integrations are between 0 and 1:

$$\begin{aligned}PS^{(4)}=&\frac{q^4(4\pi/q^2)^{3\varepsilon}}{2048\pi^5\Gamma(2-2\varepsilon)\Gamma(1-\varepsilon)}\\&\cdot\int_0^1 dy_{134}\,y_{134}^{1-2\varepsilon}(1-y_{134})^{2-3\varepsilon}\int_0^1 ds\,s^{1-2\varepsilon}(1-s)^{-\varepsilon}\\&\cdot\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}(1-zy_{134})^{-2+2\varepsilon}\\&\cdot\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\\&\cdot\int_0^\pi\frac{d\theta'}{N_{\theta'}}\sin^{-2\varepsilon}\theta'.\end{aligned} \tag{B.8}$$

Here $z=y_{13}/(y_{134}y_{123})$ and $s=y_{123}(1-zy_{134})/(1-y_{134})$. The invariants y_{ij} may be expressed with the help of variables appearing in (B.8):

$$y_{12}=(1-y_{134})sv, \tag{B.9}$$

$$y_{23}=(1-y_{134})s(1-v), \tag{B.10}$$

$$\begin{aligned}y_{14}=&y_{134}(1-zy_{123})(v(1-\gamma)+\gamma(1-v)\\&-2\cos\theta'\sqrt{v(1-v)\gamma(1-\gamma)}),\end{aligned} \tag{B.11}$$

$$\begin{aligned}y_{34}=&y_{134}(1-zy_{123})((1-v)(1-\gamma)+v\gamma\\&+2\cos\theta'\sqrt{v(1-v)\gamma(1-\gamma)}),\end{aligned} \tag{B.12}$$

$$y_{24}=(1-y_{134})(1-s), \tag{B.13}$$

where $\gamma:=zy_{24}/((1-zy_{123})(1-zy_{134}))$.

In the main text we are concentrating on the region $y_{134}<y$. There the invariants may be approximated by

$$y_{12}=y_{123}v, \tag{B.14}$$

$$y_{23}=y_{123}(1-v), \tag{B.15}$$

$$y_{24}=1-y_{123}, \tag{B.16}$$

$$\begin{aligned}y_{14}=&y_{134}(v(1-z)+z(1-v)y_{24}\\&-2\cos\theta'\sqrt{v(1-v)z(1-z)y_{24}}),\end{aligned} \tag{B.17}$$

$$\begin{aligned}y_{34}=&y_{134}((1-v)(1-z)+vzy_{24}\\&+2\cos\theta'\sqrt{v(1-v)z(1-z)y_{24}}).\end{aligned} \tag{B.18}$$

The phase space in this limit is

$$\begin{aligned}PS^{(4)}=&\frac{q^4(4\pi/q^2)^{3\varepsilon}}{2048\pi^5\Gamma(2-2\varepsilon)\Gamma(1-\varepsilon)}\int_0^y dy_{134}\,y_{134}^{1-2\varepsilon}\\&\cdot\int_0^1 dy_{24}\,y_{24}^{-\varepsilon}(1-y_{24})^{1-2\varepsilon}\\&\cdot\int_0^1 dz\,z^{-\varepsilon}(1-z)^{-\varepsilon}\int_0^1 dv\,v^{-\varepsilon}(1-v)^{-\varepsilon}\\&\cdot\int_0^\pi\frac{d\theta'}{N_{\theta'}}\sin^{-2\varepsilon}\theta'.\end{aligned} \tag{B.19}$$

If one exchanges the role of particles 2 and 4 in (B.7) and evaluates the limit $y_{134}\to 0$ one gets back (B.19). However, v now has a different meaning and the structure of the invariants differ from (B.14)–(B.18) (apart from $2\leftrightarrow 4$ interchange):

$$y_{14}=y_{134}(1-zy_{123})v, \tag{B.20}$$

$$y_{34}=y_{134}(1-zy_{123})(1-v), \tag{B.21}$$

$$y_{24}=1-y_{123}. \tag{B.22}$$

$$y_{12}=\frac{y_{123}}{1-zy_{123}}(v(1-z)+z(1-v)\,y_{24} -2\cos\theta'\sqrt{v(1-v)\,z(1-z)\,y_{24}}). \tag{B.23}$$

$$y_{23}=\frac{y_{123}}{1-zy_{123}}((1-v)(1-z)+zvy_{24} +2\cos\theta'\sqrt{v(1-v)\,z(1-z)\,y_{24}}). \tag{B.24}$$

References

1. CELLO Collab. H.-J. Behrend et al.: Phys. Lett. **138B**, 311 (1984)
2. JADE Collaboration, W. Bartel et al.: Phys. Lett. **119B**, 239 (1982); Z. Phys. C – Particles and Fields, **25**, 231 (1984)
3. MARK-J Collab. A. Adeva et al.: Phys. Rev. Lett. **50**, 2051; (1983) Phys. Rep. **109**, 131 (1984)
4. PLUTO Collab. Ch. Berger et al.: Z. Phys. C – Particles and Fields **28**, 365 (1985)
5. TASSO Collab. M. Althoff et al.: Z. Phys. C – Particles and Fields **26**, 157 (1984)
6. H. Aihara et al.: Z. Phys. C – Particles and Fields **28**, 31 (1985); E. Fernandez et al., Phys. Rev. **D31**, 2724 (1985)
7. R.K. Ellis, D.A. Ross, E.A. Terrano: Phys. Rev. Lett. **45**, 1225 (1980); Nucl. Phys. **B178**, 421 (1981)
8. J.A.M. Vermaseren, K.J.F. Gaemers, S.J. Oldham: Nucl. Phys. **B187**, 301 (1981)
9. K. Fabricius, I. Schmitt, G. Kramer, G. Schierholz: Phys. Lett. **97B**, 431 (1980); Z. Phys. C – Particles and Fields **11**, 315 (1982)
10. B. Lampe, G. Kramer: Physica Scripta **28**, 585 (1983)
11. Z. Kunszt: Phys. Lett. **99B**, 429 (1981); Phys. Lett.: **107B**, 123 (1981); A. Ali: Phys. Lett. **110B**, 67 (1982); B. Barreiro: Phys. Lett. **118B**, 155 (1982); Nucl. Phys. **B236**, 269 (1984)
12. F. Gutbrod, G. Kramer, G. Schierholz: Z. Phys. C – Particles and Fields **21**, 235 (1984)
13. G. Kramer, B. Lampe: Commun. Math. Phys. **97**, 257 (1985)
14. H.H. Schneider, G. Kramer, G. Schierholz: Z. Phys. C – Particles and Fields **22**, 201 (1984)
15. T.D. Gottschalk, M.P. Shatz: CALT-68-1172 (1984)
16. J.G. Körner, G. Schuler: Z. Phys. C – Particles and Fields **26**, 559 (1985)
17. J.G. Körner, G. Schuler, G. Kramer, B. Lampe: Z. Phys. C – Particles and Fields **32**, 181 (1981)
18. T.D. Gottschalk: Phys. Lett. **109B**, 331 (1982)
19. A. Ali et al.: Nucl. Phys. **B167**, 454 (1980)
20. G. Kramer, B. Lampe: DESY 86-119 (1986)
21. G. Sterman, S. Weinberg: Phys. Rev. Lett. **39**, 1436 (1977)
22. H.J. Daum, H. Meyer, J. Bürger: Z. Phys. C – Particles and Fields **8**, 167 (1981); J. Dorfan, Z. Phys. C – Particles and Fields **7**, 347 (1981); A. Bäcker, Z. Phys. C – Particles and Fields **12**, 161 (1982); W. Bartel et al.: Phys. Lett. **119B**, 239 (1982)
23. M. Dine, J. Sapirstein: Phys. Rev. Lett. **43**, 668 (1979); W. Celmaster, R.J. Gonsalves: Phys. Rev. Lett. **44**, 560 (1980); Phys. Rev. **D21**, 3112 (1980); K.G. Chetyrkin, A.L. Kataev, F.V. Tachov: Phys. Lett. **85**, 277 (1979)
24. G. Kramer, B. Lampe: DESY 86-038 (1986)
25. J.C. Collins: ANL-HEP-PR-84-36 and the earlier literature given there
26. R.J. Gonsalves: Phys. Rev. **D28**, 1542 (1983) and the earlier literature given there
27. R. Gastmans, W. Troost: SIMON STEVIN **55**, 205 (1981); A. Devoto, D.W. Duke: Tallahassee preprint FSU-HEP-83 1003
28. W. Celmaster, R.J. Gonsalves: Phys. Rev. **D20**, 1429 (1979)
29. G. Kramer: Springer Tracts in Modern Physics, Vol. 102. Berlin, Heidelberg, New York: Springer 1984
30. A.V. Smilga: Nucl. Phys. **B161**, 449 (1979)
31. J. Frenkel, R. Meulderman, I. Mohammad, J.C. Taylor: Nucl. Phys. **B121**, 58 (1977)