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TOPOLOGY OF SU(3) LATTICE GAUGE THEORY
First calculation of the topological susceptibility

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Using a section corresponding to Lüscher's bundle, we develop a method for computing the second Chern number of SU(3) lattice gauge fields. A key ingredient is gauge fixing, which ensures a smooth section, so that the (numerical) evaluation of winding numbers is reasonably fast. We employ the algorithm on 4^4 and 5^4 lattices and present first results for the topological susceptibility χ_t.

1. Introduction

Recent numerical calculations [1–4] of the topological susceptibility

$$\chi_t = \langle Q^2 \rangle / V \tag{1.1}$$

(where Q is the topological charge, or the second Chern number, of a gauge field configuration, and V is the volume of the space-time manifold) in SU(2) lattice gauge theory have led to a semi-quantitative resolution of the $U_A(1)$ problem. It is of

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urgent interest to extend these calculations to the physically relevant gauge group SU(3). For the SU(3) topological charge there is no combinatoric algorithm though, and the computation of Lüscher's charge [5] requires – as it stands – too much computer time.

One might be tempted to seek short cuts such as "cooling" [6]. The validity of such a procedure is, however, dubious since topological charge disappears during the cooling process. (Even semi-classical arguments suggest that meron pairs, and not instantons alone, drive confinement [7]. Meron pair configurations do not minimize the action and hence are unstable under cooling. For recent work indicating that the vacuum is indeed more complex than just a dilute gas of instantons, see ref. [8].)

In this paper we rewrite Lüscher's bundle [5] in terms of a section. Combined with appropriate gauge fixing, the section allows us to determine Q (on small lattices) reasonably fast. Since we deal directly with the second Chern number, our computation of χ_t is (in the context of numerical simulations) manifestly correct.

Our calculations are based on a hypercubic lattice. This is defined by

$$\Lambda = \left\{ s \in \mathrm{T}^4 | s_\mu \in \mathbb{Z}, \mu = 0,1,2,3 \right\}, \tag{1.2}$$

where the 4-torus T^4 is covered by hypercubes,

$$\mathrm{T}^4 = \bigcup_{s \in \Lambda} \mathrm{c}(s), \tag{1.3}$$

with

$$\mathrm{c}(s) = \left\{ x \in \mathrm{T}^4 | s_\mu \leq x_\mu \leq s_\mu + 1 \right\}. \tag{1.4}$$

In each hypercube we transform the gauge fields into a nonsingular gauge by

$$A^s_\mu = w^s (A_\mu + \partial_\mu)(w^s)^{-1}. \tag{1.5}$$

The maps w^s define a section on the boundaries $\partial \mathrm{c}(s)$ of the hypercubes. Following ref. [4] we derive

$$Q = \sum_{s \in \Lambda} Q_s, \tag{1.6}$$

with

$$Q_s = -\frac{1}{24\pi^2} \int_{\partial \mathrm{c}(s)} \mathrm{d}^3\sigma_\mu \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[(w^s)^{-1} \partial_\nu w^s (w^s)^{-1} \partial_\rho w^s (w^s)^{-1} \partial_\sigma w^s \right]. \tag{1.7}$$

Since Q_s is the winding number of w^s on $\partial \mathrm{c}(s)$, i.e.

$$Q_s \in \pi_3(\mathrm{SU}(3)) = \mathbb{Z}, \tag{1.8}$$

Q is the sum of local integers.

Under gauge transformations g the gauge field A_μ transforms as

$$\bar{A}_\mu = g(A_\mu + \partial_\mu)g^{-1}. \tag{1.9}$$

Bringing $\bar{A}_\mu$ into the same nonsingular gauge as before requires a gauge transformation $\bar{w}^s$,

$$A^s_\mu = \bar{w}^s(\bar{A}_\mu + \partial_\mu)(\bar{w}^s)^{-1}, \tag{1.10}$$

where

$$\bar{w}^s = w^s g^{-1}. \tag{1.11}$$

By inserting $w^s = \bar{w}^s g$ into eq. (1.7) one finds that the (total) charge Q is gauge invariant, while its local terms Q_s are not.

In SU(2) the integral in eq. (1.7) can be done analytically, which requires locating the gauge singularities of the 1-cochain [4, 9, 10] only. On a simplicial lattice one can use geometrical methods to do so, which has led to the fast combinatoric algorithm of Phillips and Stone [11]. In SU(3) one can proceed along the same lines: an analogous expression for the 1-cochain has been given in ref. [12]. However, it has not been possible to locate its gauge singularities geometrically, mainly because of the complicated geometry of the SU(3) group manifold. Instead, we shall evaluate eq. (1.7) by numerical integration.

An alternative to the section is to describe the bundle in terms of transition functions

$$v_{s,\mu} = w^{s-\hat{\mu}}(w^s)^{-1}, \tag{1.12}$$

which are defined on the faces

$$\mathrm{f}(s,\mu) = \mathrm{c}(s) \cap \mathrm{c}(s-\hat{\mu}). \tag{1.13}$$

On the plaquettes

$$\mathrm{p}(s,\mu,\nu) = \mathrm{c}(s) \cap \mathrm{c}(s-\hat{\mu}) \cap \mathrm{c}(s-\hat{\nu}) \tag{1.14}$$

they obey the cocycle condition

$$v_{s-\hat{\mu},\nu} v_{s,\mu} = v_{s-\hat{\nu},\mu} v_{s,\nu} \tag{1.15}$$

and lead to the expression for the charge [5, 4]

$$Q = -\frac{1}{24\pi^2} \sum_{s\in\Lambda} \left\{ \int_{\mathrm{f}(s,\mu)} \mathrm{d}^3x_\mu \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[v_{s,\mu}^{-1} \partial_\nu v_{s,\mu} v_{s,\mu}^{-1} \partial_\rho v_{s,\mu} v_{s,\mu}^{-1} \partial_\sigma v_{s,\mu} \right] + 3 \int_{\mathrm{p}(s,\mu,\nu)} \mathrm{d}^2x_{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[v_{s,\mu} \partial_\rho v_{s,\mu}^{-1} v_{s-\hat{\mu},\nu}^{-1} \partial_\sigma v_{s-\hat{\mu},\nu} \right] \right\}. \tag{1.16}$$

In contrast to eq. (1.7), Lüscher's charge (1.16) yields an integer only after summing over all hypercubes.

The rest of the paper is organized as follows. In sect. 2 we construct the section for Lüscher's bundle. Sect. 3 shows that the charge (1.6) is gauge invariant. In sect. 4 we prove that our and Lüscher's bundle are indeed the same. A necessary ingredient to our calculation is fixing the gauge, which we discuss in sect. 5. In sect. 6 we describe the method of integration and our results. The integrand involves powers of SU(3) matrices, which are computed by the algorithm outlined in the appendix. Finally, in sect. 7 we conclude with some remarks.

2. Construction of the section

On the lattice the gauge fields are represented by the parallel transporters $U(s, \hat{\mu})$. Following Lüscher [5] we transform the gauge fields to a nonsingular, complete axial gauge by

$$U^s_{xy} = w^s(x) U(x, \hat{\mu}) w^s(y)^{-1} \qquad \text{for } y = x + \hat{\mu},$$

$$U^s_{xy} = \left(U^s_{yx}\right)^{-1} \qquad \text{for } y = x - \hat{\mu}. \tag{2.1}$$

The gauge transformation is given by

$$w^s(x) = U(s,\hat{1})^{y_1} U(s + y_1\hat{1}, \hat{2})^{y_2} U(s + y_1\hat{1} + y_2\hat{2}, \hat{3})^{y_3} U(s + y_1\hat{1} + y_2\hat{2} + y_3\hat{3}, \hat{4})^{y_4}, \tag{2.2}$$

where

$$x = s + \sum_{\mu=1}^{4} y_\mu \hat{\mu}, \qquad y_\mu \in \{0,1\} \tag{2.3}$$

are the corners of the hypercube $c(s)$.

For the computation of eq. (1.7) the section w^s, which by eq. (2.2) is only given at the corners of the hypercube, needs to be interpolated throughout $\partial c(s)$. The latter consists of 8 faces, $f(s, \mu)$ and $f(s + \hat{\mu}, \mu)$, which are cubes extended in directions

$$\alpha < \beta < \gamma \in \{1,2,3,4\} \setminus \{\mu\} \tag{2.4}$$

complementary to the direction μ. The corners of the hypercube are labelled

$$s \mathrel{\hat{=}} 1, \qquad s + \hat{\alpha} \mathrel{\hat{=}} 2, \qquad s + \hat{\beta} \mathrel{\hat{=}} 3, \qquad s + \hat{\alpha} + \hat{\beta} \mathrel{\hat{=}} 4,$$

$$s + \hat{\gamma} \mathrel{\hat{=}} 5, \qquad s + \hat{\alpha} + \hat{\gamma} \mathrel{\hat{=}} 6, \qquad s + \hat{\beta} + \hat{\gamma} \mathrel{\hat{=}} 7, \qquad s + \hat{\alpha} + \hat{\beta} + \hat{\gamma} \mathrel{\hat{=}} 8 \tag{2.5}$$

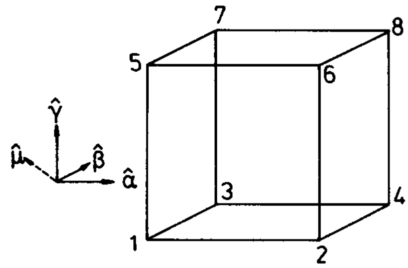

Fig. 1. Labelling of the corners of the hypercube.

as shown in fig. 1. We take the following interpolation of the section for $x \in \mathrm{f}(s, \mu)$

$$w^t(s_\alpha, s_\beta, x_\gamma) = (U^t_{51})^{y_\gamma} \left[U^t_{15} w^t(s_\alpha, s_\beta, s_\gamma + 1) w^t(s_\alpha, s_\beta, s_\gamma)^{-1} \right]^{y_\gamma} w^t(s_\alpha, s_\beta, s_\gamma),$$

$$w^t(s_\alpha + 1, s_\beta, x_\gamma) = (U^t_{62})^{y_\gamma} \left[U^t_{26} w^t(s_\alpha + 1, s_\beta, s_\gamma + 1) w^t(s_\alpha + 1, s_\beta, s_\gamma)^{-1} \right]^{y_\gamma} \times w^t(s_\alpha + 1, s_\beta, s_\gamma),$$

$$w^t(s_\alpha, s_\beta + 1, x_\gamma) = (U^t_{73})^{y_\gamma} \left[U^t_{37} w^t(s_\alpha, s_\beta + 1, s_\gamma + 1) w^t(s_\alpha, s_\beta + 1, s_\gamma)^{-1} \right]^{y_\gamma} \times w^t(s_\alpha, s_\beta + 1, s_\gamma),$$

$$w^t(s_\alpha + 1, s_\beta + 1, x_\gamma) = (U^t_{84})^{y_\gamma} \left[U^t_{48} w^t(s_\alpha + 1, s_\beta + 1, s_\gamma + 1) w^t(s_\alpha + 1, s_\beta + 1, s_\gamma)^{-1} \right]^{y_\gamma} \times w^t(s_\alpha + 1, s_\beta + 1, s_\gamma),$$

$$w^t(s_\alpha, x_\beta, x_\gamma) = \left[f^t_{s,\mu}(x_\gamma)^{-1} \right]^{y_\beta} \left[f^t_{s,\mu}(x_\gamma) w^t(s_\alpha, s_\beta + 1, x_\gamma) w^t(s_\alpha, s_\beta, x_\gamma)^{-1} \right]^{y_\beta} \times w^t(s_\alpha, s_\beta, x_\gamma),$$

$$w^t(s_\alpha + 1, x_\beta, x_\gamma) = \left[g^t_{s,\mu}(x_\gamma)^{-1} \right]^{y_\beta} \left[g^t_{s,\mu}(x_\gamma) w^t(s_\alpha + 1, s_\beta + 1, x_\gamma) \times w^t(s_\alpha + 1, s_\beta, x_\gamma)^{-1} \right]^{y_\beta} w^t(s_\alpha + 1, s_\beta, x_\gamma),$$

$$w^t(x_\alpha, x_\beta, x_\gamma) = \left[l^t_{s,\mu}(x_\beta, x_\gamma) \right]^{y_\alpha} \left[l^t_{s,\mu}(x_\beta, x_\gamma) w^t(s_\alpha + 1, x_\beta, x_\gamma) \times w^t(s_\alpha, x_\beta, x_\gamma)^{-1} \right]^{y_\alpha} w^t(s_\alpha, x_\beta, x_\gamma), \quad (2.6)$$

where $y = x - s$ and $t = s$ or $t = s - \hat{\mu}$. Eq. (2.6) interpolates first in the γ, then in the β, and finally in the α direction using interpolating functions [5]

$$f_{s,\mu}^{t}(x_\gamma) = (U_{51}^{t})^{y_\gamma} (U_{15}^{t} U_{57}^{t} U_{73}^{t} U_{31}^{t})^{y_\gamma} U_{13}^{t} (U_{37}^{t})^{y_\gamma},$$

$$g_{s,\mu}^{t}(x_\gamma) = (U_{62}^{t})^{y_\gamma} (U_{26}^{t} U_{68}^{t} U_{84}^{t} U_{42}^{t})^{y_\gamma} U_{24}^{t} (U_{48}^{t})^{y_\gamma},$$

$$h_{s,\mu}^{t}(x_\gamma) = (U_{51}^{t})^{y_\gamma} (U_{15}^{t} U_{56}^{t} U_{62}^{t} U_{21}^{t})^{y_\gamma} U_{12}^{t} (U_{26}^{t})^{y_\gamma},$$

$$k_{s,\mu}^{t}(x_\gamma) = (U_{73}^{t})^{y_\gamma} (U_{37}^{t} U_{78}^{t} U_{84}^{t} U_{43}^{t})^{y_\gamma} U_{34}^{t} (U_{48}^{t})^{y_\gamma},$$

$$l_{s,\mu}^{t}(x_\beta, x_\gamma) = \left[f_{s,\mu}^{t}(x_\gamma)^{-1} \right]^{y_\beta} \left[f_{s,\mu}^{t}(x_\gamma) k_{s,\mu}^{t}(x_\gamma) g_{s,\mu}^{t}(x_\gamma)^{-1} h_{s,\mu}^{t}(x_\gamma)^{-1} \right]^{y_\beta}$$
$$\times h_{s,\mu}^{t}(x_\gamma) \left[g_{s,\mu}^{t}(x_\gamma) \right]^{y_\beta}, \tag{2.7}$$

which ensure gauge invariance of the topological charge Q, as we show below. As one can easily check, the section $w^s(x)$ is continuous on $\partial c(s)$. Below we also show that it leads to the correct continuum limit of Q.

By virtue of eq. (2.1), eqs. (2.6) and (2.7) simplify to

$$w^t(s_\alpha, s_\beta, x_\gamma) = w^t(s_\alpha, s_\beta, s_\gamma) \left[w^t(s_\alpha, s_\beta, s_\gamma)^{-1} w^t(s_\alpha, s_\beta, s_\gamma + 1) U^+(1, \hat{\gamma}) \right]^{y_\gamma} \left[U(1, \hat{\gamma}) \right]^{y_\gamma},$$

$$w^t(s_\alpha + 1, s_\beta, x_\gamma) = w^t(s_\alpha + 1, s_\beta, s_\gamma) \left[w^t(s_\alpha + 1, s_\beta, s_\gamma)^{-1} w^t(s_\alpha + 1, s_\beta, s_\gamma + 1) U^+(2, \hat{\gamma}) \right]^{y_\gamma}$$
$$\times \left[U(2, \hat{\gamma}) \right]^{y_\gamma},$$

$$w^t(s_\alpha, s_\beta + 1, x_\gamma) = w^t(s_\alpha, s_\beta + 1, s_\gamma) \left[w^t(s_\alpha, s_\beta + 1, s_\gamma)^{-1} w^t(s_\alpha, s_\beta + 1, s_\gamma + 1) U^+(3, \hat{\gamma}) \right]^{y_\gamma}$$
$$\times \left[U(3, \hat{\gamma}) \right]^{y_\gamma},$$

$$w^t(s_\alpha + 1, s_\beta + 1, x_\gamma) = w^t(s_\alpha + 1, s_\beta + 1, s_\gamma) \left[w^t(s_\alpha + 1, s_\beta + 1, s_\gamma)^{-1} \right.$$
$$\left. \times w^t(s_\alpha + 1, s_\beta + 1, s_\gamma + 1) U^+(4, \hat{\gamma}) \right]^{y_\gamma} \left[U(4, \hat{\gamma}) \right]^{y_\gamma},$$

$$w^t(s_\alpha, x_\beta, x_\gamma) = w^t(s_\alpha, s_\beta, x_\gamma) \left[w^t(s_\alpha, s_\beta, x_\gamma)^{-1} w^t(s_\alpha, s_\beta + 1, x_\gamma) F_{s,\mu}^+(x_\gamma) \right]^{y_\beta} \left[F_{s,\mu}(x_\gamma) \right]^{y_\beta},$$

$$w^t(s_\alpha + 1, x_\beta, x_\gamma) = w^t(s_\alpha + 1, s_\beta, x_\gamma) \left[w^t(s_\alpha + 1, s_\beta, x_\gamma)^{-1} w^t(s_\alpha + 1, s_\beta + 1, x_\gamma) G_{s,\mu}^+(x_\gamma) \right]^{y_\beta}$$
$$\times \left[G_{s,\mu}(x_\gamma) \right]^{y_\beta},$$

$$w^t(x_\alpha, x_\beta, x_\gamma) = w^t(s_\alpha, x_\beta, x_\gamma) \left[w^t(s_\alpha, x_\beta, x_\gamma)^{-1} w^t(s_\alpha + 1, x_\beta, x_\gamma) L_{s,\mu}^+(x_\beta, x_\gamma) \right]^{y_\alpha}$$
$$\times \left[L_{s,\mu}(x_\beta, x_\gamma) \right]^{y_\alpha}, \tag{2.8}$$

with

$$F_{s,\mu}(x_\gamma) = [U^+(1,\hat{\gamma})]^{y_\gamma}[U(1,\hat{\gamma})U(5,\hat{\beta})U^+(3,\hat{\gamma})U^+(1,\hat{\beta})]^{y_\gamma}U(1,\hat{\beta})[U(3,\hat{\gamma})]^{y_\gamma},$$
$$G_{s,\mu}(x_\gamma) = [U^+(2,\hat{\gamma})]^{y_\gamma}[U(2,\hat{\gamma})U(6,\hat{\beta})U^+(4,\hat{\gamma})U^+(2,\hat{\beta})]^{y_\gamma}U(2,\hat{\beta})[U(4,\hat{\gamma})]^{y_\gamma},$$
$$H_{s,\mu}(x_\gamma) = [U^+(1,\hat{\gamma})]^{y_\gamma}[U(1,\hat{\gamma})U(5,\hat{\alpha})U^+(2,\hat{\gamma})U^+(1,\hat{\alpha})]^{y_\gamma}U(1,\hat{\alpha})[U(2,\hat{\gamma})]^{y_\gamma},$$
$$K_{s,\mu}(x_\gamma) = [U^+(3,\hat{\gamma})]^{y_\gamma}[U(3,\hat{\gamma})U(7,\hat{\alpha})U^+(4,\hat{\gamma})U^+(3,\hat{\alpha})]^{y_\gamma}U(3,\hat{\alpha})[U(4,\hat{\gamma})]^{y_\gamma},$$
$$L_{s,\mu}(x_\gamma) = [F^+_{s,\mu}(x_\gamma)]^{y_\beta}[F_{s,\mu}(x_\gamma)K_{s,\mu}(x_\gamma)G^+_{s,\mu}(x_\gamma)H^+_{s,\mu}(x_\gamma)]^{y_\beta} \times H_{s,\mu}(x_\gamma)[G_{s,\mu}(x_\gamma)]^{y_\beta}. \tag{2.9}$$

This is the expression used in the numerical work.

3. Gauge invariance

The section w^s constructed above leads to a gauge invariant topological charge. Under gauge transformations the link matrices transform as

$$\overline{U}(s,\hat{\mu}) = g(s)U(s,\hat{\mu})g(s+\hat{\mu})^{-1}. \tag{3.1}$$

This implies the following transformation properties of the interpolating functions (2.9):

$$\overline{F}_{s,\mu}(x_\gamma) = g(s_\alpha, s_\beta, x_\gamma)F_{s,\mu}(x_\gamma)g(s_\alpha, s_\beta+1, x_\gamma)^{-1},$$
$$\overline{G}_{s,\mu}(x_\gamma) = g(s_\alpha+1, s_\beta, x_\gamma)G_{s,\mu}(x_\gamma)g(s_\alpha+1, s_\beta+1, x_\gamma)^{-1},$$
$$\overline{L}_{s,\mu}(x_\beta, x_\gamma) = g(s_\alpha, x_\beta, x_\gamma)L_{s,\mu}(x_\beta, x_\gamma)g(s_\alpha+1, x_\beta, x_\gamma)^{-1}, \tag{3.2}$$

with

$$g(s_\alpha, s_\beta, x_\gamma) = g(s_\alpha, s_\beta, s_\gamma)\left[g(s_\alpha, s_\beta, s_\gamma)^{-1}g(s_\alpha, s_\beta, s_\gamma+1)U^+(1,\hat{\gamma})\right]^{y_\gamma}[U(1,\hat{\gamma})]^{y_\gamma},$$
$$g(s_\alpha+1, s_\beta, x_\gamma) = g(s_\alpha+1, s_\beta, s_\gamma)\left[g(s_\alpha+1, s_\beta, s_\gamma)^{-1}g(s_\alpha+1, s_\beta, s_\gamma+1)U^+(2,\hat{\gamma})\right]^{y_\gamma} \times[U(2,\hat{\gamma})]^{y_\gamma},$$
$$g(s_\alpha, s_\beta+1, x_\gamma) = g(s_\alpha, s_\beta+1, s_\gamma)\left[g(s_\alpha, s_\beta+1, s_\gamma)^{-1}g(s_\alpha, s_\beta+1, s_\gamma+1)U^+(3,\hat{\gamma})\right]^{y_\gamma} \times[U(3,\hat{\gamma})]^{y_\gamma},$$
$$g(s_\alpha+1, s_\beta+1, x_\gamma) = g(s_\alpha+1, s_\beta+1, s_\gamma)\left[g(s_\alpha+1, s_\beta+1, s_\gamma)^{-1} \times g(s_\alpha+1, s_\beta+1, s_\gamma+1)U^+(4,\hat{\gamma})\right]^{y_\gamma}[U(4,\hat{\gamma})]^{y_\gamma},$$
$$g(s_\alpha, x_\beta, x_\gamma) = g(s_\alpha, s_\beta, x_\gamma)\left[g(s_\alpha, s_\beta, x_\gamma)^{-1}g(s_\alpha, s_\beta+1, x_\gamma)F^+_{s,\mu}(x_\gamma)\right]^{y_\beta} \times[F_{s,\mu}(x_\gamma)]^{y_\beta},$$
$$g(s_\alpha+1, x_\beta, x_\gamma) = g(s_\alpha+1, s_\beta, x_\gamma)\left[g(s_\alpha+1, s_\beta, x_\gamma)^{-1} \times g(s_\alpha+1, s_\beta+1, x_\gamma)G^+_{s,\mu}(x_\gamma)\right]^{y_\beta}[G_{s,\mu}(x_\gamma)]^{y_\beta},$$
$$g(x_\alpha, x_\beta, x_\gamma) = g(s_\alpha, x_\beta, x_\gamma)\left[g(s_\alpha, x_\beta, x_\gamma)^{-1}g(s_\alpha+1, x_\beta, x_\gamma)L^+_{s,\mu}(x_\beta, x_\gamma)\right]^{y_\alpha} \times[L_{s,\mu}(x_\beta, y_\gamma)]^{y_\alpha}. \tag{3.3}$$

These formulae determine the transformation law for the section:

$$\bar{w}^{s}(x_\alpha, x_\beta, x_\gamma) = g(s) w^{s}(x_\alpha, x_\beta, x_\gamma) g(x_\alpha, x_\beta, x_\gamma)^{-1}. \tag{3.4}$$

Eq. (3.4) has exactly the form (1.11) apart from the constant factor $g(s)$, which is the gauge transformation at the origin of c(s), and it drops out in the expression (1.7) for the topological charge.

4. Equivalence to Lüscher's charge

The section w^s defines transition functions

$$v_{s,\mu}(x) = w^{s-\hat{\mu}}(x) w^{s}(x)^{-1} \tag{4.1}$$

on the faces f(s, μ). They carry the same topological information as w^s. From eq. (2.6) we derive

$$v_{s,\mu}(s_\alpha, s_\beta, x_\gamma) = v_{s,\mu}(s_\alpha, s_\beta, s_\gamma)\left[v_{s,\mu}(s_\alpha, s_\beta, s_\gamma)^{-1} v_{s,\mu}(s_\alpha, s_\beta, s_\gamma + 1) U^{s}_{51}\right]^{y_\gamma} (U^{s}_{15})^{y_\gamma},$$

$$v_{s,\mu}(s_\alpha + 1, s_\beta, x_\gamma) = v_{s,\mu}(s_\alpha + 1, s_\beta, s_\gamma)\left[v_{s,\mu}(s_\alpha + 1, s_\beta, s_\gamma)^{-1} v_{s,\mu}(s_\alpha + 1, s_\beta, s_\gamma + 1) U^{s}_{62}\right]^{y_\gamma} (U^{s}_{26})^{y_\gamma},$$

$$v_{s,\mu}(s_\alpha, s_\beta + 1, x_\gamma) = v_{s,\mu}(s_\alpha, s_\beta + 1, s_\gamma)\left[v_{s,\mu}(s_\alpha, s_\beta + 1, s_\gamma)^{-1} v_{s,\mu}(s_\alpha, s_\beta + 1, s_\gamma + 1) U^{s}_{73}\right]^{y_\gamma} (U^{s}_{37})^{y_\gamma}$$

$$v_{s,\mu}(s_\alpha + 1, s_\beta + 1, x_\gamma) = v_{s,\mu}(s_\alpha + 1, s_\beta + 1, s_\gamma)\Big[v_{s,\mu}(s_\alpha + 1, s_\beta + 1, s_\gamma)^{-1}$$
$$\times v_{s,\mu}(s_\alpha + 1, s_\beta + 1, s_\gamma + 1) U^{s}_{84}\Big]^{y_\gamma} (U^{s}_{48})^{y_\gamma},$$

$$v_{s,\mu}(s_\alpha, x_\beta, x_\gamma) = v_{s,\mu}(s_\alpha, s_\beta, x_\gamma)\left[v_{s,\mu}(s_\alpha, s_\beta, x_\gamma)^{-1} v_{s,\mu}(s_\alpha, s_\beta + 1, x_\gamma) f^{s}_{s,\mu}(x_\gamma)^{-1}\right]^{y_\beta} [f^{s}_{s,\mu}(x_\gamma)]^{y_\beta},$$

$$v_{s,\mu}(s_\alpha + 1, x_\beta, x_\gamma) = v_{s,\mu}(s_\alpha + 1, s_\beta, x_\gamma)\Big[v_{s,\mu}(s_\alpha + 1, s_\beta, x_\gamma)^{-1}$$
$$\times v_{s,\mu}(s_\alpha + 1, s_\beta + 1, x_\gamma) g^{s}_{s,\mu}(x_\gamma)^{-1}\Big]^{y_\beta} [g^{s}_{s,\mu}(x_\gamma)]^{y_\beta},$$

$$v_{s,\mu}(x_\alpha, x_\beta, x_\gamma) = v_{s,\mu}(s_\alpha, x_\beta, x_\gamma)\Big[v_{s,\mu}(s_\alpha, x_\beta, x_\gamma)^{-1}$$
$$\times v_{s,\mu}(s_\alpha + 1, x_\beta, x_\gamma) l^{s}_{s,\mu}(x_\beta, x_\gamma)^{-1}\Big]^{y_\alpha} [l^{s}_{s,\mu}(x_\beta, x_\gamma)]^{y_\alpha}. \tag{4.2}$$

It is tedious but trivial to verify that expressions (4.2) fulfill the cocycle condition (1.15).

A straightforward manipulation of eq. (4.2) gives

$$v_{s,\mu}(s_\alpha, s_\beta, x_\gamma) = (U_{51}^{s-\hat{\mu}})^{y_\gamma} v_{s,\mu}(s_\alpha, s_\beta, s_\gamma)(U_{15}^{s})^{y_\gamma},$$

$$v_{s,\mu}(s_\alpha, x_\beta, x_\gamma) = \left[f_{s,\mu}^{s-\hat{\mu}}(x_\gamma)^{-1}\right]^{y_\beta} v_{s,\mu}(s_\alpha, s_\beta, x_\gamma)\left[f_{s,\mu}^{s}(x_\gamma)\right]^{y_\beta},$$

$$v_{s,\mu}(x_\alpha, x_\beta, x_\gamma) = \left[l_{s,\mu}^{s-\hat{\mu}}(x_\beta, x_\gamma)^{-1}\right]^{y_\alpha} v_{s,\mu}(s_\alpha, x_\beta, x_\gamma)\left[l_{s,\mu}^{s}(x_\beta, x_\gamma)\right]^{y_\alpha}, \quad (4.3)$$

which immediately leads to

$$v_{s,\mu}(x_\alpha, x_\beta, x_\gamma) = S_{s,\mu}^{s-\hat{\mu}}(x_\alpha, x_\beta, x_\gamma)^{-1} v_{s,\mu}(s_\alpha, s_\beta, s_\gamma) S_{s,\mu}^{s}(x_\alpha, x_\beta, x_\gamma) \quad (4.4)$$

with

$$S_{s,\mu}^{t}(x_\alpha, x_\beta, x_\gamma) = (U_{15}^{t})^{y_\gamma}\left[f_{s,\mu}^{t}(x_\gamma)\right]^{y_\beta}\left[l_{s,\mu}^{t}(x_\beta, x_\gamma)\right]^{y_\alpha}. \quad (4.5)$$

This is exactly Lüscher's definition of the interpolated transition functions, which proves that our section w^s defines the same bundle as Lüscher's. Moreover, the equivalence shows that w^s has the correct continuum limit.

5. Gauge fixing

In order to be able to integrate eq. (1.7) numerically with a minimum of mesh points, we have to make the section w^s as smooth as possible. This may be achieved by gauge transforming the original gauge fields into a lattice Landau gauge. Quantitatively, this means minimizing

$$T = \sum_{s,\mu} \left\{1 - \tfrac{1}{3}\mathrm{Re}\,\mathrm{Tr}[U(s,\hat{\mu})]\right\} \quad (5.1)$$

by gauge transforming the link matrices. This is done iteratively by going to each lattice point and constructing a gauge transformation $g(s)$. First choose an SU(2) subgroup characterized by the indices

$$k, l \in \{1,2,3\}, \qquad k \neq l \quad (5.2)$$

such that

$$g_{ij}(s) = \begin{cases} g_{ij}^{(2)}(s) & \text{for } i, j \in \{k, l\} \\ \delta_{ij} & \text{otherwise}, \end{cases} \quad (5.3)$$

with

$$g^{(2)}(s) = \begin{pmatrix} g_{kk}(s) & g_{kl}(s) \\ g_{lk}(s) & g_{ll}(s) \end{pmatrix} \in \mathrm{SU}(2). \quad (5.4)$$

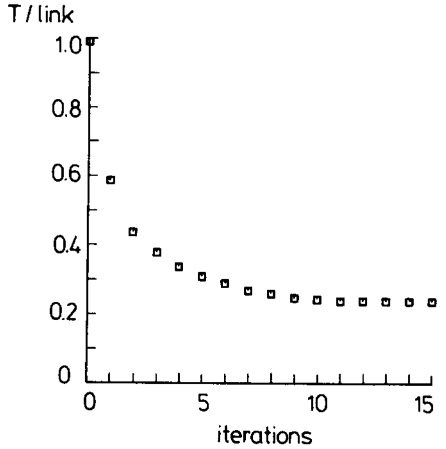

Fig. 2. T per link versus successive gauge fixing iterations for a typical gauge field configuration.

If we write

$$U^{(2)}(s,\hat{\mu}) = \begin{pmatrix} U_{kk}(s,\hat{\mu}) & U_{kl}(s,\hat{\mu}) \\ U_{lk}(s,\hat{\mu}) & U_{ll}(s,\hat{\mu}) \end{pmatrix}, \tag{5.5}$$

the variation of T under the gauge transformation (5.3) is

$$\Delta T = -\sum_{\mu} \tfrac{1}{3}\mathrm{Re\,Tr}\left[g^{(2)}(s) U^{(2)}(s,\hat{\mu}) + U^{(2)}(s-\hat{\mu},\hat{\mu}) g^{(2)}(s)^{-1} \right]. \tag{5.6}$$

To obtain $g^{(2)}(s)$ one simply has to minimize ΔT, which is elementary, and the steps will be omitted here. Minimizing T requires covering all SU(2) subgroups and sweeping through the lattice several times. In fig. 2 we have shown the history of T under successive gauge fixing iterations.

In the continuum limit this procedure reduces to $\partial \cdot A = 0$, hence the name Landau gauge. On larger lattices and at higher values of β, Landau gauge fixing is critically slowed down. However, Fourier acceleration [13] can mitigate this problem [14].

6. Method and results

We integrate eq. (1.7) numerically by covering the faces with a regular mesh of points at which we compute the section w^s, and then apply standard methods to obtain the integral. The computation of the section requires raising SU(3) matrices to powers between 0 and 1. This is done using the Cayley-Hamilton method [15], which is described in the appendix. We treat each hypercube separately and increase

TABLE 1
Compilation of $a^4\chi_t$ from this work

β	V	$a^4\chi_t$
5.5	4^4	$(4.07 \pm 2.12) \times 10^{-2}$
5.6	4^4	$(4.22 \pm 2.26) \times 10^{-3}$
5.6	5^4	$(8.40 \pm 4.48) \times 10^{-3}$
5.7	4^4	$(0.66 \pm 0.66) \times 10^{-3}$
5.7	5^4	$(2.00 \pm 0.40) \times 10^{-3}$

the number of mesh points until we can uniquely identify the integer Q_s. We emphasize that it is vital for our computation that the integral yields integers for each hypercube. In practice the integral converges very fast for most of the hypercubes, increasingly so as β increases, thanks to the gauge fixing. This algorithm demands about as much CPU time as it took to compute Lüscher's charge in SU(2) [1]. This is not especially fast, but it is sufficient for a first calculation of the SU(3) topological susceptibility. As in ref. [1], we are limited to small lattices, but the topological foundations are sound.

We have computed the topological susceptibility on 4^4 and 5^4 lattices at $\beta = 5.5$, 5.6 and 5.7. Our results are presented in table 1. Each entry is the average over 12 gauge field configurations. In fig. 3, we plot $a^4\chi_t$ as a function of β. The curve is the 2-loop renormalization group formula for the lattice spacing a,

$$a = \Lambda_L^{-1}\left(\tfrac{8}{33}\pi^2\beta\right)^{51/121}\exp\left(-\tfrac{4}{33}\pi^2\beta\right), \tag{6.1}$$

raised to the 4th power and normalized to the $V = 5^4$, $\beta = 5.6$ value of $a^4\chi_t$. We

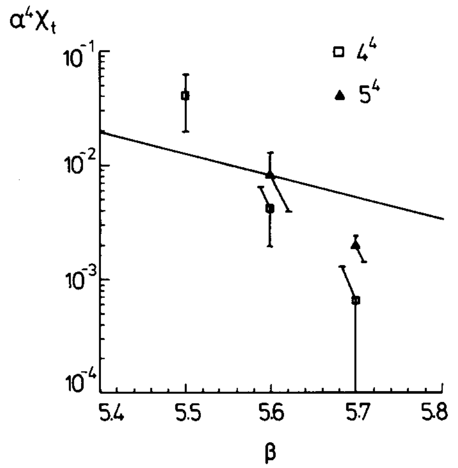

Fig. 3. The topological susceptibility $a^4\chi_t$ as a function of β.

find the β and volume dependences in agreement with what one expects on small lattices: the scaling violations are stronger than those seen in the string tension simulation [16]. With some reservation we finally offer the topological susceptibility in physical units. Using the $V = 5^4$, $\beta = 5.6$ result, the string tension calculation of ref. [16] and $\sqrt{K} = 400$ MeV we obtain

$$\chi_t = \left(247 \pm ^{28}_{43} \text{ MeV}\right)^4. \qquad (6.2)$$

This value is of the anticipated order of magnitude and in close agreement with the SU(2) result [4]. The latter is maybe not really surprising.

7. Conclusions and outlook

Two years ago the computation of the topological charge of SU(2) lattice gauge fields was barely feasible, and that of SU(3) was clearly beyond reach. Since then there has been remarkable progress for both gauge groups. In each case use of the section, i.e., eq. (1.7), rather than the transition functions, eq. (1.16), proved crucial. For SU(2) the problem has been reduced to combinatorics [11], making possible large scale studies [3,4]. Lacking the analogous algorithm for SU(3), however, forces us to tackle eq. (1.17) essentially with brute force. Previously, we [10,4] and others [17,18] have expressed the hope of applying reduction of the gauge group from SU(3) to SU(2). Unfortunately, the section is defined, here and in ref. [11], in terms of the link matrices, and interpolations based on reduction conflict with the gauge transformation law of the link field. Since this finesse fails, we instead exploit the gauge freedom, somewhat similarly to ref. [2], to construct an especially smooth w^s. Then the numerical integration of (1.7) is accurate even with a coarse mesh. At least on 4^4 and 5^4 lattices, Landau gauge yields considerable gains.

On the technical side the computation of the integrand of eq. (1.7) requires fractional powers of SU(3) matrices. We optimized this using the Cayley-Hamilton method. The method has, of course, more general application, which we should emphasize because it does not seem widely recognized by the lattice gauge theory literature.

These developments have enabled us to present first results for the topological susceptibility of SU(3) gauge theory. Our algorithm is unfortunately not fast enough to attain the statistics of ref. [4] without a supercomputer, but it studies topology directly in the quantum vacuum, and it is gauge invariant. Thus, it provides, perhaps, a basis for future progress in understanding the vacuum of gauge theories.

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Appendix

For computing a power of an SU(3) matrix U we first diagonalize it by a unitary transformation V,

$$D = V^{-1}UV = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{A.1}$$

then

$$U^x = (VDV^{-1})^x = VD^xV^{-1}, \tag{A.2}$$

with

$$D^x = \begin{pmatrix} \lambda_1^x & 0 & 0 \\ 0 & \lambda_2^x & 0 \\ 0 & 0 & \lambda_3^x \end{pmatrix}, \tag{A.3}$$

where (N.B.: the eigenvalues are, in general, complex)

$$\lambda_k^x = e^{x \ln \lambda_k}. \tag{A.4}$$

The logarithms $\ln \lambda_k$ are uniquely determined by

$$\sum_k \ln \lambda_k = 0,$$

$$|\ln \lambda_k - \ln \lambda_l| \leqslant 2\pi. \tag{A.5}$$

A particularly simple method for determining the power of U is that of Cayley-Hamilton [14]. Consider the characteristic equation of U:

$$\sum_{n=0}^{3} a_n \lambda_k^n = 0, \tag{A.6}$$

which in matrix form reads

$$\sum_{n=0}^{3} a_n D^n = 0. \tag{A.7}$$

By using (A.1) one finds

$$\sum_{n=0}^{3} a_n U^n = 0, \tag{A.8}$$

i.e. each matrix U fulfills its own characteristic equation. As a result only the powers

U^0, U^1 and U^2 are linearly independent. In particular, it follows that

$$U^x = \sum_{n=0}^{2} f_n(x) U^n, \tag{A.9}$$

where

$$f_0(x) = -\frac{\lambda_1^x \lambda_2 \lambda_3}{(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)} - \frac{\lambda_2^x \lambda_3 \lambda_1}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} - \frac{\lambda_3^x \lambda_1 \lambda_2}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)},$$

$$f_1(x) = \frac{\lambda_1^x (\lambda_2+\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)} + \frac{\lambda_2^x (\lambda_3+\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} + \frac{\lambda_3^x (\lambda_1+\lambda_2)}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)},$$

$$f_2(x) = -\frac{\lambda_1^x}{(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)} - \frac{\lambda_2^x}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} - \frac{\lambda_3^x}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}. \tag{A.10}$$

References

[1] I.A. Fox, J.P. Gilchrist, M.L. Laursen and G. Schierholz, Phys. Rev. Lett. 54 (1985) 749
[2] Y. Arian and P. Woit, Nucl. Phys. B268 (1986) 521
[3] G. Lasher, A. Phillips and D. Stone, in Quark confinement and liberation: numerical results and theory, eds. F. Klinkhammer and M.B. Halpern (World Scientific, Singapore, 1985)
[4] A.S. Kronfeld, M.L. Laursen, G. Schierholz and U.-J. Wiese, Nucl. Phys. B292 (1987) 330
[5] M. Lüscher, Comm. Math. Phys. 85 (1982) 29
[6] J. Hoek, Phys. Lett. 166B (1986) 199;
J. Hoek, M. Teper and J. Waterhouse, Phys. Lett. 180B (1986) 112
[7] C. Callan, R. Dashen and D. Gross, Phys. Rev. D17 (1978) 2717
[8] G. Schierholz, DESY report 86-99 (1986);
M.L. Laursen and G. Schierholz, DESY report, in preparation
[9] M.L. Laursen, G. Schierholz and U.-J. Wiese, Comm. Math. Phys. 103 (1986) 693
[10] M. Göckeler, M.L. Laursen, G. Schierholz and U.-J. Wiese, Comm. Math. Phys. 107 (1986) 467
[11] A. Phillips and D. Stone, Comm. Math. Phys. 103 (1986) 599
[12] C. Cronström and J. Mickelsson, J. Math. Phys. 24 (1983) 2528
[13] G.G. Batrouni, G.R. Katz, A.S. Kronfeld, G.P. Lepage, B. Svetitsky, and K.G. Wilson, Phys. Rev. D32 (1985) 2736
[14] C.T.H. Davies et al., private communication
[15] F.E. Hohn, Elementary matrix algebra (Macmillan, New York, 1964)
[16] Ph. de Forcrand, G. Schierholz, H. Schneider and M. Teper, Phys. Lett. 160B (1985) 137
[17] G. Parisi and F. Rapuano, Phys. Lett. 152B (1985) 218
[18] G. Bhanot, F. Fucito, and S. Solomon, Caltech report CALT-68-1281 (1985)