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WEAK COUPLING EXPANSION OF THE LOW-LYING ENERGY VALUES IN SU(3) GAUGE THEORY ON A TORUS

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Low-lying energy values of SU(3) gauge theory in a L^3 periodic box are calculated for small volumes using Luscher's effective hamiltonian The lowest particle state is the 2^{++} Mass ratios exhibit smooth volume dependence in the domain of validity of the perturbation expansion in particular $M(0^{++})/M(2^{++})$ remains practically constant at $\sim 1\,23$ up to $M(2^{++})L \sim 2$

1. Introduction

It is generally believed that pure SU(N) gauge theory has a mass gap m and possesses a rich low-lying spectrum of stable particles and resonances – the glueballs The calculation of this spectrum is not only a challenging theoretical problem in its own right but one of phenomenological interest since in QCD we anticipate the existence of low-lying resonances which may be interpreted as glueballs Despite the fact that mixing effects will certainly distort the quantitative relationship between the spectra of glueballs in the pure and full theories it is expected that qualitative features such as quantum numbers will survive Experimentally, there are at present resonances (ι (1440), θ (1690)) which do not seem to fit into a pure quark model framework, but their interpretation as glueballs remains controversial [1]

In the infinite-volume continuum Yang-Mills theory the glueball masses m_t are proportional to the lambda parameter $\Lambda_{\overline{\rm MS}}$

$$m_t = c_t \Lambda_{\overline{\rm MS}} \tag{1}$$

The determination of the numbers c_t is a non-perturbative problem Attempts to calculate the spectrum in this framework are all semi-phenomenological [2], or invoke other ad hoc assumptions [3] which make the control or estimation of systematical errors practically impossible and lead to varying results Reasonable

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model independent qualitative considerations by Jaffe, Johnson and Ryzak [4] favour J^{PC} quantum numbers $0^{++}, 2^{++}, 0^{-+}, 2^{-+}$ for the lowest states, and it is generally argued that they should be narrow compared to mesons with valence quarks.

Probably the most promising approach to the problem to derive results from first principles is the numerical Monte Carlo (MC) method applied to the theory formulated on a finite L^3T lattice. Most of such simulations in the past [5], however, suffered from a poor signal-to-noise ratio because of the high dimension of operators used as interpolating fields for the glueballs. In a recent calculation Berg, Billoire and Vohwinkel have obtained significantly better statistics by measuring correlations of spatial Polyakov loops in the adjoint representation. In both SU(2) [6] and SU(3) [7] their analysis favours 2^{++} to be the lowest state for all volumes.

In MC analyses, as in real experiments, one must incorporate sufficient theoretical input to extract reliable predictions from the raw data. In particular in MC simulations one must understand and isolate the cut-off effects.

In the scaling region, for sufficiently large correlation length the dependence on the lattice spacing a for mass ratios is supposed to be of the form

$$\frac{m_1}{m_2} = \frac{c_1}{c_2} + O(a^2 m^2) \tag{2}$$

The $O(a^2)$ corrections are non-universal, i.e. depend on the lattice action used. These effects should be fitted or one can attempt to suppress these effects by using Symanzik improved actions [8].

In an important paper Luscher [9] has proven* a universal formula which expresses precisely how the stable particle masses $M_i(L)$ in a finite volume with periodic boundary conditions, exponentially approach their infinite volume limit $m_i = M_i(\infty)$. For low-lying scalar glueballs, for example, the leading term is given by

$$(M_i(L) - m_i)/m_i \xrightarrow[L\to\infty]{} -C_i(mL)^{-1} e^{-\sqrt{3}mL/2}, \tag{3}$$

and where C_i is the appropriate triple-glueball coupling (in the infinite volume theory).

Lattice strong coupling (SC) expansions have been made by Munster for the glueball spectrum [10] and for the triple-glueball coupling [11]. Extrapolating the results to regions of smaller coupling suggest 0^{++} to be the lowest state with strong interactions $C \approx 100$ [11]. This order of magnitude for C is consistent with MC calculations by de Forcrand et al. [5] and would mean non-negligible finite volume corrections despite the exponential fall-off.

* The proof is to all orders of perturbation theory, but thought to be of general validity.

As a further check comparisons of MC results should be made with analytical calculations in domains where the latter can be performed. These are the regions where either the ultraviolet cut-off a^{-1} or the infrared cut-off L are small in units of m The former is the SC expansion mentioned above

In a finite volume the spectrum is discrete and moreover for small volumes the spectrum can be calculated perturbatively since then the running coupling constant is small In another pioneering work, Luscher [12] showed that the lowest-lying energies of the SU(N) theory on a torus have an expansion*

$$E = \frac{1}{L} \sum_{k=1}^{\infty} \varepsilon_k \bar{\lambda}^k , \qquad \bar{\lambda} = \left[\bar{g}(\Lambda_{\overline{\mathrm{MS}}} L) \right]^{2/3} \tag{4}$$

and that they are exactly equal to the eigenvalues of an effective quantum mechanical hamiltonian H' acting on wave functions in the space of constant gauge potentials

$$c_k^a \qquad k = 1,2,3, \qquad a = 1, \quad , N^2 - 1$$

H' has an expansion

$$H' = \frac{\bar{\lambda}}{L} \sum_{\nu=0}^{\infty} \bar{\lambda}^{\nu} H'_{\nu} , \tag{5a}$$

with

$$\begin{aligned}
H'_0 &= -\frac{1}{2} \frac{\partial^2}{\partial c_i^a \, \partial c_i^a} + \tfrac{1}{4} f^{abe} f^{ecd} c_i^a c_j^b c_i^c c_j^d , \\
H'_1 &= \kappa_1 c_i^a c_i^a , \\
H'_2 &= 0 , \\
H'_3 &= \kappa_2 H'_0 + \kappa_3 s^{abcd} c_i^a c_i^b c_j^c c_j^d + \kappa_4 s^{abcd} \left[5 c_i^a c_i^b c_i^c c_i^d - 3 c_i^a c_i^b c_j^c c_j^d \right] ,
\end{aligned} \tag{5b}$$

where the numerical constants κ_i are given by

$$\begin{aligned}
\kappa_1 &= -\frac{N}{4\pi} \times 1\ 89153165 , \\
\kappa_2 &= -\frac{11N}{9(4\pi)^2} \times 0\ 409052802 , \\
\kappa_3 &= \frac{2}{15(4\pi)^2} , \\
\kappa_4 &= -\frac{1}{5(4\pi)^2} \times 0\ 619331710
\end{aligned} \tag{5c}$$

* Most of the equations in this paper can be found in the works of Luscher we include them to make the paper self-contained

f^{abc} are the SU(N) structure constants and s^{abcd} the totally symmetric invariant tensor defined by

$$s^{abcd} = \tfrac{1}{12}N(d^{abe}d^{ecd} + d^{ace}d^{ebd} + d^{ade}d^{ebc}) + \tfrac{2}{3}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) \tag{6}$$

Luscher and Munster [13] calculated the eigenvalues of H' for the case of SU(2) Among their results they found that the 2^{++} was the lowest state in the perturbative region, however, the mass ratio $m(0^{++})/m(2^{++})$ decreased with increasing volume

In this paper we report on the analogous calculation for the phenomenologically interesting case of SU(3) During the course of completion of our work the MC computations [7] were made and hence enable a comparison

2. Aspects of the Rayleigh-Ritz variational calculation

The physical states of H' are gauge invariant

$$\Psi(Rc) = \Psi(c) \tag{7}$$

for R in the adjoint representation of SU(N) These states are classified according to their transformation properties under symmetries of H', ($c_k = \lambda^a c_k^a$),

parity $\quad c_k \to -c_k$,

charge conjugation $\quad c_k \to -(c_k)^*$,

cubic rotations $\quad c_k \to c_l S_{lk}$, $\quad S \in \mathcal{O}$

Note that in fact H_0' is invariant under the full rotation group SO(3) and its eigenstates can be classified by J, the angular momentum The SO(3) is only broken down to the cubic group $\mathcal{O}$ by H_3' As pointed out by Luscher and Munster [13] this symmetry at small $\bar{\lambda}$ is "accidental", and the rotation symmetry which is restored at large L may put states which have different J into the same multiplet The scrambling of the lowest lying states is however limited by the cubic symmetry

The method we used follows exactly that of Luscher and Munster* First, the eigenvalues of H_0' are determined by applying the Rayleigh-Ritz variational method using a basis of harmonic functions of the form

$$\Psi(c) = Q(c)\exp(-\tfrac{1}{2}\omega c_k^a c_k^a), \tag{8}$$

* The second numerical method used in [13] a "$1/n$" expansion, is not applicable to SU(N) for $N > 2$

TABLE 1
The number of generators N_g and of linearly independent invariant polynomials N_p of a given degree D

D	$N_g(D)$	$N_p(D)$	$\Sigma_{d=0}^{D} N_p(d)$
0	0	1	1
2	1	1	2
4	2	3	5
6	5	8	13
8	6	17	30
10	11	38	68

where $Q(c)$ is a polynomial in the c's having appropriate transformations for a given J^{PC}, and ω is a variational parameter In a second step the perturbations H_ν', $\nu \geqslant 1$ are treated according to standard Rayleigh-Schrodinger perturbation theory

It is in the first step that SU(3) turns out to be much more awkward to handle than SU(2) For example, for SU(2) the 3 monomials

$$g_\nu = \operatorname{Tr} G^\nu, \qquad \nu = 1,2,3, \qquad G_{kl} = c_k^a c_l^a \tag{9}$$

generate the space of polynomials invariant under the symmetries of H_0' (i e those appropriate to form a basis of the $J^P = 0^+$ states) In SU(3) the situation is much more complicated and although a general theorem of Hilbert [14] tells us that the number of (invariant) generators of the space of invariant polynomials is finite, we (the authors) do not yet know how large this number is We adopted a constructive approach to the problem For a fixed degree we first established an exhaustive list of invariants For this task, a result of Dittner [15], which says that any SU(3) numerically invariant tensor of rank > 6 can be expressed as an outer product of lower-rank invariant tensors, is particularly useful Then, starting with the lowest degree and increasing the degree in steps of 2, we formed a maximal subset S of independent generators (up to a given degree) by systematically working our way through the list and admitting a member as an element of S only in the case that it could not be expressed as a polynomial in the previously established elements of S The test of independence was done numerically using randomly generated configurations of the c's In table 1 we tabulate the number of independent generators N_g to degree 10 in the c's, and the number N_p of linearly independent polynomials to which these give rise* A list of a set of generators, complete (only) up to degree 10, is given in the appendix We see that to this degree there are already 25 generators which is greater than the amount of c's These therefore must be related in a more complicated algebraic manner We did not pursue the matter further since length 10

* The independence of our set is also confirmed during the Gram-Schmidt orthogonalisation procedure [13]

non-trivial wave functions were the longest that we could presently handle in practice (we say a wave function is non-trivial if it has no common factor $c_i^a c_i^a$) Without a finite generating basis on hand, however, we could not generate the wave function inner products and expectation values recursively as was the case for SU(2) [13] The main amount of CPU time is required for the calculation of the inner products of elements involving polynomials of the same length – the others can be related to these, and the hamiltonian matrix elements can then be related to the set of inner products from knowledge of various laplacians The CPU time required for calculating an inner product depends strongly on the permutation symmetry of the polynomials involved and in particular is greatly reduced when the number of traces of length 2 is large We required approximately 200 hours of CPU time on the DESY-IBM to generate the entire set of matrix elements required for our basis of 106 wave functions in the 0^{++} sector Although we realise that our programs were not optimised, it would require drastic improvements to tackle the computation of inner products involving polynomials of larger length, since our method would typically need a factor ~ 100 more CPU time to compute inner products involving invariants of length 12 compared to those for length 10

The spectrum of SU(3) is richer than that for SU(2) since C-invariance gives rise to an additional non-trivial quantum number, and also vector states are present Furthermore the practical treatment of J^{PC} states other than the 0^{++} is much more involved, e g the pseudoscalars are not all generated simply as products of $f^{abc}\varepsilon_{ijk}c_i^a c_j^b c_k^c$ times the scalars and the 2^{+-} states are not all of the form scalar times $G_{kl}^r - \frac{1}{3}g_r\delta_{kl}$, $(r = 1, 2)$ as is the case for SU(2) We followed an analogous approach to constructing their bases as that described above for the scalars, and this made the calculations very tedious and time consuming Moreover we were often restricted to a very small basis due to the fact that the construction of suitable wave functions required rather long polynomials, an extreme example is the 0^{+-} which for SU(3) requires at least degree 10 For the same reason we did not investigate states with $P = C = (-1)^{J+1}$ at all, but we expect them to lie quite high in the spectrum

Finally we remark that many of our results were checked by using independent programs or by performing other internal consistency checks

3 Results and discussion

The accuracy of the results obtained can be estimated by observing the dependence of the eigenvalues on the number of basis vectors Table 2 shows this dependence for the 0^{++} state The choice $N_b = 13, 30, 68$ can be understood from table 1 and the 21, 47, 106 correspond to addition of trivial wave functions to the set immediately below We estimate our result for the ground state energy of H_0' to be correct to 5 significant figures

In principle a lower bound for the ground state can be found using Temple's inequality [16] This states that if H is a self-adjoint operator bounded from below

TABLE 2
Eigenvalues ε_l of H_0' for SU(3) in the $J^{PC} = 0^{++}$ sector as a function of the number N_b of basis vectors used in the Rayleigh-Ritz variational calculation

N_b	1st state	2nd state	3rd state
13	12 61975	15 70477	17 89338
21	12 61817	15 62457	17 85260
30	12 59110	15 44853	17 34241
47	12 59079	15 42199	17 32980
68	12 58885	15 39047	17 23844
106	12 58869	15 38468	17 23411

with lowest and next lowest eigenvalues E_0, E_1 then if Ψ is a normalized state ($(\Psi, \Psi) = 1$) in the Hilbert space on which H acts and for which $(\Psi, H\Psi) < E < E_1$, then

$$(\Psi, H\Psi) - \frac{(\Psi, H^2\Psi) - (\Psi, H\Psi)^2}{E - (\Psi, H\Psi)} \leqslant E_0 \tag{10}$$

However the matrix elements of $H_0'^2$ for wave functions of degree $\geqslant 8$ are tedious to obtain The best bound we obtained so far in this way is rather poor (compared to our estimate), viz

$$\varepsilon_1(\text{ground state}) \geqslant 12\ 485$$

In table 3 our results for the ε_l for various J^{PC} states are listed We consider our results are accurate to a difference of about 1 in the last digit quoted The inaccuracy of the states other than 0^{++} and 2^{++} is due to the small dimension of the basis used However for the lowest state in each J^{PC} channel the estimates are good enough to form a qualitative picture of the spectrum which is given in fig 1 (see figure caption)

Some notable features are the following For very small volumes, the lowest state* is the 2^{++} followed by a 0^{++}** This pattern repeats itself in that e g the 1^{--} state (vector states are absent in the SU(2) theory) lies above the 3^{--} state We might also speculate that for a given spin J the lowest excitation is the one with $P = C = (-1)^J$ and the highest that with $P = C = (-1)^{J+1}$

All the low-lying eigenvalues in each case, with exception of ε_4, were rather stable to a range of the variational parameter ω lying between 1–2 To get a reliable estimate of ε_4 requires an extremely large basis and hence our results for this coefficient are the least reliable and must be considered as merely an estimate of the

* The mass of a state is given by the energy of the state minus the ground state energy

** This is in fact true for all SU(N) The qualitative features of the low-lying spectrum remain the same for general N [17]

TABLE 3
Low-lying eigenvalues of the effective hamiltonian H' for SU(3) as computed by the Rayleigh-Ritz technique (cf. eqs. (4),(5), $\varepsilon_0 = 0$ for all states listed)

J^{PC}	Γ	ε_1	ε_2	ε_3	ε_4	N_b
0^{++}	A_1	12.5887	−4.0628	−0.4280	−0.104	106
0^{++}	A_1	15.38	−4.79	−0.57	0.08	106
0^{++}	A_1	17.23	−5.28	−0.50	−0.05	106
0^{+-}	A_1	28	−6			4
0^{-+}	A_1	17.8	−4.5	−0.4		15
0^{--}	A_1	> 20				0
1	T_1	17.05	−4.67	−0.39	−0.06	29
1^{-+}	T_1	23	−5			4
1^{+-}	T_1	18	−5			4
1^{++}	T_1	> 20				0
2^{++}	E	14.854	−4.719	−0.474	0.20	69
2^{++}	T_2	14.854	−4.719	−0.474	−0.02	69
2^{++}	E, T_2	17.26	−5.27	−0.45		69
2^{+-}	E, T_2	22.1	−5.0	−0.3		17
2^{-+}	E, T_2	21	−4			4
2^{--}	E, T_2	> 20				0
3^{--}	$A_2\ T_1\ T_2$	16.5	−4.7	−0.4		7
4^{++}	A_1, E, T_1, T_2	18				3

N_b is the dimension of the basis used. Γ denotes the cubic group representation. A gap means either that the respective quantity was not measured or that we did not obtain a reliable estimate.

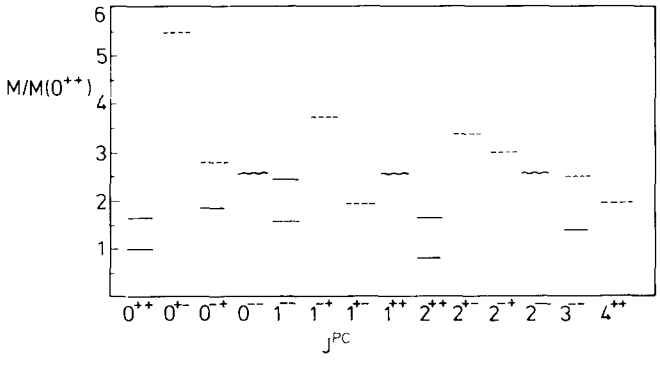

Fig. 1. Qualitative picture of the low-lying spectrum of H_0 for SU(3), corresponding to the values of ε_1 in table 3. Only at most the lowest 2 states are drawn for each J^{PC}. A dashed line indicates that the result has some error bars (in some cases quite large). The wavy lines for the $0^{-}, 1^{+}, 2^{-}$ states indicate in our view a conservative guess for a lower bound in these channels.

order of magnitude In general, even for a fixed basis dimension, our results are expected to be less accurate than the corresponding SU(2) case simply because there are many more degrees of freedom

Luscher pointed out that it was natural to plot physical quantities as functions of a variable $z = M(L)L$ In his original work Luscher hoped that combining knowledge of the small z spectrum with the known behaviour in the $z \to \infty$ limit would lead to an estimate of the infinite volume value $M(0^+)/\Lambda_{\overline{\rm MS}}$ This original optimism was encouraged by calculations in the O(N)-σ-model [18] In the Yang-Mills theory however a plot of $M(0^+)/\Lambda_{\overline{\rm MS}}$ against $z_0 = M(0^+)L$ shows a very rapid crossover in the region $z_0 \sim 1-2$ A similar behaviour occurs in the case of SU(3) In fig. 2 we plot $M(2^{++})/\Lambda_{\overline{\rm MS}}$ as a function of $z_2 = \overline{M}(2^{++})L$ Here $\overline{M}(2^{++})$ is defined as

$$\overline{M}(2^{++}) = \tfrac{1}{3}\left[M(2^{++}, E) + 2M(2^{++}, T)\right], \tag{11}$$

the combination for which the SO(3) breaking term does not contribute We chose to plot the 2^{++} not because we necessarily believe that the 2^{++} will turn out to be the lowest excited state for all volumes but because our results for the 2^{++} are probably more precise than those for the 0^{++} Our plot follows the scheme of ref

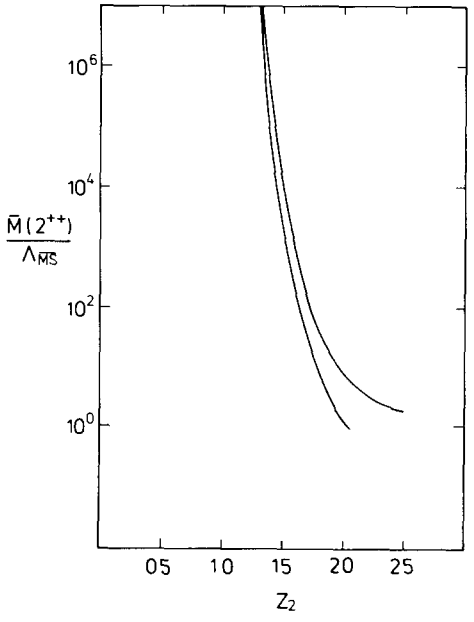

Fig 2 Energy gap $\overline{M}(2^{++})$ in the 2^{++} sector as a function of z_2 for SU(3) The meaning of the curves is explained in the main text

[13] From the definition of the running coupling $\bar{g}(\Lambda_{\overline{\rm MS}})$ it follows that

$$\ln\frac{\overline{M}(2^{++})}{\Lambda_{\overline{\rm MS}}} = \ln(z_2) + \frac{1}{2b_0\bar{\lambda}^3} + \frac{b_1}{2b_0^2}(\ln b_0\bar{\lambda}^3) - \tfrac{1}{2}(\ln(4\pi) + \Gamma'(1)) + O(\bar{\lambda}^3), \tag{12}$$

where b_0, b_1 are the Callan-Symanzik β-function coefficients for SU(N) The perturbative expansion for z_2

$$z_2 = \sum_{\nu=1}^{\infty} c_\nu \bar{\lambda}^\nu \tag{13}$$

is then inverted

$$\bar{\lambda} = \sum_{\nu=1}^{\infty} d_\nu z_2^\nu \tag{14}$$

The right curve represents eq (12) where $\bar{\lambda}$ is evaluated numerically by truncating the expansion (14) after the first four terms The left curve is obtained by eliminating $\bar{\lambda}$ from (12) using (14) to obtain

$$\ln\frac{\overline{M}(2^{++})}{\Lambda_{\overline{\rm MS}}} = \tfrac{274}{121}\ln(z_2) + \sum_{\nu=-3}^{\infty} a_\nu z_2^\nu \tag{15}$$

and including coefficients $\nu \leqslant 0$ Here we observe a rapid crossover region at $z_2 \sim 1\,5$–2 Around these values the two curves deviate considerably and we conclude that the results for this ratio should not be trusted beyond $z_2 \sim 1\,3$ In fig 2 we have not included a MC estimate for the infinite volume limit, (which is approached from below, cf eq (3)), since there appears, at present, to be no definite consensus in the literature The crossover seems to set in at slightly higher z values than for SU(2) Indeed we might expect the crossover behaviour to become smoother as N increases because the triple-glueball coupling vanishes as $N \to \infty$, causing a more rapid approach to the $z_2 \to \infty$ limit However the triple-glueball coupling may still be large for comparatively low values of N such as $N = 3$ [11]

The behaviour of mass-ratios on the other hand is expected to be smoother In table 4 we give the coefficients r_ν occurring in the expansion

$$\frac{M_i}{\overline{M}(2^{++})} = \sum_{\nu=0}^{\infty} r_\nu z_2^\nu, \tag{16}$$

and the corresponding curves are plotted in fig 3 This smooth behaviour would suggest that the results can be trusted at least right up to the crossover region Surprisingly, we find (in contrast to the case of SU(2)) that the mass ratio

TABLE 4
The first four coefficients in the z_2 expansions of mass ratios (cf eq (16))

J^{PC}	Γ	r_0	r_1	r_2	r_3
0^{++}	A_1	1 23	0 017	−0 003	−0 001
0^{++}	A_1	2 05	0 025	0 01	
0^{-+}	A_1	2 3	0 2		
1^{--}	T_1	1 97	0 13	−0 05	
2^{++}	E	1	0	0	−0 01
2^{++}	T_2	1	0	0	0 005
2^{++}	E, T_2	2 06	0 029	0 01	

$M(0^{++})/M(2^{++})$ initially increases with z_2, however very slowly. The ratio is initially 1 23 (very close to the SU(2) value), and it only increases by $< 1\%$ up to $z_2 \sim 2$ The splitting between the E and T states, (for which we again stress our estimate is not very accurate), remains small even at $z_2 \sim 1.5$ and leads to the hope that the curves merge again in the $z_2 \to \infty$ limit

In fig 3 we have also included the data points extracted from the SU(3) MC measurements by Berg et al [7] We note very good agreement in the region $z_2 \sim 1\,5$–2, however for the lower z values, where the perturbative results become more and more reliable, the MC data points are systematically higher than the weak

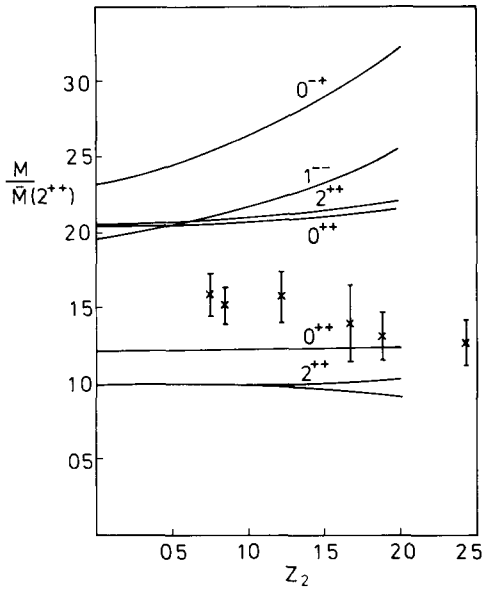

Fig 3 Mass ratios as a function of z_2 The crosses are estimates for $M(0^{++})/M(2^{++})$ extracted from SU(3) MC measurements by Berg et al [7]

coupling curve (a similar but less accentuated trend appears in SU(2)) At least two explanations of this deviation are possible Either the small z results obtained above for the mass ratios are valid in a region much smaller than we would like to expect or the tensor mass has been underestimated in ref [7] A possible source of such an underestimation, which has been pointed out to us by Luscher, is as follows First recall that the Hilbert space of physical states of the finite volume theory divides into N^3 sectors of definite "electric flux" [12] The states in these sectors are degenerate to all orders of perturbation theory, however, the degeneracy is lifted due to tunneling The intricate analytical calculation of the ground state splitting ΔE at weak coupling has been performed in SU(2) by Koller and van Baal [19]* They find that tunneling contributions become appreciable for $z \sim 1$** Numerical calculations of ΔE have also been performed by Berg et al [6,7] In the large L limit ΔE is related to the string tension σ, $\Delta E \sim \sigma L$

The true ground state having zero electric flux will be invariant under rotations and hence operators belonging to non-trivial representations of the cubic group will have vanishing expectation values in this state However the other states carrying electric flux, which are nearly degenerate with the vacuum for small z, belong to non-trivial representations of the cubic group and there is no a priori reason for the adjoint Polyakov loop to have vanishing expectation values in these states In the latter case the correlation function of two adjoint Polyakov loops would contain terms which would be independent of their "time" separation Certainly when the "time" extent T of the lattice is made large these contributions are damped exponentially $\exp(-\Delta E T)$, however as stressed above, for small z, ΔE is very small and hence the t-independent terms could be significant unless T is made sufficiently large Omission of t-independent terms in fitting the adjoint Polyakov correlation function MC data would result in underestimating the tensor mass

The origin of the discrepancy between the perturbative and MC results concerning the mass ratio $M(0^{++})/M(2^{++})$ discussed above and concerning the ground state splitting ΔE [7,19], are questions that should be resolved in the near future If it turns out that mass ratios do indeed show only weak volume dependence, as the present evidence indicates, then perturbative results certainly yield reliable information on the spectrum which can be used profitably in conjunction with MC analyses In that case it would be worth the effort to extend the computation made in this paper to include larger bases and to investigate higher states The more pressing problem however is to further improve the MC spectrum calculations

We are indebted to M Luscher for many helpful discussions and suggestions We also profitted from discussions with P van Baal and G Munster Finally we appreciated the help of R Wohlert in preparing the manuscript

* The calculation for SU(3) is under way [20]

** The same region where the perturbation theory becomes unreliable

Appendix

Below we list a set of polynomial generators of O(3) and SU(3) invariants complete up to degree 10. We use the notation $(ij \quad k)$ to denote $\mathrm{tr}(c_i c_j \cdot \quad c_k)$

degree 2	(ii),
degree 4	(ij)(ij)
	(iji),
degree 6	(ij)(jk)(ki)
	(iij)(jkk)
	(ijk)(ijk)
	(ijk)(ikj)
	(ij)(ikjk),
degree 8	(ij)(ikk)(jll)
	(ij)(ikl)(jlk)
	(ij)(ijk)(kll)
	(ij)(jk)(ikll)
	(iik)(jkll)
	(iik)(jllk),
degree 10	(ij)(jk)(ikl)(lmm)
	(ij)(jk)(ill)(kmm)
	(ij)(jk)(ilm)(klm)
	(ij)(kl)(ijk)(lmm)
	(ij)(kl)(ijm)(klm)
	(ij)(ikk)(jllmm)
	(ij)(ijk)(kllmm)
	(ij)(klm)(ijklm)
	(iij)(kkl)(jlmm)
	(iij)(jkl)(kmlm)
	(ijk)(ijl)(klmm)

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