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INFLATION FROM HIGHER DIMENSIONS

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We argue that an inflationary phase in the very early universe is related to the transition from a higher dimensional to a four-dimensional universe. We present details of a previously considered model which gives sufficient inflation without fine tuning of parameters.

1. Introduction

An inflationary phase [1] at an early stage of the evolution of our universe has become a paradigm of modern cosmology. (By inflation we mean an exponential expansion of the Robertson-Walker scale factor $R_3(t)$.) The horizon problem can be solved if the inflationary phase lasts long enough. At the same time the inflationary cosmology could explain the flatness, age, homogeneity and isotropy of our universe. In some of the first attempts inflation was related to the phase transition of grand unified theories. This scenario seems to be difficult to realize and attention has been turned to models with a scalar singlet field [2] responsible for inflation, sometimes called the "inflaton". Although useful for purposes of demonstration that inflation could work, the existence of a special inflaton field only motivated by cosmology seems not very satisfactory, especially since these models need very accurate fine tuning of parameters. It would be more natural to relate the inflationary phase to some transition period where the properties of the universe change qualitatively.

Could unification of gravity (strings) be the underlying physics for inflation? It was proposed [3] that inflation describes the transition from higher dimensional cosmology to an effective four-dimensional cosmology*. In this context inflation could be a purely (classical) gravitational mechanism. Higher dimensional theories have become candidates for a unification of all interactions, including gravity. The

* For alternative ideas in this context see ref. [4].

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simplest version would be riemannian geometry in more than four dimensions, in which case four-dimensional gauge interactions arise from isometries of internal space [5]. The model which so far is perhaps closest to observation [6] starts from $d = 18$ gravity coupled to a Majorana-Weyl spinor [7]. Since models of simple riemannian gravity coupled to fermions are not renormalizable and in general not anomaly free, modifications and generalizations are needed. Most popular today are string theories [8]. In their most ambitious version they may be considered as a purely bosonic unification of all forces in 26 dimensions [9].

All realistic higher dimensional models must have a ground state solution with spontaneous compactification of the extra dimensions. The characteristic length scale L of the internal space must be of the order of the Planck length or somewhat larger, whereas the observed four spacetime dimensions are flat. If this ground state solution is classically stable, there always exist Friedmann-type cosmologies at late times where $L(t)$ is almost constant whereas the expansion of the four-dimensional Robertson-Walker scale factor $R_3(t)$ is well described by the standard four-dimensional hot big bang cosmology. However, in the very early universe L and R_3 are expected to be of the same order of magnitude. The static approximation for L breaks down and we have to consider the coupled system for the time evolution of $L(t)$ and $R_3(t)$. The basic question of Kaluza-Klein cosmology [10] is: How did $L(t)$ and $R_3(t)$ become separated by so many orders of magnitude?

We relate inflation and this asymmetric evolution of $L(t)$ and $R_3(t)$. The basic mechanism can be understood in a four-dimensional language. We choose the d-dimensional cosmological constant (or some other appropriate parameter) so that the four-dimensional cosmological constant vanishes for the ground state with $L = L_0$. However, an internal radius $L(t)$ different from L_0 induces a positive effective cosmological constant in four dimensions. This will be responsible for a phase of exponential expansion of $R_3(t)$.

Two new features are important for the inflationary phase of higher dimensional cosmology:

(i) There is a natural scalar singlet with an associated exponentially flat potential. This scalar singlet φ is related to the internal space. In the simplest version, the deviation of the overall internal scale from its ground state value, $L(t) - L_0$, leads after dimensional reduction to a four-dimensional scalar field $\varphi(t)$. A change of volume does not change the symmetries and φ must be a singlet under the four-dimensional gauge transformations. (In more complicated models, other characteristic scales of the internal manifold may also play this role.) The potential terms in the action are typically polynomials in L^{-2}, whereas kinetic terms have the form $\sim (\dot{L}/L)^2$ which reflects their gravitational origin. For a standard normalization of the kinetic term one has $\varphi \sim \ln(L/L_0)$ and the potential is $W(\varphi) \approx c + b\exp(-a\varphi)$ for large L. This exponentially flat tail of the potential is relevant for the inflationary period [11, 12] since it corresponds to a very slow time evolution of $\varphi(t)$. On the other hand, near the ground state $L \approx L_0$ the potential has a relatively

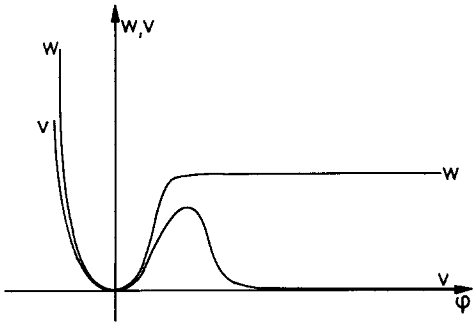

Fig. 1. Difference between the potentials $w(\varphi)$ and $v(\varphi)$.

large quadratic term $\sim \mu^2\varphi^2$ ($\mu^2 \approx (10^{17}$ GeV$)^2$) which facilitates sufficient heating at the end of the inflationary period.

(ii) The four-dimensional equivalence principle is broken as a consequence of higher dimensional unification. The scalar particle corresponding to φ does not move on four-dimensional geodesics in the absence of non-gravitational interactions. It feels additional gravitational forces. A coherent mode $\varphi(t)$ receives additional gravitational contributions to its evolution equation. As a consequence, the potential $W(\varphi)$ which determines the time evolution of $\varphi(t)$ is different [11,12] from the potential $V(\varphi)$ appearing in the energy momentum tensor and acting as a cosmological constant for the gravitational field. A typical form for the potentials $W(\varphi)$ and $V(\varphi)$ is depicted in fig. 1. The "cosmological constant" $V(\varphi)$ vanishes exponentially for large φ (large L). This leads to a comparatively small Hubble constant (typically $H \approx 10^{12}$ GeV) during inflation. As a consequence, the density fluctuations $\Delta\rho/\rho$ are naturally small without fine tuning of parameters. This was demonstrated [12] in a simple model [13] where an adjustment of one parameter within 10% was sufficient to obtain a long enough inflationary period and a satisfactory value for the density fluctuations. The small Hubble constant during inflation is also effective in avoiding problems with particle production during inflation and unacceptable temperature fluctuations in the background radiation by anisotropies in the gravitational field. This is remarkable since the other relevant mass scale during inflation, the inverse radius $L^{-1}(t)$, is typically of order 10^{16}–10^{17} GeV.

The breaking of the four-dimensional equivalence principle and the difference between $W(\varphi)$ and $V(\varphi)$ seems crucial to obtain enough inflation and small $\Delta\rho/\rho$ without fine tuning of parameters. For this it seems essential that the higher dimensional gravity theory contains higher derivative terms such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ etc. Indeed, contributions to the four-dimensional action of the form $f(\varphi)R$, which at first sight seem to violate the equivalence principle for the motion of the scalar φ, can always be removed by an appropriate Weyl scaling of the four-dimensional metric. With this scaling one has $W = V$ and classical physics is of course independent of the choice of variables. (We are not concerned here by additional gravitational interactions of other fields.) In contrast, terms like $F(\varphi)R^2$ cannot be scaled away simultaneously. Here the breakdown of the four-dimensional equivalence principle is a genuine physical effect and not a mere artefact of an inappropriate scaling of the metric. The use of R^2 terms for compactification [13] and cosmology [3] has been proposed earlier, but it has sometimes been criticized. Let us therefore briefly review some arguments why higher derivative terms are expected in generic gravitational theories.

For any discussion of the ground state (compactification) and cosmology the relevant classical field equations are those derived from the *effective action*, which includes the effects of quantum fluctuations. The full effective action is often not known, but information on its general form can be obtained from symmetry and scale arguments. In any quantum theory of gravity, including string theories, the effective action should be invariant under general coordinate transformations*. We expect the effective action to be a complicated function of the various gravitational invariants R, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R_{\mu\nu}R^{\nu}_{\rho}R^{\rho\mu}$, etc. In string theories, such terms appear even in the tree approximation [14]. The full classical field equations will be rather complicated. For large enough length scales one may use an approximation where only terms up to a certain number of derivatives are included, resulting in an action which is a polynomial in the gravitational invariants. Of course, such an approximation is in general expected to break down somewhere near the Planck scale. A second derivative approximation which only involves R is often insufficient to give a satisfactory description of the ground state. R^2-type terms seem to be needed for compactification of pure gravitational theories [3] as well as for string theories [15] and therefore cannot be neglected for cosmology.

The main objection against R^2-type terms concerns classical stability. Indeed, higher derivative actions lead to unstable higher poles in the propagators [16]. However, the location of the higher poles indicates [17] the breakdown of the polynomial approximation for the effective action rather than a genuine classical instability. As long as all relevant energy scales are below the higher poles we may simply neglect them. In particular, the effective action need not be a generalized Euler form [18,19].

So far our remarks have been rather general and we would like to demonstrate them in a specific model. We want to describe inflationary cosmology which finally changes to a four-dimensional Friedmann universe**. We therefore require that the model has a satisfactory ground state with vanishing four-dimensional cosmological

* We neglect here corrections arising in perturbation theory from fluctuations with length scale larger than the compactification scale L.

** Models which are only valid for the inflationary phase [4] can demonstrate a fast time evolution of $R_3(t)$, but they are not suitable for more precise questions like heating after inflation, the evolution of density fluctuations etc.

constant and static internal space. It would be interesting to study some of the superstring compactifications [15]. However, the form of the higher derivative terms in the effective action is not yet settled. In addition, the compactifications discussed so far lead to massless scalar fields which could ruin any sensible cosmology by modifications of the Friedmann universe long after the inflationary period. Waiting for a solution to these problems we come back to our toy model [3] based on pure gravity in $4 + D$ dimensions with the most general four-derivative approximation for the effective action. We believe that this model reflects the main features of inflationary cosmology for more realistic models. The main results are already published [3, 11, 12], but a more detailed exposure of results and the various steps in the calculation seems useful in view of their applications and generalizations to more realistic models. In this paper we establish the existence of inflationary solutions in a higher dimensional language. An alternative equivalent description in terms of four-dimensional fields will be given in forthcoming publications.

In sect. 2 we describe our model and derive the field equations for an ansatz which is separately homogeneous and isotropic in internal dimensions and in the usual three-dimensional space. In sect. 3 we discuss exact de Sitter solutions with static internal space as a prototype for exponential expansion of $R_3(t)$. More general inflationary scenarios are described in sect. 4. We explain the approximations valid for arbitrary inflationary periods and solve the field equations within these approximations. We describe two scenarios with sufficient inflation: One is realized whenever the internal radius $L(t)$ has grown sufficiently larger than L_0 whereas the other corresponds to solutions in the vicinity of exact de Sitter solutions. For both scenarios no fine tuning of parameters is necessary. The discussion in sect. 5 briefly addresses the problem of how the universe could enter the inflationary phase.

2. Cosmological equations in fourth order gravity

In this paper we will illustrate the possible gravitational origin of an inflationary phase [3] in the evolution of the early universe within a simple model [13] of pure d-dimensional gravity. The action of this model is

$$S = -\frac{1}{V_\mathrm{D}}\int \mathrm{d}^d\hat{x}\,\hat{g}^{1/2}\Big(\alpha\hat{R}^2 + \beta\hat{R}_{\hat{\mu}\hat{\nu}}\hat{R}^{\hat{\mu}\hat{\nu}} + \gamma\hat{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\hat{R}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \delta\hat{R} + \varepsilon\Big). \qquad (1)$$

Here $\hat{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$, $\hat{R}_{\hat{\mu}\hat{\nu}}$ and $\hat{R}$ are the d-dimensional curvature tensor, Ricci tensor and curvature scalar defined as usual and we use conventions with signature $(+, -, -, - \ldots)$ where the curvature scalar is negative for spacelike dimensions forming a sphere. The d-dimensional metric is $\hat{g}_{\hat{\mu}\hat{\nu}}$ with $\hat{g} = |\det \hat{g}_{\hat{\mu}\hat{\nu}}|$. In general we denote d-dimensional objects and indices by hats. For convenience we have extracted a normalization factor V_D which is the volume of the internal space in the ground state. In these conventions the parameters α, β and γ are dimensionless and $\delta(\varepsilon)$ has dimension $(\text{mass})^2$ $((\text{mass})^4)$.

The action (1) should be considered as an approximation to the effective action at the length scale L_0 of spontaneous compactification, where all quantum fluctuations with length scales shorter than L_0 are included. Contributions from additional terms with more than two powers of the curvature tensor are assumed to be small. The field equations derived from this effective action are

$$\begin{aligned} &2\alpha\hat{R}\hat{R}_{\hat{\mu}\hat{\nu}} + 2\beta\hat{R}_{\hat{\mu}\hat{\sigma}}\hat{R}_{\hat{\nu}}{}^{\hat{\sigma}} + 2\gamma\hat{R}_{\hat{\mu}\hat{\sigma}\hat{\tau}\hat{\rho}}\hat{R}_{\hat{\nu}}{}^{\hat{\sigma}\hat{\tau}\hat{\rho}} + \delta\hat{R}_{\hat{\mu}\hat{\nu}} \\ &-\tfrac{1}{2}\hat{g}_{\hat{\mu}\hat{\nu}}\left(\alpha\hat{R}^2 + \beta\hat{R}_{\hat{\sigma}\hat{\tau}}\hat{R}^{\hat{\sigma}\hat{\tau}} + \gamma\hat{R}_{\hat{\sigma}\hat{\tau}\hat{\rho}\hat{\lambda}}\hat{R}^{\hat{\sigma}\hat{\tau}\hat{\rho}\hat{\lambda}} + \delta\hat{R} + \varepsilon\right) \\ &+2\alpha\left(\hat{g}_{\hat{\mu}\hat{\nu}}\hat{R}_{;\hat{\sigma}}{}^{\hat{\sigma}} - \hat{R}_{;\hat{\mu}\hat{\nu}}\right) + \beta\left(\hat{g}_{\hat{\mu}\hat{\nu}}\hat{R}^{\hat{\sigma}\hat{\tau}}{}_{;\hat{\sigma}\hat{\tau}} + \hat{R}_{\hat{\mu}\hat{\nu};\hat{\sigma}}{}^{\hat{\sigma}} - \hat{R}_{\hat{\mu}}{}^{\hat{\sigma}}{}_{;\hat{\nu}\hat{\sigma}} - \hat{R}_{\hat{\nu}}{}^{\hat{\sigma}}{}_{;\hat{\mu}\hat{\sigma}}\right) \\ &+2\gamma\left(\hat{R}_{\hat{\mu}}{}^{\hat{\sigma}}{}_{\hat{\nu}}{}^{\hat{\tau}}{}_{;\hat{\sigma}\hat{\tau}} + \hat{R}_{\hat{\mu}}{}^{\hat{\sigma}}{}_{\hat{\nu}}{}^{\hat{\tau}}{}_{;\hat{\tau}\hat{\sigma}}\right) = \tfrac{1}{2}\hat{T}_{\hat{\mu}\hat{\nu}}. \end{aligned} \tag{2}$$

The quantity $\hat{T}_{\hat{\mu}\hat{\nu}}$ on the right-hand side of eq. (2) is the d-dimensional generalization of the energy-momentum tensor. It includes all contributions to the field equations from incoherent excitations and vanishes for zero entropy. Formally we may write

$$\hat{T}_{\hat{\mu}\hat{\nu}} = 2\hat{g}^{-1/2}\left\langle \frac{\delta S}{\delta\hat{g}^{\hat{\mu}\hat{\nu}}} \right\rangle_{\text{incoherent}}. \tag{3}$$

We assume for the ground state that incoherent excitations can be neglected to a good approximation ($\hat{T}_{\hat{\mu}\hat{\nu}} = 0$). In the parameter range

$$\zeta = D(D-1)\alpha + (D-1)\beta + 2\gamma > 0, \tag{4}$$

$$\delta > 0, \tag{5}$$

the field equations (2) admit a solution [13] where the ground state is a direct product of four-dimensional Minkowski space $\mathscr{M}^4$ and a D-dimensional "internal" sphere S^D, provided we adjust the cosmological constant ε:

$$\varepsilon = \tfrac{1}{4}\delta^2\frac{D(D-1)}{\zeta}. \tag{6}$$

The radius L_0 of S^D is determined by

$$L_0^{-2} = \frac{\delta}{2\zeta}. \tag{7}$$

The effective Newton's constant governing four-dimensional gravity is positive for

$$\chi = (D-1)\beta + 2\gamma > 0 \tag{8}$$

and the Planck mass is given by

$$M_{\mathrm{P}}^2 = 16\pi \frac{\chi}{\zeta} \delta . \tag{9}$$

Throughout this paper we always consider parameters fulfilling the constraints (4), (5), (6) and (8).

The adjustment (6) of the cosmological constant ε is crucial for this solution. Otherwise, no static ground state of the form $\mathscr{M}^4 \times S^D$ can be obtained. Rather one would be left with a non-vanishing effective four-dimensional constant or with solutions which have not the direct product form "four-dimensional spacetime $\times$ internal space" and where the internal space is not compact [20]. The adjustment of ε seems not natural. This is the cosmological constant problem in the context of higher dimensional gravity. Any explanation of why the characteristic length scale for the D internal space dimensions is today so different from the characteristic length scale for the usual three space dimensions requires a vanishing four-dimensional cosmological constant for the ground state. Internal length scales of order M_{P}^{-1} are very natural in a theory with parameters of order M_{P}. The puzzling question is rather: Why is the characteristic length scale of our four-dimensional world so large? In our context, this includes the problem: Why is ε so near the value given by (6)? We will not address the cosmological constant problem in this paper but we note that the inflationary period is insensitive to ε somewhat different from (6). In any case, we note that for a given adjustment (6), $\mathscr{M}^E \times S^{d-E}$ is a solution of the field equations only for $E = 4$. Any other number of flat dimensions would require a different adjustment.

If the above ground state is classically stable*, we can immediately conclude that there are Friedmann-type cosmologies:

$$\hat{g}_{\alpha\beta} = \mathring{g}_{\alpha\beta}, \qquad \hat{g}_{\alpha\mu} = 0,$$

$$\hat{T}_{\alpha\beta} = 0, \qquad \hat{T}_{\alpha\mu} = 0, \tag{10}$$

$$\hat{g}_{00} = 1, \qquad \hat{g}_{i0} = 0,$$

$$\hat{g}_{ij} = R_3^2(t)\bar{g}_{ij}, \tag{11}$$

$$\hat{T}_{00} = \rho(t), \qquad \hat{T}_{i0} = 0,$$

$$\hat{T}_{ij} = -p(t)\hat{g}_{ij}. \tag{12}$$

* Even without classical stability the Friedmann-type solutions exist if the subsector of $SO(D+1)$ singlets is stable. For an extensive discussion of classical stability of our ground state see ref. [17].

(We use conventions where $\hat{\mu}, \hat{\nu} \dots$ are d-dimensional indices, $\alpha, \beta \dots$ are internal indices, $\mu, \nu \dots$ are indices of usual four-dimensional spacetime and $i, j \dots$ denote the three usual spacelike coordinates.) The metric $\hat{g}_{\mu\nu}$ is the Robertson-Walker metric of standard big bang cosmology with $R_3(t)$ the Robertson-Walker scale factor. The time evolution of $R_3(t)$, energy density $\rho(t)$, and pressure $p(t)$ with the associated temperature $T(t)$ is the standard evolution of big bang cosmology for an unbroken SO$(D+1)$ gauge theory coupled to gravity. The cosmology described by (10)–(12) is not an exact solution of the field equations (2), but is a very good approximation for $T^2 L_0^2 \ll 1$. It should adequately describe cosmology sufficiently well after the Planck era. However, for T^2 of order L_0^{-2} the approximation (10)–(12) becomes meaningless and one has to study how the higher dimensional cosmology evolved towards the Friedmann solution, which is the main purpose of this paper.

Let us then investigate the cosmology obtained from the effective action (1) before the transition to the Friedmann solution (10)–(12). For a first study, we make an ansatz of separate homogeneity and isotropy in the D internal and the three usual spacelike dimensions. This amounts to an ansatz for the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ where the "Minkowski" components $\hat{g}_{\mu\nu}$ are given by a Robertson-Walker metric (12) and the off-diagonal components $\hat{g}_{\mu\alpha}$ vanish. However, the internal metric $\hat{g}_{\alpha\beta}$ is now characterized by a time dependent radius $L(t)$ of S^D:

$$\hat{g}_{\alpha\beta} = \frac{L(t)}{L_0} \mathring{g}_{\alpha\beta} . \tag{13}$$

The curvature tensor is easily calculated for this ansatz:

$$\begin{aligned}
\hat{R}_{\alpha\beta\gamma\delta} &= -(L^{-2} + \dot{L}^2 L^{-2})(\hat{g}_{\alpha\gamma}\hat{g}_{\beta\delta} - \hat{g}_{\alpha\delta}\hat{g}_{\beta\gamma}) \\
&= c_1(\hat{g}_{\alpha\gamma}\hat{g}_{\beta\delta} - \hat{g}_{\alpha\delta}\hat{g}_{\beta\gamma}) , \\
\hat{R}_{ijkl} &= -(kR_3^{-2} + \dot{R}_3^2 R_3^{-2})(\hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{il}\hat{g}_{jk}) \\
&= c_2(\hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{il}\hat{g}_{jk}) , \\
\hat{R}_{\alpha i \sigma j} &= -\dot{L}L^{-1}\dot{R}_3 R_3^{-1}\hat{g}_{\alpha\beta}\hat{g}_{ij} = c_3 \hat{g}_{\alpha\beta}\hat{g}_{ij} , \\
\hat{R}_{\alpha 0 \beta 0} &= -\ddot{L}L^{-1}\hat{g}_{\alpha\beta} = c_4 \hat{g}_{\alpha\beta} , \\
\hat{R}_{i0j0} &= -\ddot{R}_3 R_3^{-1}\hat{g}_{ij} = c_5 \hat{g}_{ij} .
\end{aligned} \tag{14}$$

The curvature tensor vanishes for all index combinations except those obtained by permutations of the above expressions. As usual, $k = +1(-1)$ denotes the closed (open) Friedmann universe with $k=0$ the border line. The dots denote time derivatives.

The requirement of separate homogeneity and isotropy implies for the d-dimensional energy-momentum tensor $\hat{T}_{\hat{\mu}\hat{\nu}}$:

$$\hat{T}_{00} = \hat{\rho}(t)\hat{g}_{00},$$
$$\hat{T}_{ij} = -\hat{p}(t)\hat{g}_{ij},$$
$$\hat{T}_{\alpha\beta} = -\hat{q}(t)\hat{g}_{\alpha\beta}, \tag{15}$$

with vanishing $\hat{T}_{\hat{\mu}\hat{\nu}}$ for all other index combinations. We introduce the dimensionless variable s by

$$s(t) = \ln(L(t)/L_0). \tag{16}$$

with

$$\dot{s} = \dot{L}/L \tag{17}$$

and the Hubble "constant"

$$H(t) = \dot{R}_3(t)/R_3(t). \tag{18}$$

With these definitions the functions c_i in (14) read

$$c_1 = -(L^{-2} + \dot{L}^2L^{-2}) = -\left(L_0^{-2}\exp(-2s) + \dot{s}^2\right),$$
$$c_2 = -\left(kR_3^{-2} + \dot{R}_3^2R_3^{-2}\right) = -\left(kR_3^{-2} + H^2\right),$$
$$c_3 = -\dot{L}L^{-1}\dot{R}_3R_3^{-1} = -\dot{s}H,$$
$$c_4 = -\ddot{L}L^{-1} = -(\ddot{s} + \dot{s}^2),$$
$$c_5 = -\ddot{R}_3R_3^{-1} = -(\dot{H} + H^2), \tag{19}$$

and we can now give the field equations (2) for our ansatz:

$$
\begin{aligned}
&\alpha\left\{D^2(D-1)^2c_1^2+36c_2^2+36D^2c_3^2+12D(D-1)c_1c_2\right.\\
&\quad\left.+12D^2(D-1)c_1c_3+72Dc_2c_3-4D^2c_4^2-36c_5^2-24Dc_4c_5\right\}\\
&\quad+\beta\left\{D(D-1)^2c_1^2+12c_2^2+3D(D+3)c_3^2+6D(D-1)c_1c_3\right.\\
&\qquad+12Dc_2c_3+2D(D-1)c_1c_4+12c_2c_5+6Dc_3c_4\\
&\qquad\left.+6Dc_3c_5-18Dc_4c_5-D(3D-1)c_4^2-24c_5^2\right\}\\
&\quad+\gamma\left\{2D(D-1)c_1^2+12c_2^2+12Dc_3^2-4Dc_4^2-12c_5^2\right\}\\
&\quad+\delta\left\{D(D-1)c_1+6c_2+6Dc_3\right\}+\varepsilon\\
&\quad-4\alpha(D\dot{s}+3H)\left[D(D-1)\dot{c}_1+6\dot{c}_2+6D\dot{c}_3+2D\dot{c}_4+6\dot{c}_5\right]\\
&\quad-2\beta\left\{(D\dot{s}^2+\ddot{s}+3H\dot{s})\left[D(D-1)(c_4-c_1)+3D(c_5-c_3)\right]\right.\\
&\qquad+(3H^2+\dot{H}+DH\dot{s})\left[6(c_5-c_2)+3D(c_4-c_3)\right]+D(D-1)\dot{s}\dot{c}_1+6H\dot{c}_2\\
&\qquad\left.+3D(H+\dot{s})\dot{c}_3+\left[3DH+D(D+1)\dot{s}\right]\dot{c}_4+(3D\dot{s}+12H)\dot{c}_5\right\}\\
&\quad-8\gamma\left\{D(D-1)\dot{s}^2(c_4-c_1)+6H^2(c_5-c_2)\right.\\
&\qquad\left.+D\dot{s}\dot{c}_4+3H\dot{c}_5+3DH\dot{s}(c_4-c_3)+3DH\dot{s}(c_5-c_3)\right\}=-\hat{\rho}\,,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&D(D-1)\zeta c_1^2-4(3\alpha+\beta+\gamma)c_2^2+D\left[12D\alpha-(D-9)\beta+4\gamma\right]c_3^2\\
&\quad+D\left[4D\alpha+(D+1)\beta+4\gamma\right]c_4^2+4(3\alpha+2\beta+\gamma)c_5^2+4D(D-1)\alpha c_1c_2\\
&\quad+2D(D-1)(4D\alpha+3\beta)c_1c_3+2D(D-1)(2D\alpha+\beta)c_1c_4\\
&\quad+8D(D-1)\alpha c_1c_5-4D\beta c_2c_3-8D\alpha c_2c_4-4\beta c_2c_5\\
&\quad+2D(8D\alpha+3\beta)c_3c_4+2D(12\alpha-\beta)c_3c_5+2D(8\alpha+3\beta)c_4c_5\\
&\quad+\delta\left\{D(D-1)c_1+2c_2+4Dc_3+2Dc_4+4c_5\right\}+\varepsilon\\
&\quad-4\alpha(D\dot{s}+2H)\left[D(D-1)\dot{c}_1+6\dot{c}_2+6D\dot{c}_3+2D\dot{c}_4+6\dot{c}_5\right]\\
&\quad-2\beta\left\{D(D-1)(D\dot{s}^2+\ddot{s}+3H\dot{s})(c_4-c_1)+(6H^2+2\dot{H}+2DH\dot{s})(c_5-c_2)\right.\\
&\qquad+(3DH^2+D\dot{H}+D^2H\dot{s})(c_4-c_3)+3D(D\dot{s}^2+\ddot{s}+3H\dot{s})(c_5-c_3)\\
&\qquad-D(D-1)\dot{s}\dot{c}_1+2D\dot{s}\dot{c}_2+D(D-3)\dot{s}\dot{c}_3\\
&\qquad\left.+\left[4DH+D(2D-1)\dot{s}\right]\dot{c}_4+(12H+7D\dot{s})\dot{c}_5\right\}\\
&\quad-8\gamma\left\{(4H^2+2\dot{H}+2DH\dot{s})(c_5-c_2)+D(D\dot{s}^2+\ddot{s}+2H\dot{s})\right.\\
&\qquad\qquad\left.\times(c_5-c_3)-2H\dot{c}_2-D\dot{s}\dot{c}_3+(4H+2D\dot{s})\dot{c}_5\right\}\\
&\quad-4D(D-1)\alpha\ddot{c}_1-4(6\alpha+\beta)\ddot{c}_2-2D(12\alpha+\beta)\ddot{c}_3\\
&\quad-2D(4\alpha+\beta)\ddot{c}_4-8(3\alpha+\beta+\gamma)\ddot{c}_5=\hat{p}\,,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&(D-1)(D-4)\zeta c_1^2+12(3\alpha+\beta+\gamma)c_2^2\\
&\quad+3\left[12D(D-2)\alpha+(D^2+3D-12)\beta+4(D-2)\gamma\right]c_3^2\\
&\quad+\left[4D(D-2)\alpha+(D^2+D-4)\beta+4(D-2)\gamma\right]c_4^2+12(3\alpha+\beta+\gamma)c_5^2\\
&\quad+12(D-1)(D-2)\alpha c_1c_2+6(D-1)[2D(D-3)\alpha+(D-4)\beta]c_1c_3\\
&\quad+2(D-1)[2D(D-3)\alpha+(D-4)\beta]c_1c_4+12(D-1)(D-2)\alpha c_1c_5\\
&\quad+12[6(D-1)\alpha+D\beta]c_2c_3+24(D-1)\alpha c_2c_4+12(6\alpha+\beta)c_2c_5\\
&\quad+6[4D(D-2)\alpha+(D-4)\beta]c_3c_4\\
&\quad+6[12(D-1)\alpha+D\beta]c_3c_5+6[4(D-1)\alpha+D\beta]c_4c_5\\
&\quad+\delta\{(D-1)(D-2)c_1+6c_2+6(D-1)c_3+2(D-1)c_4+6c_5\}+\varepsilon\\
&\quad-4\alpha[(D-1)\dot{s}+3H][D(D-1)\dot{c}_1+6\dot{c}_2+6D\dot{c}_3+2D\dot{c}_4+6\dot{c}_5]\\
&\quad-2\beta\{(D-1)(D-2)(\ddot{s}+D\dot{s}^2+3H\dot{s})(c_4-c_1)\\
&\qquad+6(3H^2+\dot{H}+DH\dot{s})(c_5-c_2)+3D(3H^2+\dot{H}+DH\dot{s})(c_4-c_3)\\
&\qquad+3(D-2)(\ddot{s}+D\dot{s}^2+3H\dot{s})(c_5-c_3)+3(D-1)H\dot{c}_1-6H\dot{c}_2-3(D-3)\\
&\qquad\quad\times H\dot{c}_3+[2D(D-1)\dot{s}+3(2D+1)H]\dot{c}_4+[6(D-1)\dot{s}+15H]\dot{c}_5\}\\
&\quad-8\gamma\{(D-1)[(D-1)\dot{s}^2+\ddot{s}+3H\dot{s}](c_4-c_1)+3[3H^2+\dot{H}+(D-1)H\dot{s}]\\
&\qquad\qquad\times(c_4-c_3)-(D-1)\dot{s}\dot{c}_1-3H\dot{c}_3+[2(D-1)\dot{s}+6H]\dot{c}_4\}\\
&\quad-2(D-1)(2D\alpha+\beta)\ddot{c}_1-24\alpha\ddot{c}_2-6(4D\alpha+\beta)\ddot{c}_3\\
&\quad-2[4D\alpha+(D+1)\beta+4\gamma]\ddot{c}_4-6(4\alpha+\beta)\ddot{c}_5=\hat{q}\,. \qquad (22)
\end{aligned}
$$

At first sight these equations look hopelessly complicated. We will see, however, that suitable approximations which apply to our inflationary solutions will simplify them considerably.

Conservation of the energy-momentum tensor

$$\hat{T}^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}}=0 \qquad (23)$$

follows directly from the field equations (2) by the use of suitable Bianchi identities. This is consistent with the possibility to introduce additional matter fields whose equations of motion obey the principle of equivalence in d dimensions. With our ansatz, the conservation of the energy-momentum tensor reads

$$\dot{\hat{\rho}} + 3H(\hat{\rho} + \hat{p}) + D\dot{s}(\hat{\rho} + \hat{q}) = 0. \tag{24}$$

We may use (24) to replace one of the equations of motion (20), (21) or (22) or alternatively as a consistency check of our calculations.

3. Exact de Sitter solutions in higher dimensional gravity

Let us now come to the main subject of this paper, the investigation of inflationary solutions of the gravitational field equations (2). Such solutions are characterized by a four-dimensional metric well approximated by de Sitter space with exponential expansion of $R_3(t)$ and almost constant Hubble "constant" H. For inflation to last long enough and thereby solve the cosmological problems, the time evolution of H and correspondingly of s must be slow. This means that the time derivatives of s and H must be small compared with the characteristic scale given by H for a sufficiently long time:

$$|\dot{s}| \ll H, \tag{25}$$

$$|\dot{H}| \ll H^2. \tag{26}$$

We will make this statement more precise below.

We first observe that, depending on the parameters, the system of field equations (20)–(22) admits exact de Sitter solutions with exponential expansion of R_3:

$$\begin{aligned} L(t) &= L_{\mathrm{H}}, \\ \dot{R}_3^2 R_3^{-2} &= \overline{H}^2 - kR_3^{-2}, \\ \hat{\rho} = \hat{p} &= \hat{q} = 0. \end{aligned} \tag{27}$$

The constant internal radius L_{H} is different from L_0 and $\overline{H}$ is a positive constant. For this solution, the c_i defined in (19) are as follows:

$$\begin{aligned} c_1 &= -L_{\mathrm{H}}^{-2} = -\bar{y}, \\ c_2 &= c_5 = -\overline{H}^2, \\ c_3 &= c_4 = 0. \end{aligned} \tag{28}$$

The field equations reduce to two coupled equations for $\bar{H}^2$ and $\bar{y}$:

$$6[2D(D-1)\alpha\bar{y}-\delta]\bar{H}^2+D(D-1)\zeta\bar{y}^2-D(D-1)\delta\bar{y}+\varepsilon=0, \qquad (29)$$

$$(D-1)(D-4)\zeta\bar{y}^2+(D-1)(D-2)(24\alpha\bar{H}^2-\delta)\bar{y}$$
$$+12\bar{H}^4[\zeta+(12-D(D-1))\alpha-(D-4)\beta]-12\bar{H}^2\delta+\varepsilon=0. \qquad (30)$$

Inserting the adjusted value of ε from eq. (6) and defining

$$\bar{z}=\frac{L_0^2}{L_{\rm H}^2}=\bar{y}L_0^2, \qquad (31)$$

with L_0 the ground state internal radius (7), we can use (29) to calculate the Hubble constant $\bar{H}$ as a function of $\bar{z}$:

$$\bar{H}^2=\frac{1}{24}\frac{D(D-1)}{\zeta}\frac{(1-\bar{z})^2}{(1+\sigma\bar{z})}\delta. \qquad (32)$$

Here we have introduced the parameter combination

$$\sigma=-\frac{D(D-1)\alpha}{\zeta}=\frac{(D-1)\beta+2\gamma}{\zeta}-1=\frac{\chi}{\zeta}-1. \qquad (33)$$

(Our constraints imply $\sigma\geqslant -1$.) The internal radius for the de Sitter solution is determined by (30):

$$a_1\bar{z}^3+a_2\bar{z}^2+a_3\bar{z}+a_4=P(\bar{z})=0, \qquad (34)$$

with

$$a_1=D(D-1)+D(D-1)\tau+[D(D-1)-12]\sigma-12\sigma^2,$$

$$a_2=-3D(D-1)-3D(D-1)\tau-\frac{3}{D}[D^2(D-1)-4D+16]\sigma+12\frac{D-4}{D}\sigma^2,$$

$$a_3=\frac{3}{D}[D^2(D-1)-4D-16]+3D(D-1)\tau+\frac{3}{D}[D^2(D-1)-4D-16]\sigma,$$

$$a_4=-D(D-1)+12-D(D-1)\tau-[D(D-1)-12]\sigma \qquad (35)$$

and

$$\tau = -(D-4)\frac{\beta}{\zeta}. \tag{36}$$

Comparing the Hubble constant for the de Sitter solution with the Planck mass determined in (9), one finds

$$\frac{\bar{H}^2}{M_P^2} = \frac{1}{384\pi}\frac{D(D-1)}{\chi}\frac{(1-\bar{z})^2}{(1+\sigma\bar{z})}. \tag{37}$$

For not too small values of χ, $\bar{H}^2$ is considerably smaller than M_P^2. If $\bar{z}$ is also small, all relevant mass scales of such a de Sitter solution are well below the Planck mass. This enhances our confidence in the classical treatment used in this paper. We note that for $D \neq 4$ a value of γ near γ_c,

$$\gamma_c = -\frac{3}{2}\frac{(D-1)}{(D+3)}\beta, \tag{38}$$

implies that a_4 approaches zero which therefore leads to solutions with very small values of $\bar{z}$. For $D=4$, one has $a_4 = 0$ independent of α, β and γ. For a given set of parameters α, β, γ there are up to three exact de Sitter solutions corresponding to the solutions of (34).

4. Approximate de Sitter solutions and the inflationary universe

There is no reason to assume that the universe was ever described exactly by one of these exact de Sitter solutions. We are more interested in general solutions characterized by a slow time evolution of $L(t)$ and an exponential expansion of $R_3(t)$ during an inflationary phase which, after a transition period, approach the Friedmann solution (eqs. (10)–(12)). During the inflationary phase, such solutions are *approximate* de Sitter solutions.

Such a scenario may be realized if the internal radius $L(t)$ grows until some time t_0 after which gravitational damping stops its further increase. The radius $L(t)$ subsequently decreases. If $L(t_0)$ is sufficiently near L_H the evolution of $L(t)$ in the vicinity of t_0 may be described by small deviations from the above exact de Sitter solutions. In this case the evolution of $L(t)$ after t_0 very much resembles the "rolling down" of a scalar field from a potential maximum in models of "new inflation". As for these models, gravitational damping should be the dominant contribution to the kinetic terms for s for a certain period of time after t_0. During the inflationary period we have

$$|\ddot{s}| \ll |H\dot{s}|, \qquad |\ddot{H}| \ll |H\dot{H}|, \tag{39}$$

and similar inequalities hold for all other higher derivatives of s and H. At some time t_1 the approximations (25), (26), (39) characterizing the slow time evolution of an approximate de Sitter solution will break down. Sufficient inflation will be obtained if

$$t_1 - t_0 \gtrsim 60H^{-1}. \tag{40}$$

Other scenarios for an inflationary period can be imagined. For example, $L(t_0)$ must not necessarily be near the radius L_H of an exact de Sitter solution, provided (25), (26), (39), and (40) hold. Also, the universe may go to an approximate de Sitter solution by tunnelling effects and subsequently be slowly moving away from such a solution. In fact, all solutions obeying (25), (26), (39), and (40) typically lead to an inflationary phase in the evolution of the universe.

For a study of possible inflationary solutions we may therefore linearize the field equations in $\dot{s}$ and $\dot{H}$ and neglect all second and higher time derivatives of s and H. All terms proportional R_3^{-1} also can be omitted due to the exponential expansion of $R_3(t)$. In this approximation one has

$$\begin{aligned} c_1 &= -y = -L_0^{-2}z = -L_0^{-2}\exp(-2s), \\ c_2 &= -H^2, \\ c_3 &= -H\dot{s}, \\ c_4 &= 0, \\ c_5 &= -H^2 - \dot{H}, \end{aligned} \tag{41}$$

and the field equations (20)–(22) simplify

$$\begin{aligned} &D(D-1)\zeta y^2 - D(D-1)\,\delta y + \varepsilon - 6H^2[\delta - 2D(D-1)\alpha y] \\ &\quad + 72(3\alpha+\beta+\gamma)H^2\dot{H} + 6D(12\alpha+3\beta+4\gamma)H^3\dot{s} \\ &\quad + 12D(D-1)(D-2)\alpha H y\dot{s} - 6D\delta H\dot{s} = -\hat{\rho}, \end{aligned} \tag{42}$$

$$\begin{aligned} &D(D-1)\zeta y^2 - D(D-1)\delta y + \varepsilon - 6H^2[\delta - 2D(D-1)\alpha y] \\ &\quad + 72(3\alpha+\beta+\gamma)H^2\dot{H} + 8D(D-1)\alpha y\dot{H} - 4\delta\dot{H} + 2D(12\alpha+3\beta+8\gamma)H^3\dot{s} \\ &\quad + 8D(D-1)(D-2)\alpha H y\dot{s} - 4D\delta H\dot{s} = \hat{p}, \end{aligned} \tag{43}$$

$$\begin{aligned} &(D-1)(D-4)\zeta y^2 - (D-1)(D-2)\,\delta y + \varepsilon - 12H^2 \\ &\quad \times[\delta - 2(D-1)(D-2)\alpha y - (12\alpha+3\beta+2\gamma)H^2] + 12(36\alpha+9\beta+2\gamma)H^2\dot{H} \\ &\quad + 12(D-1)(D-2)\alpha y\dot{H} - 6\delta\dot{H} + 18[8(D-1)\alpha + (D-2)\beta - 4\gamma]H^3\dot{s} \\ &\quad + 12(D-1)[D(D-5)\alpha - 2\beta - 2\gamma]H y\dot{s} - 6(D-1)\,\delta H\dot{s} = \hat{q}. \end{aligned} \tag{44}$$

As a check of our calculations we verified that the left-hand sides of (42)–(44) indeed obey the relation (24) up to terms involving more than one time derivative of H or s.

To proceed further, we need to make some assumptions on the energy-momentum tensor $\hat{T}_{\hat{\mu}\hat{\nu}}$. We will suppose that $|\hat{q}|$ is roughly of the same order as $\hat{\rho}$ or smaller and that $\hat{p}$ is positive and at most of the same order of magnitude as $\hat{\rho}$. No further specifications of the equation of state are needed. In this case we can neglect the third term in (24) since $|D\dot{s}| \ll 3H$ according to the approximation (25). Therefore $\hat{\rho}$ decreases exponentially during the inflationary period

$$\hat{\rho}(t) < \hat{\rho}(t_0)\exp[-3H(t-t_0)] \tag{45}$$

and we can neglect all effects from incoherent excitations during the inflationary period. As a consequence, the inflationary period is rather insensitive to the state of the universe before inflation: The universe may have been hot or cold – in the limiting case containing almost no entropy!

We now have to solve eqs. (42) and (44) with vanishing right-hand side. (Eq. (43) is automatically fulfilled since it follows from (24), (42), and (44).) This is a system of two differential equations for two functions s and H. Consistency requires that possible solutions obey the approximations (25), (26), and (39) which in general will be the case only for a certain range of values of the parameters α, β and γ. To leading order, the Hubble constant H_0 is determined by neglecting time derivatives of H and s in (42)

$$H^2 = H_0^2 + \Delta_{\mathrm{H}},$$

$$H_0^2 = \frac{1}{24}\frac{D(D-1)}{\zeta}\frac{(1-z)^2}{(1+\sigma z)}\delta. \tag{46}$$

Here we have inserted eqs. (6) and (7) for ε and L_0^{-2} (compare eq. (32)). Contributions to the Hubble constant due to the time variation of H and s are given by

$$\Delta_{\mathrm{H}} = \frac{D(D-1)}{2\zeta}(3\alpha+\beta+\gamma)\frac{(1-z)^2}{(1+\sigma z)^2}\dot{H}$$

$$+\left\{\frac{D^2(D-1)}{24\zeta}(12\alpha+3\beta+4\gamma)\frac{(1-z)^2}{(1+\sigma z)^2} - \frac{[D+(D-2)\sigma z]}{(1+\sigma z)}\right\}H_0\dot{s}. \tag{47}$$

According to our approximations we have $\Delta_{\mathrm{H}} \ll H_0^2$ and $\dot{H}$ can be obtained to leading order by taking a time derivative of (46):

$$\dot{H} = z\left\{\frac{2}{1-z} + \frac{\sigma}{1+\sigma z}\right\}H_0\dot{s} \equiv g_1(z)H_0\dot{s}. \tag{48}$$

Inserting in (47) we can write Δ_H in the form

$$\Delta_H = g_2(z) H_0 \dot{s},$$

$$g_2(z) = \frac{D(D-1)}{2\zeta} \frac{1}{(1+\sigma z)^2} \times \left\{ -D\alpha + \frac{D-8}{4}\beta + \left(\frac{D}{3} - \frac{4}{D-1}\right)\gamma + z\left[(2D-1)\alpha + \left(1 - \tfrac{1}{2}D\right)\beta + \left(1 - \tfrac{2}{3}D\right)\gamma\right] + z^2\left[(D-3+2(D-2)\sigma)\alpha + \left(\tfrac{1}{4}D - 1\right)\beta + \left(\tfrac{1}{3}D - 1\right)\gamma\right] + \frac{z(1-z)}{1+\sigma z}(1+\sigma)(3\alpha + \beta + \gamma) \right\}. \tag{49}$$

Eqs. (46)–(49) determine H and $\dot{H}$ as functions of s and $\dot{s}$. The evolution equation for s is now obtained by inserting these equations into eq. (44):

$$g_3(z) H_0 \dot{s} + g_4(z) \dot{H} + g_5(z)\, \Delta_H = k(z) H_0^2, \tag{50}$$

with

$$g_3(z) = \left(\frac{D}{8} - \tfrac{5}{4} + \frac{1}{D}\right)\beta - \left(\tfrac{1}{2} + \frac{2}{D}\right)\gamma - z\left\{4\alpha + \left(\frac{D}{4} - \tfrac{1}{2} + \frac{2}{D}\right)\beta - \left(1 - \frac{2}{D}\right)\gamma\right\} + z^2\left\{(D-1+(D-5)\sigma)\alpha + \left(\frac{D-2}{8} - \frac{2}{D}\sigma\right)\beta - \left(\tfrac{1}{2} + \frac{2}{D}\sigma\right)\gamma\right\}, \tag{51}$$

$$g_4(z) = 2\alpha + \left(\tfrac{3}{4} - \frac{1}{D}\right)\beta + \left(\tfrac{1}{6} - \frac{2}{D(D-1)}\right)\gamma - z\left\{\left(4 + \frac{2}{D}\right)\alpha + \tfrac{3}{2}\beta + \tfrac{1}{3}\gamma\right\} + z^2\left\{\left(3 + \sigma - \frac{2}{D}\sigma\right)\alpha + \tfrac{3}{4}\beta + \tfrac{1}{6}\gamma\right\}, \tag{52}$$

$$g_5(z) = \left(\tfrac{1}{2} - \frac{2}{D}\right)\beta + \left(\tfrac{1}{3} - \frac{4}{D(D-1)}\right)\gamma - z\left\{\frac{4}{D}\alpha + \beta + \tfrac{2}{3}\gamma\right\} + z^2\left\{\left(2 + 2\sigma - \frac{4}{D}\sigma\right)\alpha + \tfrac{1}{2}\beta + \tfrac{1}{3}\gamma\right\}, \tag{53}$$

$$k(z) = \frac{1}{1-z}\left\{\left(\frac{1}{D} - \tfrac{1}{4}\right)\beta + \left(\frac{2}{D(D-1)} - \tfrac{1}{6}\right)\gamma + z\left[\left(\tfrac{3}{4} - \frac{1}{D} - \frac{4}{D^2}\right)\beta + \left(\tfrac{1}{2} - \frac{2}{D(D-1)} - \frac{8}{D^2(D-1)}\right)\gamma\right] + z^2\left[\left(-1 + \frac{4}{D}\right)(1+\sigma)\alpha - \tfrac{3}{4}\beta - \tfrac{1}{2}\gamma\right] + z^3\left[(1+\sigma)\alpha + \tfrac{1}{4}\beta + \tfrac{1}{6}\gamma\right]\right\}. \tag{54}$$

One finds

$$\frac{\dot{s}}{H_0} = \frac{k(z)}{g_3(z) + g_1(z)g_4(z) + g_2(z)g_5(z)} = w(z). \tag{55}$$

This is the key equation for the inflationary period.

The inflationary period is characterized by a slow time evolution of s compared with the Hubble "constant" characterizing the expansion rate of the usual three space dimensions. This requires

$$|w(z)| \ll 1. \tag{56}$$

Our approximations (26), (39) hold provided

$$|g_1(z)w(z)| \ll 1,$$

$$\left| g_1(z)w(z) - 2z\frac{\mathrm{d}w}{\mathrm{d}z} \right| \ll 1,$$

$$\left| 2g_1(z)w(z) - 2z\frac{\mathrm{d}w}{\mathrm{d}z} - 2z\frac{w(z)}{g_1(z)}\frac{\mathrm{d}g_1(z)}{\mathrm{d}z} \right| \ll 1. \tag{57}$$

In the region of interest $0 \leqslant z < 1$ the function $g_1(z)$ (eq. (48)) (as well as $(z/g_1)(\mathrm{d}g_1/\mathrm{d}z)$) is of order unity or smaller provided z is not too near the pole at $z = 1$. Solutions consistent with our approximations therefore exist provided in addition to (56) one has

$$\left| z\frac{\mathrm{d}w}{\mathrm{d}z} \right| \ll 1. \tag{58}$$

Assume now that at some time t_0 the universe is characterized by an inflationary solution obeying (46), (55), (56), and (58). As we will see, such solutions often exist for an appropriate range of z. The critical question is: How long will the unverse stay in this regime? Is the time of inflation sufficient to produce the 60 or more e-foldings in $R_3(t)$ required to solve the cosmological horizon and flatness problems? We will discuss two different circumstances where sufficient inflation can be realized.

4.1 INFLATION FOR LARGE INTERNAL RADIUS

Assume that at t_0 the internal radius $L(t_0)$ is much larger than L_0 so that $z \ll 1$. In this case $w(z)$ approaches a constant

$$k(0) = -\frac{1}{12}\frac{D-4}{D(D-1)}\{3(D-1)\beta + 2(D+3)\gamma\},$$

$$g_1(0) = 0,$$

$$g_2(0) = -\tfrac{1}{2}D^2(D-1)\frac{\alpha}{\zeta} + \tfrac{1}{8}D(D-1)(D-8)\frac{\beta}{\zeta} + \tfrac{1}{6}D(D+3)(D-4)\frac{\gamma}{\zeta},$$

$$g_3(0) = \frac{1}{8D}\{(D^2 - 10D + 8)\beta - 4(D+4)\gamma\},$$

$$g_5(0) = -2k(0),$$

$$w(0) = \frac{k(0)}{g_3(0) - 2k(0)g_2(0)}. \tag{59}$$

We choose parameters so that $k(0)$ is negative and $g_3(0) - 2k(0)g_2(0)$ positive. In this case the internal radius decreases slowly after t_0 with s almost constant. The evolution of s is driven by a constant force $\sim k(0)$ and damped by a force $\sim g_3(0) - 2k(0)g_2(0)$. (This motion is a good approximation if $w(0)$ is sufficiently small; condition (58) holds trivially for small z.) The inflationary period ends once $L(t)$ becomes so small that the approximation $w(z) \approx w(0)$ (or (58)) breaks down. We denote by s_1 the value of $s(t)$ where this happens, and $s_0 = s(t_0)$, $\Delta s = s_0 - s_1$. The duration of inflation is given by

$$t_1 - t_0 \approx \frac{\Delta s}{w(0)H_0(0)} \tag{60}$$

and sufficient inflation is obtained for

$$\frac{w(0)}{\Delta s} \lesssim \frac{1}{60}. \tag{61}$$

It is obviously possible to obtain an arbitrarily long inflation time starting with very large s_0. This, however, corresponds to an enormous internal radius $L(t_0) = L_0 \exp s_0$

which seems unlikely to be realized by the evolution of the universe prior to t_0. We rather concentrate on moderate values of $L(t_0)$ roughly an order of magnitude bigger than L_0, implying a range $1 \lesssim |\Delta s| \lesssim 3$. Sufficient inflation requires then

$$|w(0)| \lesssim \tfrac{1}{20}. \tag{62}$$

This can be realized if $k(0)$ is sufficiently small compared to $g_3(0)$. The term $\sim k(0) g_2(0)$ in (59) is small and eq. (62) is fulfilled [12] for $g_2(0) \gtrsim -5$ and

$$\frac{2}{3}\frac{D-4}{D-1}\frac{3(D-1)\beta + 2(D+3)\gamma}{(D^2-10D+8)\beta - 4(D+4)\gamma} \lesssim \frac{1}{30}. \tag{63}$$

For suitable choices of β and γ this condition can be easily fulfilled and sufficient inflation is realized.

As an example, we may consider, for $D = 9$, the range $-\beta \leqslant \gamma \lesssim -0.85\beta < 0$, $\alpha \gtrsim \beta$. (The special case $D = 4$ will be discussed in a subsequent paper of this series.) In contrast to many four-dimensional models, sufficient inflation occurs quite naturally, without any extreme fine tuning of parameters!

4.2. INFLATION NEAR EXACT DE SITTER SOLUTIONS

Let us now discuss solutions in the vicinity of the exact de Sitter solutions presented in sect. 3: Assume that at t_0 the internal radius is near a critical value corresponding to an exact de Sitter solution

$$|s(t_0) - \bar{s}| \ll 1 \tag{64}$$

with $\bar{z} = \exp(-2\bar{s})$ a zero of the polynomial $P(z)$ (eq. (34)). Since

$$k(z) = \frac{\zeta}{12D(D-1)}\frac{1}{(1-z)}P(z), \tag{65}$$

one has $w(\bar{z}) = 0$ and we can expand the equation of motion (55) around $\bar{z}$

$$\dot{s} \approx \frac{\mathrm{d}w}{\mathrm{d}z}(\bar{z}) H_0(\bar{z})(z - \bar{z}) \tag{66}$$

or

$$\dot{\tilde{s}} \approx -2\bar{z}\frac{\mathrm{d}w}{\mathrm{d}z}(\bar{z}) H_0(\bar{z})\tilde{s}, \tag{67}$$

with $\tilde{s} = s - \bar{s}$. This approximation is valid as long as $|\tilde{s}|$ is sufficiently small compared to one and leads to the solution

$$s(t) - \bar{s} = (s(t_0) - \bar{s})\exp(\omega(t - t_0)), \tag{68}$$

with

$$\omega = -2\bar{z}\frac{\mathrm{d}w}{\mathrm{d}z}(\bar{z})H_0(\bar{z}). \tag{69}$$

For a small positive ω this corresponds to a slow evolution away from the exact de Sitter solution. If $s(t_0)$ is not extremely close to $\bar{s}$ this inflationary period ends at t_1 with $t_1 - t_0 \approx \omega^{-1}$. Sufficient inflation is therefore obtained if

$$\frac{\omega}{H_0} = -2\bar{z}\frac{\mathrm{d}w}{\mathrm{d}z}(\bar{z}) \lesssim \frac{1}{60}\,. \tag{70}$$

Replacing the parameters α, β and γ by ζ, σ and $\bar{z}$ one obtains

$$\frac{\omega}{H_0} = \frac{4\bar{z}}{(1-\bar{z})^2}\frac{(1+\sigma)}{D\tilde{B}}\left\{\frac{2}{D}(1+2(1+\sigma)\bar{z}+\sigma\bar{z}^2)-(1-\bar{z})(1+\sigma\bar{z})\right\}, \tag{71}$$

with

$$\tilde{B}(\sigma,\zeta,\bar{z}) = \frac{(D-1)}{\zeta}(g_3(\bar{z})+g_1(\bar{z})g_4(\bar{z})+g_2(\bar{z})g_5(\bar{z})), \tag{72}$$

a function of σ, ζ and $\bar{z}$ which is of order one for a wide range of parameters. This is the result of ref. [3] and sufficient inflation can be obtained for a suitable choice of parameters.

5. Conclusion

We have found two scenarios for a phase of exponential expansion of three space dimensions, during which the volume of the internal space remains almost constant. Both are described by approximate de Sitter solutions. One scenario describes the vicinity of exact de Sitter solutions, whereas the other more generic scenario applies to situations where the radius of the internal space has grown sufficiently, say an order of magnitude, larger than its ground state value and starts to subsequently decrease. The inflationary phase achieves the separation of length scales between the internal space and our observed four-dimensional world. After inflation, a four-dimensional description of the universe becomes more appropriate for questions like the heating of the universe, entropy production, and the evolution of density fluctuations. We will give more details of the four-dimensional treatment [11, 12] of our model in subsequent publications.

Let us finally turn to the question: How did the inflationary phase start? For our solutions with two degrees of freedom $R_3(t)$, $L(t)$ to apply, one needs at least one region of the universe which is sufficiently homogeneous and isotropic separately for internal space and the observable three dimensions during the whole inflationary period. At some time t_0 this region must have evolved into one of the approximate de Sitter solutions. In particular, the energy-momentum tensor on the right-hand side of the gravitational equations must have become negligible. If this region is large enough so that boundary effects can be neglected, it can evolve to the entire presently observed universe. We emphasize that inflation is described here by pure classical gravity for the evolution of $R_3(t)$ and $L(t)$ which should be reliable if all length scales are sufficiently larger than the Planck length. Ambiguities such as how to determine the cosmological constant in quantum theory are not immediately relevant in this scenario – except for the unnatural fine tuning (6) of the cosmological constant for the ground state which hopefully will be replaced by some mechanism of dynamical adjustment. We note that small changes of the four-dimensional or higher dimensional cosmological constant do not modify the inflationary phase.

The question if the conditions for a beginning of inflation are fulfilled involves cosmology at length scales of the order of the compactification scale or shorter. This is a difficult subject since one moves quickly into a region where quantum gravity should play an important role. It is conceivable that the approximate de Sitter state at t_0 is a direct result of a genuine quantum process ("tunnelling from nothing" [21]). Alternatively, it is possible that this state is a result of previous (classical) evolution of the universe. In this case one first has to explain the split in topology into three and D space dimensions (a typical topology for the spacelike dimensions would be $S^D \times S^3$). This needs a description of transitions between different topologies. Continuous transitions of this type require "non-compact" internal space [22] at intermediate steps. The next question concerns the validity of a description by two degrees of freedom $R_3(t)$ and $L(t)$. The likelihood for a region with sufficient homogeneity and isotropy depends to a large extent on whether additional degrees of freedom have a tendency to be damped and die out. (This is partly related to classical stability of our approximate de Sitter solutions, which is certainly a necessary condition for our scenario to work.) If a description by R_3 and L is appropriate, it is plausible that at some moment R_3 and L have grown relatively large, then L stops growing due to gravitational damping and subsequently decreases because of potential terms. Entropy will be diluted in such a process so that neglecting the energy-momentum tensor becomes justified. Our scenario is therefore rather insensitive to the amount of entropy before inflation: The universe could have been hot or cold – in the extreme case even without any incoherent excitations and vanishing entropy.

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