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HIGH-ENERGY, LARGE-MOMENTUM-TRANSFER PROCESSES: LADDER DIAGRAMS IN φ^3 THEORY (I)

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Relativistic quantum field theories may give us useful guidance to understanding high-energy, large-momentum-transfer processes, where the center-of-mass energy is much larger than the transverse momentum transfers, which are in turn much larger than the masses of the participating particles. With this possibility in mind, we study the ladder diagrams in φ^3 theory. In this paper, some of the necessary techniques are developed and applied to the simplest cases of the fourth- and sixth-order ladder diagrams.

1. Introduction

During the sixties and early seventies, there was a systematic effort to study the asymptotic behavior of scattering amplitudes in relativistic quantum field theories. At that time, the limit of interest was large s (the square of the center-of-mass energy) with fixed transverse momentum transfers. One of the deep and unexpected results is that both the total hadron-hadron cross section σ_{tot} and the ratio $\sigma_{\text{el}}/\sigma_{\text{tot}}$ of the integrated elastic cross section to the total cross section must increase at high energies [1]. This was later confirmed both at the CERN ISR [2] and more dramatically at the CERN $\bar{\text{p}}\text{p}$ Collider [3].

It is the purpose of this paper to initiate an attempt to extend this previous study to the case where both the center-of-mass energy and the transverse momentum transfers are large. Such problems have been studied before [4], but the present point of view is slightly different in the realization that, in the physically interesting cases, the transverse momentum transfers, although large, may nevertheless be much smaller than the center-of-mass energy. For example, at the CERN $\bar{\text{p}}\text{p}$ Collider and

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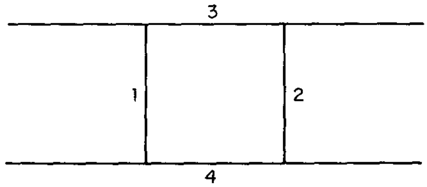

Fig. 1. The box diagram, or the fourth-order ladder diagram in φ^3 theory.

the Fermilab Tevatron Collider, the center-of-mass energy is of the order of 1 TeV, while the transverse momentum transfer is perhaps of the order of 30 GeV/c. Theoretically, it may be added that it is sometimes more profitable to attempt to enlarge the range of applicability of an approach instead of jumping to a different case.

For two-body elastic amplitudes, the case of interest is therefore

$$s \gg |t| \gg m^2, \tag{1.1}$$

where m is the mass of the heaviest particle in the theory. It goes without saying that this case (1.1) cannot be analyzed by simply taking the previous results of large s and fixed t and then letting t be large. For example, s may be smaller than t^2/m^2, or more generally $\ln(s/m^2)$ and $\ln(|t|/m^2)$ may be comparable.

In the study of the sixties on the case of fixed momentum transfers, the first step is to understand the ladder diagrams [5] in φ^3 theory, even though this is not a "good" theory. It is therefore a natural proposal to consider first the same diagrams in the more complicated situation of (1.1). In this paper, we restrict ourselves to the two simplest cases of the fourth- and the sixth-order ladders of figs. 1 and 2. The eighth-order ladder of paper II is already much richer in structure. In particular, we shall see that it is asymptotically given by two different expressions, depending on whether $\ln(|t|/m^2)$ is less than or larger than $\frac{1}{2}\ln(s/m^2)$.

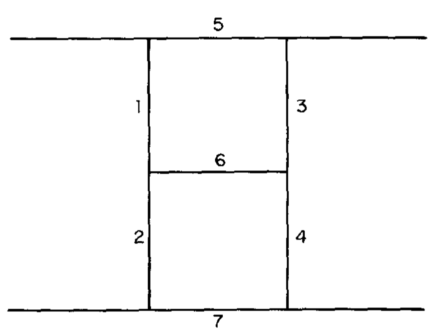

Fig. 2. The sixth-order ladder diagram in φ^3 theory.

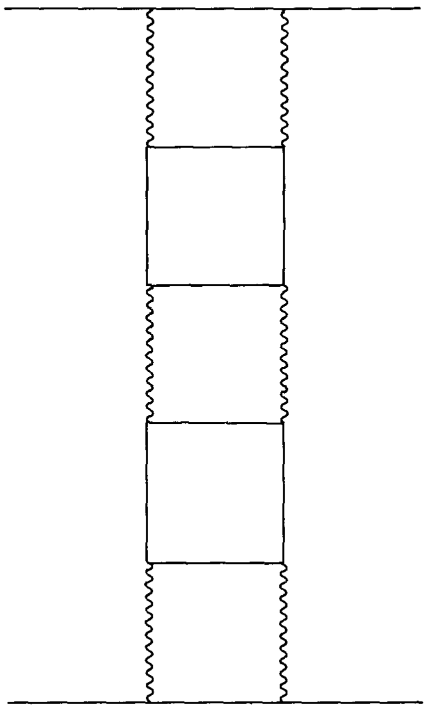

Fig. 3. An example of tower diagrams in gauge theory.

There is a more speculative motivation for the present program. When applied to gauge theories, the important diagrams for large s but fixed t are the tower and multitower diagrams [6]. An example of the low-order tower diagrams is shown in fig. 3. Roughly speaking, the tower diagrams are one-dimensional. For (1.1), we speculate that, again when applied to gauge theories, the important diagrams are two-dimensional. If so, a low-order example of such diagrams is perhaps the one of fig. 4. A further speculation is then that these "castle" diagrams may be related to strings [7–9]. For both the one-dimensional tower diagrams and the two-dimensional castle diagrams, the sums of the contributions, but not the individual ones, are gauge invariant. Consequently, most of the castle diagrams, including the one shown in fig. 4, are not planar.

2. Method of approach

In studying the asymptotic behavior of scattering amplitudes for large s but fixed t, at least two distinct approaches have been used: in one approach Feynman parameters are employed, while in the other one the integrals are considered directly in momentum space. Generally speaking, the first approach requires less *a priori*

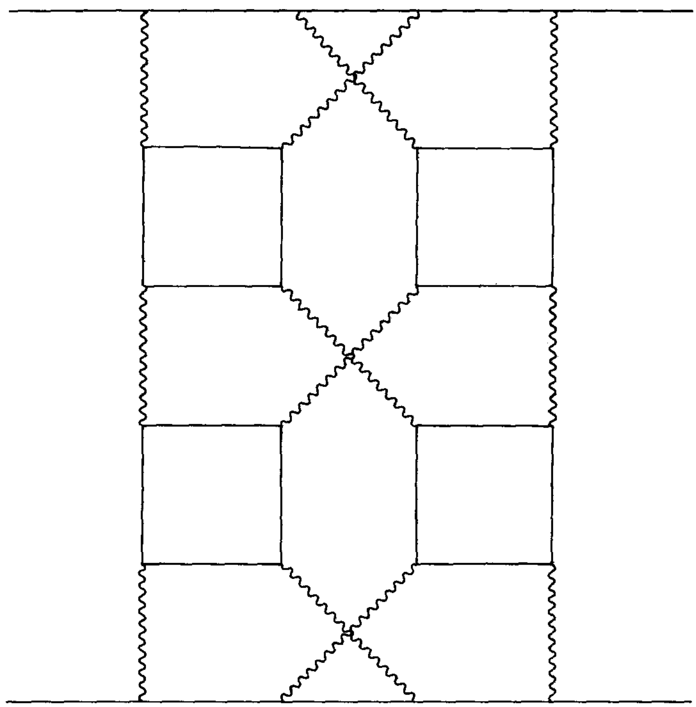

Fig. 4. An example of castle diagrams in gauge theory.

knowledge about the asymptotic behavior, while the latter leads more readily to physical insight.

At present very little is known about the corresponding asymptotic behavior of scattering amplitudes for (1.1). Under this circumstance, it is natural to try first the Feynman-parameter approach. This is the approach to be used here.

In the Feynman-parameter approach for large s but fixed t, it is most convenient to start with the Mellin transform formula [10]

$$\int_0^{\infty}(As+B+i\varepsilon)^{-a}s^{-\zeta}\,\mathrm{d}s=\frac{\Gamma(1-\zeta)\Gamma(a-1+\zeta)}{\Gamma(a)}(A+i\varepsilon)^{-1+\zeta}(B+i\varepsilon)^{1-a-\zeta}, \tag{2.1}$$

where $\varepsilon \to 0^+$ and Γ is the gamma function. A corresponding formula is needed for the case (1.1). For this purpose, let

$$t=-t_0 s^{\gamma}, \tag{2.2}$$

where $t_0>0$ and $0<\gamma<1$. Then the generalization of (2.1) to be used as the starting point here is

$$\int_0^\infty \mathrm{d}s\, s^{-\zeta}(As+Bt+C+i\varepsilon)^{-a}\Big|_{t=-t_0 s^\gamma}$$
$$=[\Gamma(a)]^{-1}(2\pi i)^{-1}\int_{c_1-i\infty}^{c_1+i\infty}\mathrm{d}z_1\int_{c_2-i\infty}^{c_2+i\infty}\mathrm{d}z_2\,\Gamma(z_1)\Gamma(z_2)\Gamma(a-z_1-z_2)$$
$$\times\delta_c(1-\zeta-z_1-\gamma z_2)(A+i\varepsilon)^{-z_1}(-B+i\varepsilon)^{-z_2}(C+i\varepsilon)^{-a+z_1+z_2}t_0^{-z_2}. \quad (2.3)$$

The r.h.s. of (2.3) needs some explanation. The contours of integration for z_1 and z_2 are parallel to the imaginary axis. The delta function $\delta_c(1-\zeta-z_1-\gamma z_2)$ is somewhat different from the conventional one since the argument is complex. It means that, for ζ real

(i) the contours are chosen so that

$$1-\zeta-c_1-\gamma c_2=0; \quad (2.4)$$

(ii) with (2.4), the δ_c is

$$\delta_c(1-\zeta-z_1-\gamma z_2)=i^{-1}\delta(-\operatorname{Im} z_1-\gamma\operatorname{Im} z_2), \quad (2.5)$$

where the δ function on the r.h.s. has a real argument and is the usual Dirac delta function. Eq. (2.3) holds when $a>0$, with c_1 and c_2 chosen so that

$$c_1>0, \qquad c_2>0, \qquad a-c_1-c_2>0. \quad (2.6)$$

The relation (2.3) can be proved in various ways. One way makes use of the representation

$$\mathrm{e}^{iT}=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\mathrm{d}z\,\Gamma(z)\mathrm{e}^{i\pi z/2}T^{-z}, \quad (2.7)$$

where $c>0$. With (2.7), it is a straightforward calculation to find that

$$\text{l.h.s. of (2.3)}=\int_0^\infty \mathrm{d}s\,s^{-\zeta}[\Gamma(a)]^{-1}\mathrm{e}^{-i\pi a/2}\int_0^\infty \mathrm{d}y\,y^{a-1}\exp(iy(As-Bt_0s^\gamma+C+i\varepsilon))$$
$$=[\Gamma(a)]^{-1}\mathrm{e}^{-i\pi a/2}\int_0^\infty\mathrm{d}y\,y^{a-1}\mathrm{e}^{iy(C+i\varepsilon)}\int_0^\infty\mathrm{d}s\,s^{-\zeta}$$
$$\times(2\pi i)^{-2}\int_{c_1-i\infty}^{c_1+i\infty}\mathrm{d}z_1\int_{c_2-i\infty}^{c_2+i\infty}\mathrm{d}z_2\,\Gamma(z_1)\Gamma(z_2)\mathrm{e}^{i\pi(z_1+z_2)/2}$$
$$\times[(A+i\varepsilon)ys]^{-z_1}[(-B+i\varepsilon)t_0ys^\gamma]^{-z_2}$$
$$=[\Gamma(a)]^{-1}\mathrm{e}^{-i\pi a/2}(2\pi i)^{-2}\int_{c_1-i\infty}^{c_1+i\infty}\mathrm{d}z_1\int_{c_2-i\infty}^{c_2+i\infty}\mathrm{d}z_2\,\Gamma(z_1)\Gamma(z_2)$$
$$\times\mathrm{e}^{i\pi(z_1+z_2)/2}(A+i\varepsilon)^{-z_1}(-B+i\varepsilon)^{-z_2}t_0^{-z_2}(C+i\varepsilon)^{-a+z_1+z_2}$$
$$\times\Gamma(a-z_1-z_2)\mathrm{e}^{i\pi(a-z_1-z_2)/2}\int_0^\infty\mathrm{d}s\,s^{-\zeta-z_1-\gamma z_2}. \quad (2.8)$$

In order for the integration over s to be meaningful, the contours of integration in z_1 and z_2 must have been chosen to satisfy (2.4). With this choice and since (with $s = e^\tau$)

$$\int_0^\infty \mathrm{d}s\, s^\lambda \Big|_{\mathrm{Re}\,\lambda=-1} = \int_{-\infty}^\infty \mathrm{d}\tau\, e^{i\tau\,\mathrm{Im}\,\lambda} = 2\pi\delta(\mathrm{Im}\,\lambda), \tag{2.9}$$

it follows from (2.5) that the right-hand sides of (2.3) and (2.8) are the same. This proves (2.3).

In this paper, this fundamental representation (2.3) is to be applied to the diagrams of figs. 1 and 2. An alternative way of studying these ladder diagrams in φ^3 theory is to use the Fredholm integral equation [5].

3. Box diagram

For the φ^3 theory, there is only one mass, which can be set to be 1 without loss of generality. Furthermore, in (2.3) t_0 appears in a rather trivial way. Hence, again without loss of generality, t_0 can be set to be 1.

The scattering amplitude due to the box diagram of fig. 1 is proportional to [11]

$$I_2(s,t) = \int_0^1 \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\,\delta\left(1 - \sum_{j=1}^4 \alpha_j\right)$$
$$\times\left\{\alpha_3\alpha_4 s + \alpha_1\alpha_2 t - \left[1 - (\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\right] + i\varepsilon\right\}^{-2}. \tag{3.1}$$

The choice of the α's is also shown in fig. 1. In order to study the behavior of this I_2 for the case of (1.1) where $s \gg -t \gg 1$, define the Mellin transform

$$\tilde{I}_2(\zeta) = \int_0^\infty \mathrm{d}s\, s^{\gamma-\zeta} I_2(s, -s^\gamma), \tag{3.2}$$

where the choice of the exponent $\gamma - \zeta$ is motivated by the observation that $I_2(s,t)$ is very roughly (i.e. modulo logarithms) of the order of $(s|t|)^{-1} = s^{-(1+\gamma)}$. We want to evaluate this $\tilde{I}_2(\zeta)$ for ζ small and positive.

By the fundamental formula (2.3), an alternative expression for this $\tilde{I}_2(\zeta)$ is

$$\tilde{I}_2(\zeta) = (2\pi i)^{-1}\int_{c_1-i\infty}^{c_1+i\infty}\mathrm{d}z_1\int_{c_2-i\infty}^{c_2+i\infty}\mathrm{d}z_2\,\Gamma(z_1)\Gamma(z_2)\Gamma(2-z_1-z_2)$$
$$\times\delta_c(1+\gamma-\zeta-z_1-\gamma z_2)\int_0^1\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\,\delta\left(1-\sum_{j=1}^4\alpha_j\right)(\alpha_3\alpha_4)^{-z_1}$$
$$\times(-\alpha_1\alpha_2+i\varepsilon)^{-z_2}\left[-1+(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)+i\varepsilon\right]^{-2+z_1+z_2}. \tag{3.3}$$

The α integrations are convergent if $c_1 < 1$ and $c_2 < 1$. Since ζ is small and, by (2.4),

$$1 + \gamma - \zeta - c_1 - \gamma c_2 = 0, \tag{3.4}$$

let

$$z_1 = 1 - \zeta y_1,$$

$$z_2 = 1 - \zeta y_2. \tag{3.5}$$

Then

$$\Gamma(z_1) = \Gamma(1 - \zeta y_1) \sim 1,$$

$$\Gamma(z_2) = \Gamma(1 - \zeta y_2) \sim 1,$$

$$\Gamma(2 - z_1 - z_2) = \Gamma(\zeta y_1 + \zeta y_2) \sim [\zeta(y_1 + y_2)]^{-1},$$

and

$$\delta_c(1 + \gamma - \zeta - z_1 - \gamma z_2) = \zeta^{-1}\delta_c(1 - y_1 - \gamma y_2). \tag{3.6}$$

Therefore, to leading order for small ζ,

$$\tilde{I}_2(\zeta) \sim -(2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 (y_1 + y_2)^{-1} \delta_c(1 - y_1 - \gamma y_2)$$
$$\times \int_0^1 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \delta\left(1 - \sum_{j=1}^{4} \alpha_j\right) (\alpha_3\alpha_4)^{-1+\zeta y_1} (\alpha_1\alpha_2)^{-1+\zeta y_2}, \tag{3.7}$$

where $c_1 > 0$ and $c_2 > 0$. The c_1 and c_2 in (3.7) are, of course, not the same as the c_1 and c_2 of (3.3) and (3.4). The evaluation of the α integrations is completely straightforward; the leading contributions for small ζ come from the four regions:

(i) α_2, α_3, and α_4 are small, i.e. $\alpha_1 \sim 1$;
(ii) α_1, α_3, and α_4 are small, i.e. $\alpha_2 \sim 1$;
(iii) α_1, α_2, and α_4 are small, i.e. $\alpha_3 \sim 1$; and
(iv) α_1, α_2, and α_3 are small, i.e. $\alpha_4 \sim 1$. (3.8)

By symmetry, the contributions from the first two regions are identical, and so are those from the last two regions. Adding together these four contributions, the result is

$$\tilde{I}_2(\zeta) \sim -(2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 \delta_c(1 - y_1 - \gamma y_2) 2\zeta^{-3} y_1^{-2} y_2^{-2}$$
$$= -4\gamma\zeta^{-3}. \tag{3.9}$$

This means that

$$I_2(s, -s^\gamma) \sim -2\gamma s^{-1-\gamma}(\ln s)^2 \tag{3.10}$$

for large s, and hence

$$I_2(s, t) \sim -2s^{-1}|t|^{-1}\ln s \ln|t| \tag{3.11}$$

for large, physical values of s and t. This is the desired answer.

4. Three-rung ladder diagram

The same procedure is now to be applied to the three-rung, i.e. sixth-order, ladder diagram of fig. 2, where the s channel is from left to right, while the t channel is from bottom to top. The scattering amplitude is proportional to [11]

$$I_3(s, t) = 2! \int_0^1 \left(\prod_{j=1}^{7} \mathrm{d}\alpha_j \right) \delta\left(1 - \sum_{j=1}^{7} \alpha_j\right) \Lambda (D_s s + D_t t + D_m + i\varepsilon)^{-3}, \tag{4.1}$$

where

$$\Lambda = (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) - \alpha_6^2, \tag{4.2}$$

$$D_s = \alpha_5 \alpha_6 \alpha_7, \tag{4.3}$$

$$\begin{aligned} D_t = {} & \alpha_1\alpha_3(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \\ & + \alpha_6(\alpha_1\alpha_4 + \alpha_2\alpha_3), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} D_m = {} & -\Lambda + \alpha_5(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_7(\alpha_2 + \alpha_4) \\ & \times (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_5\alpha_6(\alpha_2 + \alpha_4) + \alpha_6\alpha_7(\alpha_1 + \alpha_3). \end{aligned} \tag{4.5}$$

Since this $I_3(s, t)$ is expected to be very roughly of the order of $(st^2)^{-1}$, define the Mellin transform by

$$\tilde{I}_3(\zeta) = \int_0^\infty \mathrm{d}s\, s^{2\gamma-\zeta} I_3(s, -s^\gamma). \tag{4.6}$$

By (2.3), this is given alternatively by

$$\begin{aligned} \tilde{I}_3(\zeta) = {} & (2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}z_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}z_2\, \Gamma(z_1)\Gamma(z_2)\Gamma(3 - z_1 - z_2) \\ & \times \delta_c(1 + 2\gamma - \zeta - z_1 - \gamma z_2) \int_0^1 \left(\prod_{j=1}^{7} \mathrm{d}\alpha_j \right) \delta\left(1 - \sum_{j=1}^{7} \alpha_j\right) \\ & \times (\alpha_5\alpha_6\alpha_7)^{-z_1}(-D_t + i\varepsilon)^{-z_2}(D_m + i\varepsilon)^{-3+z_1+z_2}\Lambda. \end{aligned} \tag{4.7}$$

For this three-rung diagram, the change of variable (3.5) takes the form

$$z_1 = 1 - \zeta y_1,$$

$$z_2 = 2 - \zeta y_2. \tag{4.8}$$

The choice of the additive constants 1 and 2 in eqs. (4.8) is obvious from an inspection of the δ_c function in (4.7). More generally, it follows from a study of the way in which the integrand diverges as one or more of the α_j becomes small.

The leading term for small positive ζ is then given by

$$\tilde{I}_3(\zeta) \sim (2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 (y_1 + y_2)^{-1}$$

$$\times \delta_c(1 - y_1 - \gamma y_2) \int_0^1 \left(\prod_{j=1}^{7} \mathrm{d}\alpha_j \right) \delta\left(1 - \sum_{j=1}^{7} \alpha_j\right)$$

$$\times (\alpha_5\alpha_6\alpha_7)^{-1+\zeta y_1} D_t^{-2+\zeta y_2} (-D_m)^{-\zeta(y_1+y_2)} \Lambda. \tag{4.9}$$

The important task is to ascertain the regions in the six-dimensional α-space which give the leading contribution to $\tilde{I}_3(\zeta)$ for small ζ. From the box diagram, it is seen from (3.8) that there are four such regions, namely, the four corners of the tetrahedron. We therefore expect the major contributions to (4.9) to come from the seven corners, i.e. the seven regions $\alpha_1 \sim 1$, $\alpha_2 \sim 1$, $\alpha_3 \sim 1$, $\alpha_4 \sim 1$, $\alpha_5 \sim 1$, $\alpha_6 \sim 1$, and $\alpha_7 \sim 1$. Again by symmetry, the contributions from the first four regions are identical, and so are those from the fifth and seventh regions.

In the integrand of (4.9), the presence of the factor $D_s^{-1+\zeta y_1} = (\alpha_5\alpha_6\alpha_7)^{-1+\zeta y_1}$ means that a factor of ζ^{-1} shows up when integrated over the region $\alpha_5 \sim 0$, or $\alpha_6 \sim 0$, or $\alpha_7 \sim 0$. On the other hand, the D_t of (4.4) is more complicated: $D_t = 0$ when $\alpha_1 = \alpha_2 = 0$, or $\alpha_3 = \alpha_4 = 0$. In other words, in order to produce a factor of ζ^{-1} from $D_t^{-2+\zeta y_2}$, two α's must be small. Since, at most, only six of the seven α's can be small, the maximum power of ζ^{-1} is thus ζ^{-5}. Conversely, in order to get ζ^{-5} for small ζ, the region of contribution must be one where six of the α's are small, i.e. one of the corners listed above.

It follows from (4.5) that

$$D_m \sim -\Lambda, \tag{4.10}$$

valid in each of the seven corners. Indeed, this is a general feature of all diagrams, not limited to ladder diagrams. Let $\tilde{I}_{31}(\zeta)$, $\tilde{I}_{32}(\zeta)$, and $\tilde{I}_{33}(\zeta)$ be the contributions from the corners $\alpha_4 \sim 1$, $\alpha_6 \sim 1$, and $\alpha_7 \sim 1$, respectively. Then

$$\tilde{I}_3(\zeta) \sim 4\tilde{I}_{31}(\zeta) + \tilde{I}_{32}(\zeta) + 2\tilde{I}_{33}(\zeta), \tag{4.11}$$

where

$$\tilde{I}_{31}(\zeta) = (2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 (y_1 + y_2)^{-1} \delta_c(1 - y_1 - \gamma y_2)$$

$$\times \int_0^1 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_5 \mathrm{d}\alpha_6 \mathrm{d}\alpha_7 (\alpha_5\alpha_6\alpha_7)^{-1+\zeta y_1} [\alpha_1(\alpha_3 + \alpha_6)$$

$$+ \alpha_2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)]^{-2+\zeta y_2} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{1-\zeta(y_1+y_2)}, \quad (4.12)$$

$$\tilde{I}_{32}(\zeta) = (2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 (y_1 + y_2)^{-1} \delta_c(1 - y_1 - \gamma y_2)$$

$$\times \int_0^1 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \mathrm{d}\alpha_5 \mathrm{d}\alpha_7 (\alpha_5\alpha_7)^{-1+\zeta y_1} [(\alpha_1 + \alpha_2)$$

$$\times (\alpha_3 + \alpha_4)]^{-2+\zeta y_2} (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7)^{1-\zeta(y_1+y_2)}, \quad (4.13)$$

and

$$\tilde{I}_{33}(\zeta) = (2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 (y_1 + y_2)^{-1} \delta_c(1 - y_1 - \gamma y_2)$$

$$\times \int_0^1 \left(\prod_{j=1}^{6} \mathrm{d}\alpha_j \right) (\alpha_5\alpha_6)^{-1+\zeta y_1} [\alpha_1\alpha_3 + \alpha_2\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)$$

$$+ \alpha_6(\alpha_1\alpha_4 + \alpha_2\alpha_3)]^{-2+\zeta y_2} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{1-\zeta(y_1+y_2)}. \quad (4.14)$$

In these three integrals, the upper limit of integration for the α variables has been set arbitrarily to be 1. So far as the leading terms for small ζ are concerned, it makes no difference what this upper limit is, so long as it is positive and independent of ζ. For example, it may be $\frac{1}{6}$, in which case there is no overlap in the various regions.

It may be instructive to contrast (4.12) with (4.14). The main difference is the following: when $\alpha_4 \sim 1$, D_t simplifies to a polynomial which is homogeneous in the remaining α's; on the other hand, when $\alpha_7 \sim 1$, there is very little simplification in D_t and the resulting expression contains both quadratic and cubic terms in α's. In either case, the neglected terms are each much smaller than at least one of the terms retained. For example, in the former case, $\alpha_6\alpha_2\alpha_3 \ll \alpha_2\alpha_3$. In the latter case of $\alpha_7 \sim 1$, although α_j, $j = 1, \dots, 6$, are all small, there is no reason why $\alpha_2\alpha_4\alpha_6$, for example, should be smaller than $\alpha_1\alpha_3$. It is for this reason that the cubic terms have to be kept in (4.14), and the presence of these terms leads to essential complications.

These three integrals on the right-hand sides of (4.12)–(4.14) are evaluated approximately for small ζ in appendices A, B, and C, respectively. Because of the cubic terms, the evaluation of $\tilde{I}_{33}(\zeta)$ is by far the most complicated one. The results are

$$\tilde{I}_{31}(\zeta) \sim 2\zeta^{-5}\gamma^2(6-8\gamma+3\gamma^2), \tag{4.15}$$

$$\tilde{I}_{32}(\zeta) = \mathrm{O}(\zeta^{-3}), \tag{4.16}$$

and

$$\tilde{I}_{33}(\zeta) \sim 8\zeta^{-5}\gamma^3(2-\gamma). \tag{4.17}$$

Substitution into (4.11) gives

$$\tilde{I}_{3}(\zeta) \sim 8\zeta^{-5}\gamma^2(6-4\gamma+\gamma^2) \tag{4.18}$$

for small ζ. This means that

$$I_3(s,-s^\gamma) \sim \tfrac{1}{3}s^{-1-2\gamma}\gamma^2(6-4\gamma+\gamma^2)(\ln s)^4 \tag{4.19}$$

for large s, and hence

$$I_3(s,t) \sim \tfrac{1}{3}s^{-1}t^{-2}(\ln|t|)^2\left[6(\ln s)^2 - 4(\ln s)(\ln|t|) + (\ln|t|)^2\right], \tag{4.20}$$

for large, physical values of s and t. This is the desired answer.

Note that, in evaluating the y integration, $\gamma < 1$ has been used. Thus, (4.20) does not hold in the case $-t \gg s \gg 1$.

5. Discussion

For the purpose of comparison, it is useful to write down the known asymptotic expansions of $I_2(s,t)$ and $I_3(s,t)$ for large s and fixed $t \leqslant 0$ [5]:

$$I_2(s,t) \sim -s^{-1}(\ln s)\int_0^1 \mathrm{d}\alpha\left[\alpha(1-\alpha)|t|+1\right]^{-1}, \tag{5.1}$$

$$I_3(s,t) \sim \tfrac{1}{2}s^{-1}(\ln s)^2\left\{\int_0^1 \mathrm{d}\alpha\left[\alpha(1-\alpha)|t|+1\right]^{-1}\right\}^2. \tag{5.2}$$

If we take these results for fixed t, and evaluate them asymptotically for large $|t|$, the results are, from the right-hand sides of (5.1) and (5.2),

$$I_2(s,t) \sim -2s^{-1}|t|^{-1}\ln s\,\ln|t|, \tag{5.3}$$

$$I_3(s,t) \sim 2s^{-1}|t|^{-2}(\ln s)^2(\ln|t|)^2. \tag{5.4}$$

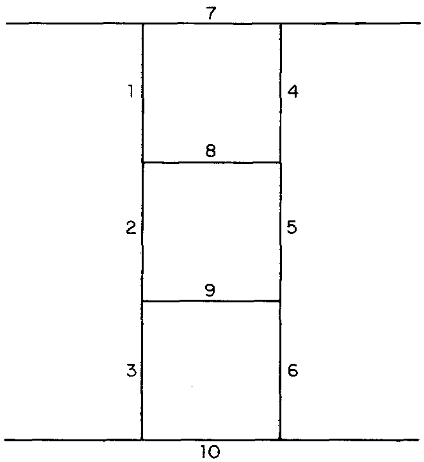

Fig. 5. The eighth-order ladder diagram in φ^3 theory.

The result (5.3) agrees with (3.11), but (5.4) fails to give all the terms in (4.20). In other words, (5.4) holds under the condition

$$\ln s \gg \ln|t| \gg 1\,, \tag{5.5}$$

which is a much stronger condition than (1.1). This is the explicit justification for the discussion following (1.1).

A simplifying feature of the three-rung ladder diagram is that the corner $\alpha_6 \sim 1$ does not contribute, as shown explicitly by (4.16). This feature is quite general; for example, for the four-rung ladder diagram of fig. 5, neither corner $\alpha_8 \sim 1$ nor $\alpha_9 \sim 1$ contributes to the leading term, as shown in paper II. Indeed, it will be seen there that this diagram of fig. 5 is much richer in structure.

It is too early to tell how the present program of studying the asymptotic region (1.1) will develop. It is clear from sect. 4, together with the appendices, that technically this is much more complicated than the previous case of large s with fixed values of t. If the present development does follow that of the previous, simpler case, then it is necessary to attack the following types of Feynman diagrams:

(A) higher-order diagrams in φ^3 theory;
(B) diagrams in abelian gauge theory; and
(C) diagrams in non-abelian gauge theory, i.e. Yang-Mills theory.

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Appendix A

EVALUATION OF $\tilde{I}_{31}(\zeta)$ FOR SMALL ζ

In this appendix, the integral $\tilde{I}_{31}(\zeta)$, as defined by (4.12), is evaluated for small ζ. In this case, the α_2 and α_7 integrations can be carried out trivially, leading to

$$\tilde{I}_{31}(\zeta) \sim (2\pi i)^{-1}\zeta^{-1}\int_{c_1-i\infty}^{c_1+i\infty} \mathrm{d}y_1 \int_{c_2-i\infty}^{c_2+i\infty} \mathrm{d}y_2 [y_1(y_1+y_2)]^{-1}\delta_c(1-y_1-\gamma y_2)$$
$$\times \int_0^1 \mathrm{d}\alpha_1 \mathrm{d}\alpha_3 \mathrm{d}\alpha_5 \mathrm{d}\alpha_6 (\alpha_5\alpha_6)^{-1+\zeta y_1}[\alpha_1(\alpha_3+\alpha_6)]^{-1+\zeta y_2}$$
$$\times (\alpha_1+\alpha_3+\alpha_5+\alpha_6)^{-\zeta(y_1+y_2)}. \tag{A.1}$$

Since the integrand is homogeneous in the four α's, the standard scaling procedure,

$$\alpha_i = \rho\alpha_i',$$

$$\int_0^1 \prod_{j=1}^N \mathrm{d}\alpha_j f(\alpha_i) = \int_0^1 \mathrm{d}\rho\,\rho^{N-1}\int_0^1 \prod_{j=1}^N \mathrm{d}\alpha_j' \delta\left(1-\sum_{k=1}^N \alpha_k'\right) f(\rho\alpha_i'), \tag{A.2}$$

gives a factor of $(y_1+y_2)^{-1}\zeta^{-1}$ together with a delta function $\delta(1-\alpha_1-\alpha_3-\alpha_5-\alpha_6)$. Similar to the development of sect. 3, the leading contributions come from the four corners of the tetrahedron, the results being

(i) $\alpha_1 \sim 1$: $\zeta^{-3}y_1^{-2}(y_1+y_2)^{-1}$;

(ii) $\alpha_3 \sim 1$: $\zeta^{-3}y_1^{-2}y_2^{-1}$;

(iii) $\alpha_5 \sim 1$: $\zeta^{-3}y_1^{-1}y_2^{-1}(y_1+y_2)^{-1}$;

(iv) $\alpha_6 \sim 1$: 0.

Therefore,

$$\tilde{I}_{31}(\zeta) \sim (\pi i)^{-1}\zeta^{-5}\int_{c_1-i\infty}^{c_1+i\infty} \mathrm{d}y_1 \int_{c_2-i\infty}^{c_2+i\infty} \mathrm{d}y_2\, y_1^{-3}y_2^{-1}(y_1+y_2)^{-2}\delta_c(1-y_1-\gamma y_2)$$
$$= (\pi i)^{-1}\zeta^{-5}\int_{c_2-i\infty}^{c_2+i\infty} \mathrm{d}y_2\, y_2^{-1}(1-\gamma y_2)^{-3}[1+(1-\gamma)y_2]^{-2}. \tag{A.3}$$

Since $\gamma^{-1} > c_2 > 0$, the poles of the integrand are at

$$y_2 = 0, \qquad \gamma^{-1}, \qquad -(1-\gamma)^{-1}, \tag{A.4}$$

where the first and third poles are to the left of the contour, while the second one is on the right. This is because $0 < \gamma < 1$. Closing on the pole γ^{-1}, for example, gives (4.15). This condition on γ is also needed to get (4.17) from (C.6).

Appendix B

EVALUATION OF $\tilde{I}_{32}(\zeta)$ FOR SMALL ζ

In this appendix, it is shown that $\tilde{I}_{32}(\zeta)$, as defined by (4.13), is small. The reason is that the integrand is homogeneous of order $-5 + \zeta(y_1 + y_2)$ in the α's. This is to be contrasted with that of $\tilde{I}_{31}(\zeta)$ of (4.12), where the order is $-6 + \zeta(2y_1 + y_2)$.

To see more explicitly the consequences of this important difference, carry out the scaling (cf. eq. (A.2))

$$\rho = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7, \tag{B.1}$$

$$\alpha_j = \rho\alpha_j', \tag{B.2}$$

for $j = 1, 2, 3, 4, 5$ and 7. Then it follows from (4.13) that

$$\begin{aligned}\tilde{I}_{32}(\zeta) \sim (2\pi i)^{-1}\int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 (y_1 + y_2)^{-1}\delta_c(1 - y_1 - \gamma y_2)\\ \times \int_0^1 \mathrm{d}\rho\, \rho^{\zeta(y_1 + y_2)} \int_0^1 \mathrm{d}\alpha_1' \mathrm{d}\alpha_2' \mathrm{d}\alpha_3' \mathrm{d}\alpha_4' \mathrm{d}\alpha_5' \mathrm{d}\alpha_7'\, \delta(1 - \alpha_1' - \alpha_2' - \alpha_3'\\ - \alpha_4' - \alpha_5' - \alpha_7')(\alpha_5'\alpha_7')^{-1+\zeta y_1}[(\alpha_1' + \alpha_2')(\alpha_3' + \alpha_4')]^{-2+\zeta y_2}.\end{aligned} \tag{B.3}$$

Therefore, the ρ integration fails to give a factor of ζ^{-1}, and (4.16) follows from (B.3).

Appendix C

EVALUATION OF $\tilde{I}_{33}(\zeta)$ FOR SMALL ζ

In this appendix, the integral $\tilde{I}_{33}(\zeta)$, as defined by (4.14), is evaluated asymptotically for small ζ. This task is much more difficult than that of appendix A. The techniques developed for this purpose will also be useful in the study of higher-order ladder diagrams, including the eighth-order one of paper II. The complication is due to the fact that the integrand is not homogeneous in the six α's, as already emphasized in sect. 4.

The first step is to integrate over α_2 to get

$$\tilde{I}_{33}(\zeta) \sim (2\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} \mathrm{d}y_1 \int_{c_2 - i\infty}^{c_2 + i\infty} \mathrm{d}y_2 (y_1 + y_2)^{-1} \delta_c(1 - y_1 - \gamma y_2)$$
$$\times \int_0^1 \mathrm{d}\alpha_1 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \mathrm{d}\alpha_5 \mathrm{d}\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{1-\zeta(y_1+y_2)}$$
$$\times [\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_3\alpha_6]^{-1} \{ [\alpha_1(\alpha_3 + \alpha_4\alpha_6)]^{-1+\zeta y_2}$$
$$- [\alpha_1\alpha_3 + \alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_3\alpha_6]^{-1+\zeta y_2} \} . \quad \text{(C.1)}$$

In the last factor of (C.1), a term $\alpha_6\alpha_1\alpha_4$ has been omitted because it is small compared with the second term $\alpha_4\alpha_1$. Due to the deletion of this term, the integrand of (C.1) is homogeneous in the α's except for the factor $\alpha_3 + \alpha_4\alpha_6$ in the next-to-last term. If the variable α_4 is shifted,

$$\alpha_4' = \alpha_4 + \frac{\alpha_3\alpha_6}{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6} , \quad \text{(C.2)}$$

where the shift is always small when the α's are all small, then the approximate formula (D.1) derived in appendix D can be applied. For the two terms of (C.1), the A's of (D.1) are, respectively,

$$A = \begin{cases} \dfrac{\alpha_6}{\alpha_3} \\ \dfrac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6}{\alpha_1\alpha_3} , \end{cases} \quad \text{(C.3)}$$

with the corresponding $A\delta$:

$$A\delta = \begin{cases} \dfrac{\alpha_6^2}{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6} \\ \dfrac{\alpha_6}{\alpha_1} . \end{cases} \quad \text{(C.4)}$$

When α_1, α_3, α_5, and α_6 are all small, the second A is large, and the first $A\delta$ is small. The other two values, α_6/α_3 and α_6/α_1, may be of any size.

The α_4 integration is carried out next by applying (D.1). The result is

$$\tilde{I}_{33}(\zeta) \sim (2\pi i)^{-1}\int_{c_1-i\infty}^{c_1+i\infty} \mathrm{d}y_1 \int_{c_2-i\infty}^{c_2+i\infty} \mathrm{d}y_2 (y_1+y_2)^{-1}\delta_{\mathrm{c}}(1-y_1-\gamma y_2)$$

$$\times \int_0^1 \mathrm{d}\alpha_1\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_5\,\mathrm{d}\alpha_6 (\alpha_5\alpha_6)^{-1+\zeta y_1}(\alpha_1\alpha_3)^{-1+\zeta y_2}(\alpha_1+\alpha_3+\alpha_5+\alpha_6)^{-\zeta(y_1+y_2)}$$

$$\times\left[\ln\frac{\alpha_1+\alpha_3+\alpha_5+\alpha_6}{\alpha_6(\alpha_3+\alpha_6)} - \alpha_1^{-\zeta y_2}(\alpha_1+\alpha_6)^{\zeta y_2}\ln\frac{\alpha_1+\alpha_6}{\alpha_6}\right]. \tag{C.5}$$

The α integrations can now be treated by the standard scaling procedure on the four α's. The calculation is still tedious but fairly straightforward. The result is

$$\tilde{I}_{33}(\zeta) \sim (2\pi i)^{-1}\int_{c_1-i\infty}^{c_1+i\infty} \mathrm{d}y_1 \int_{c_2-i\infty}^{c_2+i\infty} \mathrm{d}y_2 (y_1+y_2)^{-1}\delta_{\mathrm{c}}(1-y_1-\gamma y_2)$$

$$\times\left[4\zeta^{-5}y_1^{-2}y_2^{-2}(y_1+y_2)^{-1}\right], \tag{C.6}$$

from which (4.17) follows.

Appendix D

AN APPROXIMATE INTEGRAL

In this appendix, we obtain the following approximate formula

$$\int_\delta^1 \frac{\mathrm{d}x}{x}(1+Ax)^{-1+\zeta} \sim (1+A\delta)^\zeta \ln\frac{1+A\delta}{(1+A)\delta}, \tag{D.1}$$

valid for δ and ζ both positive and small, and A positive but arbitrary in magnitude. This (D.1) is needed to obtain (C.5). Actually, the range of validity of (D.1) is somewhat larger.

This (D.1) can be obtained in various ways. One derivation is as follows. Consider first the case where

$$\zeta \ln(1+A\delta) \ll 1. \tag{D.2}$$

In this case, $(1+Ax)^\zeta$ is close to 1 in the vicinity of the lower limit of integration. Therefore, it may be verified that

$$\text{l.h.s. of (D.1)} \sim \int_\delta^1 \frac{\mathrm{d}x}{x}(1+Ax)^{-1}$$

$$= \ln\frac{1+A\delta}{(1+A)\delta}. \tag{D.3}$$

Next consider the case where $\zeta \ln(1+A\delta)$ is not small. Since ζ is small, $A\delta$ must be large, and hence

$$\begin{aligned}\text{l.h.s. of (D.1)} &\sim \int_\delta^1 \frac{dx}{x}(Ax)^{-1+\zeta}\\ &= (1-\zeta)^{-1}\left[(A\delta)^{-1+\zeta} - A^{-1+\zeta}\right]\\ &\sim (A\delta)^{-1+\zeta}. \end{aligned}\tag{D.4}$$

Since (D.1) reduces correctly to (D.3) and (D.4) in their respective ranges of validity, and (D.3) and (D.4) cover all cases, we obtain the approximate formula (D.1).

In our application of this formula (D.1), the upper limit of integration will not be 1, but rather some unspecified value $\gamma = O(1)$. By a simple rescaling, it follows from (D.1) that

$$\int_\delta^\gamma \frac{dx}{x}(1+Ax)^{-1+\zeta} \sim (1+A\delta)^\zeta \ln\frac{1+A\delta}{(1+\gamma A)\delta}, \tag{D.5}$$

with $\gamma = O(1)$. The dependence on the upper limit of integration is only through the logarithm. Therefore, up to terms of O(1), which we are not interested in, the integral (D.5) is independent of the upper limit of integration, $\gamma = O(1)$.

References

[1] H. Cheng and T.T. Wu, Phys. Rev. Lett. 24 (1970) 1456
[2] U. Amaldi et al., Phys. Lett. 44B (1973) 112;
S.R. Amendolia et al., Phys. Lett. 44B (1973) 119;
H. Cheng, J.K. Walker, and T.T. Wu, Phys. Lett. 44B (1973) 97
[3] UA4 Collaboration, R. Battiston et al., Phys. Lett. 115B (1982) 333; 117B (1982) 126; 127B (1983) 472;
UA1 Collaboration, G. Arnison et al., Phys. Lett. 121B (1983) 77;
C. Bourrely, J. Soffer, and T.T. Wu, Phys. Lett. 121B (1983) 284; Phys. Rev. Lett. 54 (1985) 757;
G. Matthiae et al., *in* Proc. Int. Europhysics Conf. on High Energy Physics, Brighton, England (1983) ed. J. Guy and C. Costain (Rutherford Appleton Lab., 1983);
UA4 Collaboration, M. Bozzo et al., Phys. Lett. 147B (1984) 385, 392
[4] J.L. Cardy, Nucl. Phys. B17 (1970) 493; B33 (1971) 139;
H.M. Fried and T.K. Gaisser, Phys. Rev. 179 (1969) 1491;
H.M. Fried, B. Kirby, and T.K. Gaisser, Phys. Rev. D8 (1973) 3210;
I.G. Halliday, J. Huskins, and C.T. Sachrajda, Nucl. Phys. B83 (1974) 189;
J.M. Cornwall and G. Tiktopoulos, Phys. Rev. Lett. 35 (1975) 338; Phys. Rev. D13 (1976) 3370;
E.C. Poggio and H.R. Quinn, Phys. Rev. D12 (1975) 3279
[5] D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cim. 26 (1962) 896;
B.W. Lee and R.F. Sawyer, Phys. Rev. 127 (1962) 2266;
M. Gell-Mann and M.L. Goldberger, Phys. Rev. Lett. 9 (1962) 275;
P.G. Federbush and M.T. Grisaru, Ann. Phys. 22 (1963) 263, 299;

J.C. Polkinghorne, J. Math. Phys. 4 (1963) 503; 5 (1964) 431, 1491;
R.J. Eden, P.V. Landshoff, D.I. Olive, and J.C. Polkinghorne, The analytic S-matrix (Cambridge University Press, 1966), ch. 3
[6] H. Cheng and T.T. Wu, Phys. Rev. D1 (1970) 2775
[7] M. Jacob, ed., Dual theory, Physics Report reprint vol. I (North-Holland, 1974)
[8] G. Veneziano, Nuovo Cim. 57A (1968) 190;
L. Caneschi, A. Schwimmer, and G. Veneziano, Phys. Lett. 30B (1969) 351;
Y. Nambu, Proc. Int. Conf. on Symmetries and quark models, Wayne State Univ. (Gordon and Breach, 1969), p. 269;
M.A. Virasoro, Phys. Rev. D1 (1970) 2933;
H. Nielsen, Proc. 15th Int. Conf. on High Energy Physics (Kiev, 1970);
L. Susskind, Nuovo Cim. 69 (1970) 457;
P. Ramond, Phys. Rev. D3 (1971) 2415;
A. Neveu and J.H. Schwarz, Nucl. Phys. B31 (1971) 86; Phys. Rev. D4 (1971) 1109;
P. Goddard, J. Goldstone, C. Rebbi, and C.B. Thorn, Nucl. Phys. B56 (1973) 109
[9] M.B. Green and J.H. Schwarz, Nucl. Phys. B181 (1981) 502; B198 (1982) 252, 441; B218 (1983) 43; Phys. Lett. 109B (1982) 444; 149B (1984) 117; 151B (1985) 21;
M.B. Green, J.H. Schwarz, and L. Brink, Nucl. Phys. B198 (1982) 474; B219 (1983) 437;
L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269;
E. Witten, Phys. Lett. 149B (1984) 351; 153B (1985) 243;
D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253;
P. Candelas, G.T. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. B258 (1985) 46
[10] J.D. Bjorken and T.T. Wu, Phys. Rev. 130 (1963) 2566
[11] K. Symanzik, Prog. Theor. Phys. 20 (1958) 690