TIME EVOLUTION OF THE COSMOLOGICAL "CONSTANT"

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We discuss cosmologies where the cosmological constant λ depends on time. The requirements of realistic cosmology impose restrictions on the functional dependence of $\dot{\lambda}$ on the Hubble parameter H. We show that for a wide class of functions with $\dot{\lambda}$ of the order H^3 the system of field equations leads to a stable fix-point behaviour with λ naturally very small today. The age of the universe, critical matter density and deceleration parameter may be modified.

Today's value of the cosmological constant is at most of the order

$$|\lambda| \lesssim (10^{-2}\ \mathrm{eV})^4. \tag{1}$$

This is extremely small compared to other mass scales of the standard model like the Fermi scale φ_L or Λ_{QCD}. The tiny ratio $|\lambda|^{1/4}/\varphi_L \lesssim 10^{-13}$ is difficult to understand since phase transitions in the early universe (weak symmetry breaking, chiral symmetry breaking and confinement of QCD) presumably could induce a change in the cosmological constant of the order of the fourth power of the relevant mass scale (φ_L or Λ_{QCD}). For the cosmology after these phase transitions, there exist in principle two possibilities: either λ was tiny [obeying (1)] immediately after the phase transition or it must have evolved with time in order to reach its very small value today. In this letter we explore the second alternative, i.e., cosmologies with variable cosmological constant $\mathrm{d}\lambda/\mathrm{d}t \neq 0$. We concentrate on the kinematics of the problem: given some mechanism for a time variation of λ, what will be the consequence for the evolution of the universe?

Suppose that the effective action (which includes quantum fluctuations) for the metric and matter fields would be known. General coordinate invariance dictates the gravitational field equations to have the form

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2}$$

The covariantly conserved energy–momentum tensor has a variety of contributions: the effects of coherent ("background") matter fields (including their potentials and kinematic terms) are obtained by a variation of the matter part of the effective action with respect to $g^{\mu\nu}$. In the same way one finds the contributions from higher-derivative invariants in the effective action like $R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$ or, more general, the complicated (possibly non-local) terms induced by quantum effects (for example the Casimir energy for the zero point fluctuations of a matter field in a curved background). In addition there is the contribution of classical particles and fields (they are also solutions of the field equations) moving incoherently in a volume with characteristic length much larger than their wavelength. Here statistical methods can be applied to obtain energy density and pressure for a particle plasma or for dust.

We consider a homogeneous and isotropic universe with a Robertson–Walker scale factor $a(t)$. The symmetries require the energy–momentum tensor to be of the form

$$T_{00} = (\lambda + \rho)g_{00}, \quad T_{ij} = (\lambda - p)g_{ij},$$
$$T_{0i} = 0. \tag{3}$$

Covariant conservation of the energy–momentum tensor reads (with dots denoting time derivatives and $H = \dot{a}/a$)

$$\dot{\lambda} + \dot{\rho} + 3H(\rho + p) = 0. \tag{4}$$

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Obviously, the energy density $\rho(t)$, pressure $p(t)$ and cosmological "constant" $\lambda(t)$ constitute more free functions than needed for the most general description of $T_{\mu\nu}$ in a homogeneous and isotropic universe. We keep this additional freedom in order to allow for easy comparison with standard cosmology in the matter-dominated ($p=0$) or radiation-dominated ($p=1/3$) universe. Specifying a relation between p and ρ defines λ uniquely. Motivated by the success of inflation for the understanding of some of the puzzles of very early cosmology we restrict our discussion to Robertson–Walker metrics with $k=0$. (This means that a^{-2} can be neglected compared to H^2. Generalization to the $k\neq 0$ case would be straightforward.) The time component of the gravitational field equations relates the Hubble parameter to ρ and λ:

$$H^2 = \tfrac{8}{3}\pi G(\rho+\lambda) \equiv \tilde{\rho}+\tilde{\lambda}. \tag{5}$$

To specify the system completely, we need an equation of state. In addition to the relation between p and ρ defining λ, we assume that the dynamics of matter fields and gravitation gives a time evolution of λ

$$\dot{\tilde{\lambda}} = F\{H, \tilde{\lambda}, \tilde{\rho}\} \tag{6}$$

with F a functional of the functions $H(t)$, $\tilde{\lambda}(t)$ and $\tilde{\rho}(t)$ describing the system. [If there are other degrees of freedom relevant for cosmology, we assume that we can solve their field equations in dependence of H, $\tilde{\lambda}$ and $\tilde{\rho}$ and insert these solutions to obtain (6).] Eq. (4) can be rewritten in the form

$$\dot{\tilde{\lambda}} + \dot{\tilde{\rho}} + nH\tilde{\rho} = 0 \tag{7}$$

with $n=3$ and 4 for the matter- and radiation-dominated period, respectively. Our aim is to find cosmological solutions for the system of eqs. (5), (6) and (7) for given functionals F.

As a first simple exercise consider

$$F = -aH^N\tilde{\lambda}, \quad a>0. \tag{8}$$

As long as $\tilde{\lambda}$ is small compared to $\tilde{\rho}$ one has

$$(\mathrm{d}/\mathrm{d}t)(\tilde{\lambda}/\tilde{\rho}) = -(aH^{N-1}-n)H\tilde{\lambda}/\tilde{\rho}. \tag{9}$$

Consider the case $N=1$, $a>n$: if we start at some time after the last phase transition with $|\tilde{\lambda}|$ smaller than $\tilde{\rho}$ so that (9) is valid, the ratio $\tilde{\lambda}/\tilde{\rho}$ subsequently decreases continuously with the evolution of the universe. We can therefore neglect the effects of $\tilde{\lambda}$ to a good approximation and obtain an approximate Friedmann universe. (We may call this scenario a ρ-dominated universe.) Even if the cosmological "constant" was of the order of the characteristic scale immediately after the phase transition, it would be tiny today and fulfil (1). This is the type of solution of the cosmological constant problem we are looking for.

For a more general discussion we observe that $\tilde{\lambda}$ and $\tilde{\rho}$ can be expressed in terms of H and $\dot{H}$:

$$\tilde{\rho} = -(2/n)\dot{H}, \quad \tilde{\lambda} = H^2 + (2/n)\dot{H}. \tag{10}$$

Using these relations, we can treat F as a functional of H alone. Furthermore, for realistic cosmologies (after inflation) $\dot{H}$ is of the order H^2 and H is very small in units of the Planck mass. Let us assume that one can expand F in powers of H and its derivatives. We count every time derivative as a factor H:

$$F(H) = a_1H + a_2^1H^2 + a_2^2\dot{H} + a_3^1H^3 + a_3^2H\dot{H} + a_3^3\ddot{H} + \mathrm{O}(H^4), \tag{11}$$

and keep only the lowest non-vanishing terms (the non-zero a_i^j with the lowest i)[‡1]. If the dynamics leading to the time evolution of $\tilde{\lambda}$ respects time reversal symmetry all a_i^j with even i must vanish. The case $a_1\neq 0$ is highly unlikely and would lead to an unacceptable logarithmic dependence of λ on a. We take $a_1=0$. What form for $F\{H\}$ is required for realistic cosmologies?

A useful quantity for a discrimination between different cosmologies is

$$\alpha(t) \equiv (1/H)(\mathrm{d}/\mathrm{d}t)\ln(\tilde{\lambda}/\tilde{\rho}) = n + (F\{H\}/H)[1/\tilde{\rho} + 1/\tilde{\lambda}]. \tag{12}$$

[‡1] We have of course no guaranty that a polynomial expansion of F is always possible. One could imagine that F depends on $|H|$ or that the coefficients a_i^j are functions of "dimensionless" ratios like $\dot{H}/H^2$. We will not discuss these possibilities in this letter.

If α goes asymptotically to a constant, one has

$$\lim_{t\to\infty} \alpha(t) = \alpha_\infty, \quad \lim_{t\to\infty} \tilde{\lambda}(t)/\tilde{\rho}(t) \sim a(t)^{\alpha_\infty}. \tag{13}$$

For $\alpha_\infty > 0$ the ratio $\tilde{\lambda}/\tilde{\rho}$ diverges, leading to unrealistic λ-dominated cosmologies [‡2]. (This also holds for $\alpha(t) \to +\infty$.) for $\alpha_\infty = 0$ one finds asymptotically a constant ratio between λ and ρ. Finally, for $\alpha_\infty < 0$ [or $\alpha(t) \to -\infty$] the universe becomes ρ-dominated with λ/ρ going to zero.

Let us first study the conditions for a ρ-dominated universe with $\lim_{t\to\infty}\alpha(t) < 0$. In this case we can neglect the first term in the square bracket in (12) and solve (6) or (11) in the ρ-dominated background with $H = (2/n)t^{-1}$. Suppose that the leading contributions to $\tilde{\lambda}$ are of the order H^m. Unless these contributions cancel to leading order one has

$$F = -dt^{-m}, \quad d \neq 0,$$
$$\lambda = \lambda_\infty + [d/(m-1)]\, t^{-m+1}. \tag{14}$$

The appearance of an additive free integration constant λ_∞ shows that fine tuning would be needed to obtain $\lambda(t) \to 0$. Therefore the ρ-dominated universe requires for consistency that $\dot{\lambda}$ vanishes as λ goes to zero and asymptotically $\dot{\lambda} = -aH^{m-2}\lambda$ $(d=0)$. This fix-point behaviour of F was discussed before (8) and we find that a ρ-dominated universe is impossible for $m > 3$, realized for $1 < m < 3$ only if $F \sim \lambda$ and possibly realized for the interesting limiting case $m = 3$.

For the second case of interest, $\alpha_\infty = 0$, one needs asymptotically

$$\lim_{t\to\infty} \tilde{\lambda}/\tilde{\rho} = c_\infty, \quad \lim_{t\to\infty} F = [2c_\infty/(c_\infty + 1)]\, H\dot{H}. \tag{15}$$

This is again impossible for $m > 3$ and requires $m = 3$ at least asymptotically. We conclude that no realistic cosmology is possible without a special choice of initial conditions if $|\tilde{\lambda}|$ is smaller than of the order H^3. For $m > 3$ the change in λ is simply too slow – the universe is driven to an unrealistic λ-dominated cosmology with ρ decreasing faster than λ.

In the remainder of this letter we concentrate on the interesting case $F \sim H^3$ [one or several a_3^j in eq. (11) are different from zero]. The three field equations (5), (6), (7) can then be combined into the second-order differential equation

$$\ddot{H} + AH\dot{H} + BH^3 = 0, \tag{16}$$

with

$$A = \frac{n - \frac{1}{2}na_3^2}{1 - \frac{1}{2}na_3^3}, \quad B = -\tfrac{1}{2}n\frac{a_3^1}{1 - \frac{1}{2}na_3^3}. \tag{17}$$

The initial values $\tilde{\lambda}(t_0)$ and $\tilde{\rho}(t_0)$ appear here as $H_0 \equiv H(t_0)$ and $\dot{H}_0 \equiv \dot{H}(t_0)$. Eq. (16) is the master equation for cosmologies with a time evolution $\tilde{\lambda} \sim H^3$. For $B = 0$ the solutions are known explicitly. They depend critically on the ratio between $\dot{H}_0$ and H_0^2:

$$H(t) = b\frac{H_0 + b + (H_0 - b)\exp[-Ab(t-t_0)]}{H_0 + b - (H_0 - b)\exp[-Ab(t-t_0)]},$$
$$\text{for } \dot{H}_0 > -\tfrac{1}{2}AH_0^2, \quad b = [H_0^2 + (2/A)\dot{H}_0]^{1/2}, \tag{18}$$

$$H(t) = [H_0^{-1} + \tfrac{1}{2}A(t-t_0)]^{-1},$$
$$\text{for } \dot{H}_0 = -\tfrac{1}{2}AH_0^2. \tag{19}$$

$$H(t) = b\ \mathrm{tg}[\mathrm{arctg}(H_0/b) - \tfrac{1}{2}Ab(t-t_0)],$$
$$\text{for } \dot{H}_0 < -\tfrac{1}{2}AH_0^2, \quad b = [-H_0^2 - (2/A)\dot{H}_0]^{1/2}. \tag{20}$$

As a special case we recognize standard cosmology [1] [‡3] with $\tilde{\lambda}$ constant ($F \equiv 0$, $A = n$). The cosmological constant appears here as an initial value for (16) and the solutions (18), (19) and (20) correspond to positive, zero and negative values of $\tilde{\lambda}$ with $b = |\tilde{\lambda}|^{1/2}$.

For cosmologies with $B \neq 0$ it is instructive to introduce new variables:

$$h(t) = H(t)t, \quad \tau = \ln(t/t_0). \tag{21}$$

In these variables eq. (16) may be interpreted as

[‡2] We consider here cosmologies which are characterized today by their asymptotic behaviour. We do not discuss cosmologies which are similar to a Friedmann universe today but change their qualitative behaviour in the future.

[‡3] See ref. [2] for an extensive list of references.

the equation of motion for a particle in a potential with velocity-dependent damping (accelerating) forces (primes denote derivatives with respect to τ):

$$h'' + \partial V/\partial h = 3h' - Ahh',$$
$$V(h) = h^2 - \tfrac{1}{3}Ah^3 + \tfrac{1}{4}Bh^4. \tag{22}$$

Static solutions for h, i.e., solutions with $H(t) \sim t^{-1}$, exist for the extreme of V. We do not expect of course that the initial conditions exactly correspond to one of these static solutions. We need a stability analysis for the extrema: will some nearby solution asymptotically approach the static solution or will the "particle" move away from the extremum? We have performed the stability analysis at the linearized level (for small deviations from the static solutions).

Stable solutions only exist for

$$0 < B \leqslant \tfrac{1}{8}A^2. \tag{23}$$

In this case there is a second minimum of V at $h = \eta$ (besides the trivial minimum at $h = 0$ which does not lead to stable solutions). We find the asymptotic solution [‡4]

$$H(t) = t^{-1}\left(\eta + c_1 t^{-1} + c_2 t^{-\delta}\right),$$
$$\eta = (A/2B)\left(1 + \sqrt{1 - 8B/A^2}\right),$$
$$\delta = A\eta - 4 > 0, \tag{24}$$

with integration constants c_i fixed by the initial values H_0 and $\dot{H}_0$. All other extrema of V are unstable. In particular, for $B \leqslant 0$ the potential has only maxima (besides the minimum at $h = 0$). For all maxima there exist ingoing and outgoing solutions. Unless the initial conditions are fine tuned so that the "particle" asymptotically stops at the maximum [this is the case for standard cosmology with vanishing $\tilde{\lambda}$, compare (19)], the "particle" will move away from the maximum for large τ. Such solutions do not correspond to realistic cosmologies – except the limiting cases where the "particle" stays very long near the maximum which again needs fine tuning of initial values.

‡4 For the limiting case $B = \frac{1}{8}A^2$ the minimum becomes a (stable) saddle point with a corresponding logarithmic modification of (24).

Let us concentrate on the state solutions for $0 < B \leqslant \frac{1}{8}A^2$. [This case includes our example (8) with $N = 1$, $A = n + a$, $B = \frac{1}{2}na$. Realistic cosmologies require $A > 0$ so that η is positive. Asymptotically the solution (25) is given independently of the exact initial conditions for $\tilde{\rho}(t_0)$ and $\tilde{\lambda}(t_0)$ by

$$H(t) = \eta t^{-1}, \quad a(t) = a_0 t^{\eta}. \tag{25}$$

The ratio between $\tilde{\lambda}$ and $\tilde{\rho}$ approaches a constant

$$\tilde{\lambda}/\tilde{\rho} = \tfrac{1}{2}n\eta - 1. \tag{26}$$

The most striking consequence of $\tilde{\lambda}$ playing a role in cosmology is the power of the time dependence of $a(t)$ which could be different from the standard behaviour $a \sim t^{2/n}$. This has several immediate consequences for the matter-dominated evolution of the late universe.

(i) The age of the universe is given by

$$\bar{t} = \eta \bar{H}^{-1}, \tag{27}$$

with $\bar{H}$ the Hubble constant observed today. For $\eta > 2/3$ the universe would be older than in the standard model. This could be a possible explanation of the discrepancy between $\bar{H}$ and the age of globular clusters [2].

(ii) The critical energy density corresponding to $k = 0$ is now given by

$$\tilde{\rho} = \bar{H}^2 - \tilde{\lambda} = (2/3\eta)\bar{H}^2. \tag{28}$$

For $\eta > 2/3$ the critical density is smaller than in standard cosmology.

(iii) The deceleration parameter $q = -(\dot{H}/H^2 + 1)$ is given by

$$q = (1 - \eta)/\eta. \tag{29}$$

For $\eta > 2/3$ it is smaller than $1/2$ and becomes even negative for $\eta > 1$. This may restrict allowed values of η, although q is difficult to measure accurately.

What about the crucial tests of hot big bang cosmology – the background radiation and nucleosynthesis? Today's temperature of the background photon gas indirectly tests the evolution of the universe between the time of nucleosynthesis t_N and combination of electrons and protons to atoms t_{comb} [1]. For the standard radiation-dominated universe ($\eta = 1/2$) the product of temperature $T(t)$ and $a(t)$ is conserved (for $\rho = 3p = cT^4$). If this product changes by a factor of β between t_N and t_{comb},

$$a(t_{comb})T(t_{comb}) = \beta a(t_N)T(t_N), \quad (30)$$

the estimated value of today's temperature of the background radiation changes by a factor β. Thus β should not deviate from one by more than about an order of magnitude. For the asymptotic solution (25) one obtains for

$$\beta = (t_{comb}/t_N)^{\eta - 1/2}. \quad (31)$$

The ratio between t_N and t_{comb} being huge we conclude that η must be very near 1/2 for the radiation-dominated epoch! It is amazing that we have much better information on the time evolution of $a(t)$ for the radiation-dominated period than for the matter-dominated period! For general solutions one has

$$\beta = \exp\left(\int_{t_N}^{t_{comb}} dt \frac{\dot{\tilde{\lambda}}}{2\dot{H}}\right), \quad (32)$$

illustrating again that the change in $\tilde{\lambda}$ must have been very small compared to the change in H between t_N and t_{comb}.

Is there anything special about $\eta = 2/n$? Indeed, this value is always obtained if the evolution equation for $\tilde{\lambda}$ has a fix-point at $\tilde{\lambda} = 0$ and if the fix-point is approached sufficiently rapidly: $\dot{\lambda} = -aH\lambda$, $a > n$ (ρ-dominated universe). One finds for the fix-point behaviour (8)

$$\eta = 2/n \quad \text{for} \quad a \geqslant n,$$
$$= 2/a \quad \text{for} \quad 0 < a < n. \quad (33)$$

The evolution equation for $\tilde{\lambda}$ has exactly this fix-point behaviour if the coefficients a_3^j defined in eq. (11) fulfil

$$a_3^1 = \tfrac{1}{2}n(a_3^2 - na_3^3), \quad a = -2a_3^1/(2 - na_3^3). \quad (34)$$

We may conclude that for the radiation-dominated period after the QCD phase transition the only realistic evolution equation for $\tilde{\lambda}$ which can be expanded in powers of H, $\dot{H}, \ldots$ is

$$\dot{\tilde{\lambda}} = -aH\tilde{\lambda}, \quad a > 4. \quad (35)$$

(Otherwise $\tilde{\lambda}$ must have been tiny immediately after the phase transition with no contribution to $\dot{\tilde{\lambda}}$ of order H^3.) In this case, $\tilde{\lambda}$ is already much smaller than $\tilde{\rho}$ at the time of nucleosynthesis and calculations of the helium abundance remain unaffected. In the following period until recombination the ratio $\tilde{\lambda}/\tilde{\rho}$ decreases even further so that the temperature of the background radiation is not strongly modified. For the matter-dominated period there exist two alternatives: If the fix-point behaviour of F with the approach (8), $a > 3$ continues, no effects of the cosmological constant will be observable today. If, on the other hand, the matter-dominated universe reaches the asymptotic solution for a more general evolution $F \sim H^3$ (11) we expect modifications of the age of the universe, the critical matter density and the deceleration parameter according to (27), (28) and (29). Both alternatives would give an explanation why λ is so small today! In any case, the really difficult task remains to be accomplished: to find contributions to $\dot{\tilde{\lambda}}$ of order H^3! Our scenario fixes the general form of the background, namely $H = \eta t^{-1}$, expected for realistic cosmologies with λ becoming naturally very small. The various contributions to $\dot{\tilde{\lambda}}$ should be estimated on this background and then checked for consistency if there are contributions of the order H^3. As an example we take the contributions to $\dot{\tilde{\lambda}}$ from vacuum fluctuations of quantized matter fields [3]. The contribution of a minimally coupled, massless scalar field for a Robertson–Walker spacetime with $a(t) \sim t^\eta$ was calculated by Bunch and Davies [4]. From their expression for the renormalized expectation value $\langle 0 | T_{\mu\nu} | 0 \rangle_{ren}$, one obtains for the time variation of the cosmological constant

$$\begin{aligned}\dot{\tilde{\lambda}} &\equiv F(H)\\ &= -(G/9\pi\eta^4)\big[3\eta^3 - 12\eta^2(\eta-1)\\ &\quad + 3\eta(\eta-1)^2 + 4\eta(\eta-1)(\eta-2)\\ &\quad + 2(\eta-1)(\eta-2)(\eta-3)\big] H^5 \ln(H/\mu) \quad (36)\end{aligned}$$

with the renormalization scale μ chosen such that $F(\mu) = 0$. Obviously, the effect is of order H^5 and therefore too small to explain the tiny value of λ today. One also expects a contribution to $\dot{\tilde{\lambda}}$ from the continuous production of particle pairs in an expanding universe [3,5], but it seems very difficult to get a large enough effect $\tilde{\lambda} > H^3$. Most promising for a solution of our problem are perhaps effects of the time evolution of a scalar field whose evolution equation responds to the curvature of spacetime, as is suggested by higher-dimensional models where static classical solutions with arbitrary λ exist [6].

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