Parity-violating anomalies from stochastic quantization

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It is shown that the anomalous contribution to the vacuum current in $(2n+1)$-dimensional theories of massive fermions interacting with classical Yang-Mills and gravitational fields at zero and finite temperature is correctly reproduced by the stochastic quantization procedure. For massless fermions an ambiguity arises. This can be traced to the fact that the stochastic time acts as an additional IR cutoff.

Since it was first proposed by Parisi and Wu,[1] stochastic quantization has received much attention. In particular, the stochastic quantization of fermions and the derivation of, possibly anomalous, Ward identities was studied in detail.[2–4] It turned out that stochastic quantization correctly reproduces the anomalies associated with chiral fermions. As is well known,[5] these anomalies occur in any even-dimensional space since there is no regularization scheme which respects gauge invariance and chiral invariance simultaneously. As was first shown by Redlich,[6] in odd dimensions there is a similar conflict between parity and gauge invariance. If one insists on gauge invariance (which, of course, is mandatory for a fully quantized gauge theory), the vacuum current induced by an (external) gauge field contains a parity-breaking piece, which gives rise to a fractional vacuum charge and a quantum Hall effect.[7] The corresponding term in the effective (Heisenberg-Euler) action is given by the Chern-Simons term of the respective dimensionality;[8,9] in three dimensions, this is the topological mass term for the gauge field introduced by Deser, Jackiw, and Templeton.[10] In this note we show how the anomalous part of the vacuum current arises within the framework of stochastic quantization or, more precisely, stochastic regularization. For massive fermions we describe a computationally simple procedure which evaluates the anomaly in a unique way. Then, we show that in the massless case there is an ambiguity within the stochastic quantization scheme. This gives an interpretation in more physical terms of the inconsistency found in Ref. 3.

Let us consider fermions of mass m interacting with an SU(N) gauge field $A_\mu = A_\mu^a T^a$ in a $(2n+1)$-dimensional (flat) Euclidean space-time. The generators T^a are taken to be in the fundamental representation of SU(N). The action reads (our conventions are those of Ref. 9; in particular, the metric is $g_{\mu\nu} = -\delta_{\mu\nu}$)

$$S = \int d^{2n+1}x\, \bar{\psi}(i\not{D} - m)\psi \tag{1}$$

with $\not{D} = \gamma^\mu(\partial_\mu + iA_\mu)$. The Langevin equations derived from the action (1) are[11]

$$\begin{aligned} \frac{\partial}{\partial\tau}\psi(x,\tau) &= -(\not{D}^2 + m^2)\psi(x,\tau) + (i\not{D} + m)\eta(x,\tau) , \\ \frac{\partial}{\partial\tau}\bar{\psi}(x,\tau) &= -\bar{\psi}(x,\tau)(\overleftarrow{\not{D}}^2 + m^2) + \bar{\eta}(x,\tau) . \end{aligned} \tag{2}$$

Here, τ denotes the fictitious time and the random sources η and $\bar{\eta}$ satisfy the relation

$$\langle \eta_\alpha(x,\tau)\bar{\eta}_\beta(x',\tau')\rangle_\eta = 2\delta_{\alpha\beta}a_\Lambda(\tau-\tau')\delta(x-x') . \tag{3}$$

The regulator function a_Λ introduced by Breit, Gupta, and Zaks[12] has the properties

$$\begin{aligned} &a_\Lambda(\tau) = a_\Lambda(-\tau) , \\ &\int d\tau' a_\Lambda(\tau-\tau') = 1 , \\ &\lim_{\Lambda\to\infty} a_\Lambda(\tau-\tau') = \delta(\tau-\tau') . \end{aligned} \tag{4}$$

The limit $\Lambda\to\infty$ will be performed after all calculations have been done, i.e., after the $\tau\to\infty$ limit has been taken. The solutions of the Langevin equations (2) read

$$\begin{aligned} \psi_\eta(x,\tau) &= \int_0^\tau d\tau' e^{-(\not{D}^2+m^2)(\tau-\tau')}(i\not{D}+m)\eta(x,\tau') , \\ \bar{\psi}_\eta(x,\tau) &= \int_0^\tau d\tau' \bar{\eta}(x,\tau') e^{-(\overleftarrow{\not{D}}^2+m^2)(\tau-\tau')} . \end{aligned} \tag{5}$$

Any field-theory vacuum expectation value can be expressed in terms of these as

$$\langle 0 | F[\psi,\bar{\psi}] | 0\rangle = \lim_{\tau\to\infty} \langle F[\psi_\eta(\tau),\bar{\psi}_\eta(\tau)]\rangle_\eta . \tag{6}$$

The operator F we are interested in is the SU(N) current $\bar{\psi}\gamma^\mu T^a\psi$ and the U(1) current $\bar{\psi}\gamma^\mu\psi$. The former, say, gives

$$\begin{aligned} j^{\mu a}(x) &\equiv \langle 0 | \bar{\psi}(x)\gamma^\mu T^a \psi(x) | 0\rangle \\ &= \lim_{\tau\to\infty} \langle \bar{\psi}_\eta(x,\tau)\gamma^\mu T^a \psi_\eta(x,\tau)\rangle_\eta . \end{aligned} \tag{7}$$

To find the anomalous piece contained in (7) we make the assumption that the background field is purely magnetic: $A^0 = 0$, $A^k = A^k(x^i)$. [We use the convention $x^\mu = (x^0, x^k)$ with $i,j,k,\ldots = 1,2,\ldots,2n$ and $\mu,\nu,\rho,\ldots = 0,\ldots,2n$.] At the end of the calculation, the general result can be deduced from covariance arguments.[7] Hence we obtain, from (5),

$$j^{0a}(x) = \lim_{\tau\to\infty} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \langle \bar{\eta}(x,\tau_1) e^{-(\overleftarrow{\not{D}}^2+m^2)(\tau-\tau_1)} \gamma^0 T^a e^{-(\not{D}^2+m^2)(\tau-\tau_2)} (i\not{D}+m)\eta(x,\tau_2)\rangle_\eta . \tag{8}$$

Performing the η average yields, according to (3),

$$j^{0a}(x)=-2\lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\int_0^{\tau}d\tau_1\int_0^{\tau}d\tau_2 a_\Lambda(\tau_1-\tau_2)\langle x\,|\,\mathrm{tr}\gamma^0T^a e^{-(\not{D}^2+m^2)(2\tau-\tau_1-\tau_2)}\{i\gamma^0\partial_0+i\gamma^kD_k+m\}\,|\,x\rangle\ , \tag{9}$$

where tr denotes the trace in spinor and group space. To be able to combine the two exponentials we assumed that A_μ is chosen so that it commutes with T^a for a fixed value of a. Again, the general result will follow from invariance considerations.[8] Note that the first two terms in the curly brackets of (9) do not contribute. Going to momentum space (see below), the first one is odd in k_0, and the second one vanishes since γ^0 anticommutes with γ^kD_k. At this point it is advantageous to introduce new integration variables:

$$t=\tau_1-\tau_2\ ,$$
$$T=\tfrac{1}{2}(\tau_1+\tau_2)\ .$$

Thus, one gets

$$j^{0a}(x)=-2m\lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\left[\int_0^{\tau/2}dT\int_{-2T}^{2T}dt+\int_{\tau/2}^{\tau}dT\int_{-2(\tau-T)}^{2(\tau-T)}dt\right]a_\Lambda(t)\langle x\,|\,\mathrm{tr}\gamma^0T^a e^{-(-\partial_0{}^2+\not{D}_{2n}{}^2+m^2)2(\tau-T)}\,|\,x\rangle\ . \tag{10}$$

Here we introduced the $2n$-dimensional Dirac operator

$$\not{D}_{2n}=\gamma^k[\partial_k+iA_k(x^i)]\ . \tag{11}$$

Taking the properties (4) of the regulator function a_Λ into account, it is easy to evaluate the t integration for $\Lambda\to\infty$:

$$\int_{-2T}^{2T}dt\,a_\Lambda(t)=\Theta\left[T-\frac{1}{2\Lambda^2}\right]+O\left[\frac{1}{\Lambda^2}\right],$$
$$\int_{-2(\tau-T)}^{2(\tau-T)}dt\,a_\Lambda(t)=\Theta\left[\tau-\frac{1}{2\Lambda^2}-T\right]+O\left[\frac{1}{\Lambda^2}\right]. \tag{12}$$

Hence, we end up with

$$j^{0a}(x)=-2m\lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\int_{1/2\Lambda^2}^{\tau-1/2\Lambda^2}dT\langle x\,|\,\mathrm{tr}\gamma^0T^a e^{-(-\partial_0{}^2+\not{D}_{2n}{}^2+m^2)2(\tau-T)}\,|\,x\rangle\ . \tag{13}$$

This integral is convergent at the upper limit so that one now can perform the $\tau\to\infty$ limit for fixed Λ. Defining $\mathscr{D}^2\equiv-\partial_0{}^2+\not{D}_{2n}{}^2+m^2$, one has

$$\lim_{\tau\to\infty}\int_{1/2\Lambda^2}^{\tau-1/2\Lambda^2}dT\,e^{-2\mathscr{D}^2(\tau-T)}=\tfrac{1}{2}\mathscr{D}^{-2}\lim_{\tau\to\infty}e^{-2\mathscr{D}^2\tau}(e^{2\mathscr{D}^2\tau}e^{-\mathscr{D}^2/\Lambda^2}-e^{\mathscr{D}^2/\Lambda^2})$$
$$=\tfrac{1}{2}\mathscr{D}^{-2}e^{-\mathscr{D}^2/\Lambda^2}=\tfrac{1}{2}\int_0^\infty dw\,e^{-\mathscr{D}^2(w+1/\Lambda^2)}\ . \tag{14}$$

In the last line of Eq. (14) we introduced a parameter integral for the inverse of $\mathscr{D}^2$. To compute the matrix element in (13) one uses a plane-wave basis;[13] setting $z=w+(1/\Lambda^2)$, it follows that

$$\langle x\,|\,e^{-\mathscr{D}^2z}\,|\,x\rangle=\lim_{y\to x}e^{-\mathscr{D}_x{}^2z}\,\delta(x-y)$$
$$=\int\frac{d^{2n+1}k}{(2\pi)^{2n+1}}e^{ikx}e^{-\mathscr{D}_x{}^2z}e^{-ikx}\ . \tag{15}$$

After having done the trivial k_0 integration, the charge density reads

$$j^{0a}(x)=-\frac{m}{2\sqrt{\pi}}\lim_{\Lambda\to\infty}\int_{1/\Lambda^2}^{\infty}dz\,z^{-1/2}e^{-m^2z}$$
$$\times\mathrm{tr}\gamma^0T^a\int\frac{d^{2n}k}{(2\pi)^{2n}}\exp\left[-\left[-k^2+2ik_jD_{2n}^j+D_{2n}{}^2+\frac{i}{2}\gamma_i\gamma_jF^{ij}\right]z\right]. \tag{16}$$

To extract the anomalous part from (16) we rescale z by $z\to\Lambda^2z$ and k_μ according to $k_\mu\to k_\mu z^{1/2}\Lambda^{-1}$:

$$j^{0a}(x)=-\frac{m}{2\sqrt{\pi}}\lim_{\Lambda\to\infty}\Lambda^{2n-1}\int_1^\infty dz\,z^{-1/2-n}e^{-(m^2/\Lambda^2)z}\,\mathrm{tr}\gamma^0T^a$$
$$\times\int\frac{d^{2n}k}{(2\pi)^{2n}}\exp\left[k^2-2ik_jD_{2n}^j(z^{1/2}\Lambda^{-1})-\left[D_{2n}{}^2+\frac{i}{2}\gamma_i\gamma_jF^{ij}\right](z^{1/2}\Lambda^{-1})^2\right]. \tag{17}$$

Now the structure of the k integral is similar to what appears in the evaluation of the chiral anomaly using Fujikawa's method[13] or ζ-function techniques.[14] In both cases, only one term of the expansion of the last exponential in powers of $z^{1/2}\Lambda^{-1}$ gives a nonzero contribution. In the present case, all powers of $z^{1/2}\Lambda^{-1}$ contribute to the vacuum charge. However, using simple scaling arguments, one can show that higher powers of $z^{1/2}\Lambda^{-1}$ cause higher powers in $1/m$. There is only one term which does not vanish for $m\to\infty$. This is the anomaly we are interested in. As an example, let us consider the case $n=1$. Expanding to order $(z^{1/2}\Lambda^{-1})^2$ and performing the k integration, one obtains

$$\begin{aligned} j^{0a}(x) &= \frac{m}{8\pi^{3/2}}\epsilon_{ij}\mathrm{tr}(T^aF^{ij})\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\int_1^\infty dz\, z^{-1/2}e^{-(m^2/\Lambda^2)z}[1+O(z^{1/2}\Lambda^{-1})] \\ &= \frac{1}{8\pi}\frac{m}{|m|}\epsilon_{ij}\mathrm{tr}(T^aF^{ij})+O\left[\frac{1}{m}\right]. \end{aligned} \tag{18}$$

Imposing gauge and Lorentz covariance, the complete anomalous vacuum current reads

$$j^{\mu a}(x)=\frac{1}{8\pi}\frac{m}{|m|}\epsilon^{\mu\rho\sigma}\mathrm{tr}(T^aF_{\rho\sigma}) . \tag{19}$$

[From now on we ignore the regular $O(1/m)$ terms.] This is the result first obtained by Redlich.[6] In his approach the sign factor $m/|m|$ refers to the mass of the Pauli-Villars regulator field, whereas in our case m is the mass of the physical fermion field. As is well known,[8,9] the piece in the effective action belonging to (19) is, up to a normalization constant, the Chern-Simons term ω_3:

$$\Gamma_3[A]=\frac{1}{8\pi}\frac{m}{|m|}\int \mathrm{tr}(A\,dA+\tfrac{2}{3}A^3) .$$

Here, we used the standard differential form notation with $A=iA_k^aT^adx^k$. Similarly, for $n=2$, one has to expand the exponential in (17) to order $(z^{1/2}\Lambda^{-1})^4$ and one then finds the current

$$j^{\mu a}(x)=-\frac{1}{64\pi^2}\frac{m}{|m|}\epsilon^{\mu\nu\rho\sigma\lambda}\mathrm{tr}(T^aF_{\nu\rho}F_{\sigma\lambda}) \tag{20}$$

and the effective action

$$\Gamma_5[A]=\frac{i}{48\pi^2}\frac{m}{|m|}\int \mathrm{tr}[A(dA)^2+\tfrac{3}{2}A^3dA+\tfrac{3}{5}A^5] ,$$

which is proportional to the Chern-Simons term ω_5. In this way it is straightforward, though increasingly tedious, to confirm the expressions

$$j^a(x)=-\frac{m}{|m|}\frac{1}{2}\left[\frac{i}{2\pi}\right]^n\frac{1}{n!}\mathrm{tr}\left[T^a{*}F^n\right] , \tag{21}$$

$$\Gamma_{2n+1}[A]=-\frac{m}{|m|}\frac{\pi}{(n+1)!}\left[\frac{i}{2\pi}\right]^{n+1}\int\omega_{2n+1}(A) , \tag{22}$$

which originally have been derived using the chiral anomaly in $2n$ dimensions.[8,9] The result for the U(1) current is obtained from (21) by omitting the gauge group generator T^a.

A particularly interesting quantity is the U(1) vacuum charge, i.e., the space integral of (13) with T^a omitted. If one introduces in (15) a δ function for the x^0 component only and integrates over k_0, one finds

$$Q_{2n+1}=-\frac{m}{2\pi^{1/2}}\lim_{\Lambda\to\infty}\int_{1/\Lambda^2}^\infty ds\, s^{-1/2}e^{-m^2s} \times\mathrm{Tr}(\gamma^0e^{-\not{D}_{2n}^2s}) . \tag{23}$$

Here Tr denotes the sum over spinor and group indices as well as an integration over x^k. If one recalls that γ^0 anticommutes with $\not{D}_{2n}$, it is obvious that for all $s\neq0$ the trace appearing in (23) is nothing but the index of the $2n$-dimensional Dirac operator $\not{D}_{2n}$ (Ref. 15):

$$\mathrm{Tr}(\gamma^0e^{-\not{D}_{2n}s})=\mathrm{index}\,\not{D}_{2n}=\int\exp\left[\frac{i}{2\pi}F\right] . \tag{24}$$

The last equality follows from the Atiyah-Singer index theorem for the twisted spin complex, which can be considered an integrated version of the chiral anomaly in $2n$ dimensions. (To have a well-defined index problem, we could assume x^k-space to be a large $2n$-sphere.) The field-strength form

$$F=\tfrac{1}{2}iF_{kl}^aT^adx^k\wedge dx^l$$

is constructed from the spatial components of $F_{\mu\nu}$ and the integral is over $2n$-dimensional x^k space with all terms ignored which do not contain the appropriate volume form. Equation (23) with (24) implies

$$Q_{2n+1}=-\frac{1}{2}\frac{m}{|m|}\int\exp\left[\frac{i}{2\pi}F\right] . \tag{25}$$

This is the desired result. It shows that the vacuum charge produced by topologically nontrivial field configurations such as monopoles, vortices, etc., is an integer or half-integer. Equation (25) is exact to all orders in $1/m$. On the other hand, the same result also is obtained by taking the spatial integral of (21) with T^a omitted. This proves that it is the anomalous part of the current alone which causes the (fractional) vacuum charge.

The present approach can be generalized to nonzero temperatures $T=\beta^{-1}$. The spinor fields are required to be antiperiodic in ordinary Euclidean time with period β and so the trace in (23) has to be performed using a complete set of antiperiodic spinor functions. This means that the temporal δ function in (15) has to be represented by a discrete sum over momenta rather than by an integral. Evaluating this sum for $\Lambda\to\infty$ and proceeding as above yields for the temperature dependence of both the SU(N) and U(1) charges

$$Q(T)=Q(T=0)\tanh\left[\frac{|m|}{2T}\right] . \tag{26}$$

This coincides with the results of Refs. 9 and 16. Obviously, the magnitude of the vacuum charge depends on the boundary conditions imposed on ψ. We therefore conclude that Q receives contributions of fermionic field modes with arbitrarily large wavelength. This has to be contrasted with the case of chiral anomalies in even-dimensional spaces. One finds[17] that there the anomaly factor is the same for all values of β. The reason is that chiral anomalies are associated with short-distance operator-product singularities which are insensitive to global properties such as the antiperiodic boundary condition imposed here.

Another interesting generalization is the vacuum charge induced by a combined Yang-Mills and gravitational background field.[18] Then Euclidean space-time has the structure $R\times\mathscr{M}_{2n}$ with a static space $\mathscr{M}_{2n}$. Therefore the metric may be written as

$$g_{\mu\nu}(x)=\begin{bmatrix}-1 & 0\\ 0 & g_{ij}(x^k)\end{bmatrix} . \tag{27}$$

The derivation of Eq. (23) remains unaltered if we also include the spin connection constructed from g_{ij} into $\not{D}_{2n}$. (For this connection to be well defined, we assume $\mathscr{M}_{2n}$ to be a spin manifold.) It is known from the general index theorem[15] that for a curved background the index density

$$ch(F)=\exp\left[\frac{i}{2\pi}F\right] ,$$

the Chern form, is replaced by

$$ch(F)\wedge\hat{A}(\mathscr{M}_{2n})$$

with the $\hat{A}$ polynomial

$$\hat{A}(\mathscr{M}_{2n})=\left[\det\frac{\Omega/4\pi}{\sinh(\Omega/4\pi)}\right]^{1/2}$$

constructed from the curvature forms $\Omega^i_j=\frac{1}{2}R^i{}_{jkl}dx^k\wedge dx^l$, where R^i_{jkl} is the Riemann tensor belonging to g_{ij}. Therefore we expect the U(1) vacuum charge to be given by

$$Q_{2n+1}=-\frac{1}{2}\frac{m}{|m|}\int_{\mathscr{M}_{2n}}ch(F)\wedge\hat{A}(\mathscr{M}_{2n}) . \tag{28}$$

This equation can be obtained directly in the present approach if one represents the δ function appearing in (15) in a way which respects covariance under general coordinate transformations. (Just as we are not interested in regularization schemes spoiling gauge invariance, we also want general coordinate invariance to stay intact at the quantum level.) We therefore use the representation[13]

$$\delta(x,x')=\int\frac{d^{2n+1}k}{(2\pi)^{2n+1}}e^{ik_\mu D^\mu\sigma(x,x')} \tag{29}$$

in terms of the geodesic biscalar $\sigma(x,x')$ which can be considered a generalization of $\frac{1}{2}(x-x')^2$ for curved space.[19] Using (29) in (15), applying the operator $\mathscr{D}^2$ and making the rescalings which lead to (17), it is then a matter of straightforward algebra to expand the exponential to the desired order. For $n=2$ and for a pure gravitational field, say, one finds, in agreement with (28),

$$Q_5=\frac{m}{|m|}\frac{1}{1536\pi^2}\int d^4x\,\epsilon^{ijkl}R_{mnij}R^{mn}{}_{kl} . \tag{30}$$

An example of topologically nontrivial field configurations with $Q_5\neq 0$ are the monopole solutions of the five-dimensional Kaluza-Klein theory.[18,20]

Up to now we have shown that for *massive* fermions the same anomalies are obtained as by using conventional regularization schemes to evaluate the fermionic determinant (such as ζ-function or heat-kernel methods). Next let us look at the case $m=0$. In standard field theory one uses a Pauli-Villars regulator of mass M and subtracts its contribution for $M\to\infty$ from the effective action.[6] This contribution is precisely the anomalous part of the complete Heisenberg-Euler action. Thus, for $m=0$ we obtain the same anomaly as for $m\neq 0$. In stochastic quantization the situation is different. The first possibility is to again use a Pauli-Villars regulator field. Then the above calculation applies to this field and for $M\to\infty$ we again recover the anomaly. To show that there is a second possible regularization scheme we recall the form of the charge before the $\tau\to\infty$ limit has been taken:

$$\begin{aligned}Q_{2n+1}&=-\lim_{\Lambda\to\infty}\lim_{\tau\to\infty}m\int_{\Lambda^{-2}}^{2\tau-\Lambda^{-2}}ds\,\mathrm{Tr}(\gamma^0e^{-(\not{D}^2+m^2)s})\\&=-\frac{1}{2\pi^{1/2}}\lim_{\Lambda\to\infty}\lim_{\tau\to\infty}m\int_{\Lambda^{-2}}^{2\tau-\Lambda^{-2}}ds\,s^{-1/2}e^{-m^2s}\,\mathrm{index}\,\not{D}_{2n} .\end{aligned} \tag{31}$$

For any finite value of the stochastic time τ the integral is well defined even for $m=0$. We therefore may set $m=0$ in the whole expression which amounts to starting with massless fermions from the very beginning. Hence, we have a vanishing contribution for every finite value of τ which in the $\tau\to\infty$ limit implies $Q_{2n+1}=0$, i.e., there is no anomaly in this scheme. (This ambiguity is investigated in Ref. 3 using different methods.) The freedom to set $m=0$ in expressions such as (31) does not exist in conventional field theory, because there in all comparable calculations[6,9] the upper bound of the proper-time integral is equal to infinity from the outset. This ambiguity arises because the stochastic time τ acts as an additional infrared cutoff which is not present in ordinary field theory. [Note that integrals such as (31) are a proper-time representation of momentum-space integrals and $k^2\to 0$

($k^2 \to \infty$) corresponds to $s \to \infty$ ($s \to 0$).] The limit $m \to 0$ does not commute with the limit $\tau \to \infty$. In physical terms the reason is that, as we discussed in connection with Eq. (26), the vacuum charge receives contributions from fermionic field modes with arbitrarily small values of k^2, i.e., arbitrarily large values of s. For Q to become a finite quantity, an IR cutoff has to be present in order to suppress the contributions from $k^2 \to 0$. In standard field theory, this role is played by the physical fermion mass or by the mass of the regulator field. In stochastic regularization, the Langevin time τ is introduced as a new dimensionful quantity which cuts off the proper-time integral for large s. To further clarify this point let us compare the parity-violating anomalies considered here with ordinary chiral anomalies. Using the above formalism, the anomaly factor for a chiral rotation with parameter $\alpha(x)$ is essentially given by

$$\lim_{\Lambda \to \infty} \lim_{\tau \to \infty} \int_{\Lambda^{-2}}^{2\tau - \Lambda^{-2}} ds \, \mathrm{Tr}[\alpha \gamma_5 (\not{D}^2 + m^2) e^{-(\not{D}^2 + m^2)s}] \tag{32a}$$

$$= - \lim_{\Lambda \to \infty} \lim_{\tau \to \infty} \mathrm{Tr}[\alpha \gamma_5 (e^{-(\not{D}^2 + m^2)\Lambda^{-2}} - e^{-(\not{D}^2 + m^2)(2\tau - \Lambda^{-2})})] \, . \tag{32b}$$

Contrary to (31), this expression is entirely determined by the behavior of the heat-kernel $\langle x \mid \exp(-\not{D}^2 s) \mid x \rangle$ for $s \to 0$ and $s \to \infty$. For $\tau \to \infty$ the second exponential in (32b) vanishes and from the first one the anomaly is recovered by the usual Seeley—De Witt expansion. This holds for $m \neq 0$ and $m = 0$ alike [the zero modes of $\not{D}$ cancel between the two exponentials of (32b)]; i.e., this time the limits $\tau \to \infty$ and $m \to 0$ commute and stochastic quantization uniquely produces the correct answer. This reflects the well-known fact that chiral anomalies have their origin in short-distance operator-product singularities, which is not true for the parity-violating anomalies.

In conclusion, one can say that the failure of stochastic quantization to give regularization-prescription-independent anomalies in odd dimensions is due to the fact that the stochastic time τ acts as an additional regulator which has no analogue in the standard regularization prescriptions for the fermionic determinant. In the first one of the prescriptions discussed above, where a Pauli-Villars regulator was used, this fact has not been exploited and therefore the correct anomaly is obtained.

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