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Convergence of Local Charges and Continuity Properties of W^*-Inclusions*

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Abstract. The local generators of symmetry transformations which have recently been constructed from a quantum field theoretical version of Noether's theorem are shown to converge to the global ones as the volume tends to the whole space. The proof relies on the continuous volume dependence of the universal localizing maps which are associated to the local split W^*-inclusions.

1. Introduction

A new approach to a quantum Noether theorem has recently been set up [1–3] in the algebraic formulation of quantum field theory [4]. In a given theory, let $\mathscr{F}(\mathscr{O})$ be the von Neumann algebra which is generated by the field operators which are localized in the bounded space-time region $\mathscr{O}$. Let G be the group of space-time and internal symmetries, and let J_u for each u in the Lie algebra $\mathscr{G}$ of G denote the corresponding selfadjoint generator of the global symmetry transformation. The quantum Noether theorem then asserts that there exist local field operators which induce that symmetry locally.

The construction of these local generators is based on the so-called split property [5] (see below) which may be understood as the possibility to decouple a region $\mathscr{O}$ completely from any other region which is separated from $\mathscr{O}$ by a finite spacelike distance. Assume that $\mathscr{F}$ possesses the split property, and let $\mathscr{O}$ and $\hat{\mathscr{O}}$ be bounded space-time regions such that $\mathscr{O}+x\subset\hat{\mathscr{O}}$ for all x in some neighbourhood of the origin. Then for each $u\in\mathscr{G}$ there is a selfadjoint operator $J_u^{\mathscr{O},\hat{\mathscr{O}}}$ which is affiliated to $\mathscr{F}(\hat{\mathscr{O}})$ and induces on $\mathscr{F}(\mathscr{O})$ the infinitesimal symmetry transformation u, i.e.

$$e^{i\lambda J_u^{\mathscr{O},\hat{\mathscr{O}}}}\in\mathscr{F}(\hat{\mathscr{O}}),\qquad \lambda\in\mathbb{R}, \tag{1.1}$$

and for sufficiently small λ

$$e^{i\lambda J_u^{\mathscr{O},\hat{\mathscr{O}}}}Fe^{-i\lambda J_u^{\mathscr{O},\hat{\mathscr{O}}}}=e^{i\lambda J_u}Fe^{-i\lambda J_u}. \tag{1.2}$$

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Actually there is a canonical choice for $J_u^{\mathcal{O},\hat{\mathcal{O}}}$ which depends only on the triple $\Lambda=(\mathcal{F}(\mathcal{O}),\mathcal{F}(\hat{\mathcal{O}}),\Omega)$, where Ω is the vector representing the vacuum [2].

The local generators $J_u^{\mathcal{O},\hat{\mathcal{O}}}$ are analogues of the (regularized) integrals of the zero-component of the Noether current associated to u in a Lagrangian field theory. Similarly to them they transform covariantly under global symmetry transformations. It is a major open problem in the general setting to recover such a current from the correspondence $\mathcal{O},\hat{\mathcal{O}}\to J_u^{\mathcal{O},\hat{\mathcal{O}}}$.

However, the correspondence $u\to J_u^{\mathcal{O},\hat{\mathcal{O}}}$ has some unexpected remarkable properties which are not shared by its analogue in Lagrangian field theory:

(i) It is a representation of $\mathcal{G}$. This leads to a rigorous variant of current algebra, which relies on first principles only [2].

(ii) It is quasi-equivalent to the global representation $u\to J_u$ of $\mathcal{G}$, in particular, the local energy momentum operators have the same spectrum as the global ones [3].

The construction of $J_u^{\mathcal{O},\hat{\mathcal{O}}}$ is rather indirect, and it seems to be very important for further applications to understand the functional dependence of J_u on the pair $\mathcal{O},\hat{\mathcal{O}}$ in more detail.

In this paper we want to study the question whether $J_u^{\mathcal{O},\hat{\mathcal{O}}}$ tends to J_u as $\mathcal{O}$ and $\hat{\mathcal{O}}$ tend to $\mathbb{R}^4$ in a suitable way. Actually, for the integrated densities this question is nontrivial either; in this case, it has been answered in a satisfactory way first by Requardt [6].

In Sect. 2 we prove a very general result: for every non-decreasing sequence of regions $\mathcal{O}_n$ there is a sequence $\hat{\mathcal{O}}_n\supset\mathcal{O}_n$ such that

$$J_u^{\mathcal{O}_n,\hat{\mathcal{O}}_n}\to J_u \tag{1.3}$$

in the strong resolvent sense.

Actually we find a convergence which is even somewhat stronger. To explain this recall that

$$J_u^{\mathcal{O},\hat{\mathcal{O}}}=\psi_\Lambda(J_u), \tag{1.4}$$

where ψ_Λ is the "universal localizing map" associated to the triple $\Lambda=(\mathcal{F}(\mathcal{O}),\mathcal{F}(\hat{\mathcal{O}}),\Omega)$ [5, 3]. ψ_Λ is a *-isomorphism which maps $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ being the Hilbert space on which the von Neumann algebras $\mathcal{F}(\mathcal{O})$ act, onto the canonical type I factor $\mathcal{N}_\Lambda$ between $\mathcal{F}(\mathcal{O})$ and $\mathcal{F}(\hat{\mathcal{O}})$ [2, 5]. For the sequence $\mathcal{O}_n$, $\hat{\mathcal{O}}_n$ alluded to before we show that ψ_{Λ_n}, $\Lambda_n=(\mathcal{F}(\mathcal{O}_n),\mathcal{F}(\hat{\mathcal{O}}_n),\Omega)$ converges pointwise strongly to the identity,

$$\psi_{\Lambda_n}(B)\xrightarrow[n\to\infty]{} B \quad \text{strongly for all} \quad B\in\mathcal{B}(\mathcal{H}). \tag{1.5}$$

Now let A be a (possibly unbounded) selfadjoint operator on $\mathcal{H}$. Then $A_n=\psi_{\Lambda_n}(A)$ is a selfadjoint operator with the same basic measure class[1] as A and

$$g(A_n)\xrightarrow[n\to\infty]{} g(A) \quad \text{strongly} \tag{1.6}$$

for each bounded Borel function g on $\mathbb{R}$.

The convergence (1.5) of the localizing maps follows from a generalized cluster property. Let ω_0 denote the vacuum and φ_n the product state on $\mathcal{F}(\mathcal{O}_n)\cdot\mathcal{F}(\hat{\mathcal{O}}_n)'$,

[1] By the split property $\mathcal{H}$ is automatically separable [5]

which coincides with ω_0 on $\mathscr{F}(\mathcal{O}_n)$ and on $\mathscr{F}(\hat{\mathcal{O}}_n)'$. Then (1.5) is a consequence of

$$\lim_{n\to\infty} \|\varphi_n - \omega_0\| = 0 . \tag{1.7}$$

The generalized cluster property (1.7) is always fulfilled in theories with the split property if $\hat{\mathcal{O}}_n$ tends to $\mathbb{R}^4$ sufficiently fast. In general, there is no control on the required increase of $\hat{\mathcal{O}}_n$. In the special case of a dilation invariant theory, with $\mathcal{O}_n$ and $\hat{\mathcal{O}}_n$ denoting the double cones centered at the origin with radii r_n and R_n, respectively, (1.7) holds if and only if R_n/r_n tends to infinity (Sect. 2); in the case of a massive scalar free field with mass m it suffices that $r_n \to \infty$ and $R_n - r_n \geqq \frac{c}{m} \log r_n m$ for some constant c which depends on the space-time dimension (Sect. 4).

In more general theories information can be obtained from the behavior of a generalized "partition function" $Z_{BW}(\beta, \mathcal{O})$. This quantity has been introduced in general quantum field theories by Buchholz and Wichmann [7] and is defined as the nuclearity index of the set $e^{-\beta H}\mathscr{F}(\mathcal{O})_1\Omega$, where H denotes the Hamiltonian, Ω the vacuum vector and $\mathscr{F}(\mathcal{O})_1$ the unit ball in $\mathscr{F}(\mathcal{O})$. If the Buchholz-Wichmann partition function is finite for finite volumes and does not increase too fast with the temperature, the theory has the split property [7, 8]. Moreover, if the volume dependence is reasonable, one expects that KMS-states for all positive temperatures exist [9], and one may hope that the theory has a complete particle interpretation (asymptotic completeness) [10]. In Sect. 3 we show that the generalized cluster property (1.7) and therefore also the convergence of local generators of symmetry transformations can be controlled by the Buchholz-Wichmann partition function. This yields the desired convergence whenever $r_n \to \infty$ and $R_n - r_n \geqq \frac{c'}{m}(\log m r_n)^2$, for a suitable constant c' depending upon the "effective" space dimension.

In the second part of Sect. 4 the estimates for the norm difference between the product state and the vacuum in the case of the free scalar massive field are used to construct theories which have a minimal splitting distance.

We close this section with some comments on our basic assumptions. They concern the local field algebras $\mathscr{F}(\mathcal{O})$ which, in presence of superselection rules, are not generated by observable quantities. However, it is worth stressing that the existence of the compact group G of gauge transformations of the first kind, and of the net $\mathcal{O} \to \mathscr{F}(\mathcal{O})$ of algebras of field operators with normal commutation properties [so that the G-invariant part of $\mathscr{F}(\mathcal{O})$ coincides with an appropriate representation of the algebra $\mathfrak{A}(\mathcal{O})$ of all observables that can be measured within $\mathcal{O}$] can be derived from few basic principles on the net of local observables $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ given as von Neumann algebras on the vacuum sector Hilbert space $\mathscr{H}_0$ [11]; when G is commutative, see [12].

Our main assumption is that, whenever $\mathcal{O}, \hat{\mathcal{O}}$ are double cones and $\mathcal{O}$ lies in the interior of $\hat{\mathcal{O}}$, $(\mathscr{F}(\mathcal{O}), \mathscr{F}(\hat{\mathcal{O}}), \Omega)$ is a standard split W^*-inclusion.

Recall that a standard W^*-inclusion is a triple $\Lambda = (\mathfrak{A}, \mathfrak{B}, \Omega)$ consisting of von Neumann algebras $\mathfrak{A} \subset \mathfrak{B}$ acting on a Hilbert space $\mathscr{H}$ and a vector $\Omega \in \mathscr{H}$ which is cyclic and separating for $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{A}' \wedge \mathfrak{B}$, $\mathfrak{A}'$ denoting the commutant of $\mathfrak{A}$. Moreover Λ is split if there is a type I factor $\mathscr{N}$ so that $\mathfrak{A} \subset \mathscr{N} \subset \mathfrak{B}$ [5].

Under the split assumption, by the normal commutation properties of spacelike separated field operators, the standard character of the vacuum vector Ω follows from the Reeh-Schlieder theorem [2]. The most characteristic assumption is the split property, which as discussed above seems to be characteristic of reasonable theories.

The split property for the net of observables $\mathfrak{A}$ can be interpreted as a principle of local preparation of states [3, 27]. The split property for $\mathscr{F}$ implies the split property for $\mathfrak{A}$; the converse holds in special cases, e.g. if G is finite abelian, and it is an interesting open problem whether it holds more generally [1].

2. Cluster Properties and Continuity of W^*-Inclusions

In this section we shall investigate the connection between clustering and continuity properties of inclusions of von Neumann algebras.

Given a standard split W^*-inclusion Λ there is a normal state φ_Λ (the product state) on $\mathfrak{A} \vee \mathfrak{B}'$ (the von Neumann algebra generated by $\mathfrak{A}$ and by $\mathfrak{B}'$) with

$$\varphi_\Lambda(AB') = \omega_0(A)\omega_0(B'), \quad A \in \mathfrak{A}, \quad B' \in \mathfrak{B}', \tag{2.1}$$

where ω_0 denotes the state on $\mathscr{B}(\mathscr{H})$ induced by Ω, and a unique vector η_Λ in the natural cone $P^\natural_{\Omega, \mathfrak{A} \vee \mathfrak{B}'}$ which induces φ_Λ. η_Λ is cyclic and separating for $\mathfrak{A} \vee \mathfrak{B}'$ (hence for $\mathfrak{A}' \cap \mathfrak{B}$).

The vector η_Λ can be used to define a unitary operator U_Λ from $\mathscr{H}$ onto the tensor product space $\mathscr{H} \otimes \mathscr{H}$,

$$U_\Lambda AB'\eta_\Lambda = A\Omega \otimes B'\Omega, \quad A \in \mathfrak{A}, \quad B' \in \mathfrak{B}'. \tag{2.2}$$

The *universal localizing map* ψ_Λ is now defined by [5, 3]

$$\psi_\Lambda(T) = U_\Lambda^{-1}(T \otimes 1)U_\Lambda. \tag{2.3}$$

ψ_Λ is a *-isomorphism of $\mathscr{B}(\mathscr{H})$ onto the canonical type I factor $\mathscr{N}_\Lambda$ between $\mathfrak{A}$ and $\mathfrak{B}$, and it acts trivially on $\mathfrak{A}$.

Crucial for the following discussion is the fact that the distance of vectors in the natural cone can be estimated by the distance of the induced states, i.e. one has (see e.g. [13])

$$\|\eta_\Lambda - \Omega\|^2 \leqq \|\varphi_\Lambda - \omega_0\|. \tag{2.4}$$

This is the basis for a connection between the convergence of the universal localizing maps and the generalized cluster property (1.7).

2.1. Theorem. *Let $\Lambda_n = (\mathfrak{A}_n, \mathfrak{B}_n, \Omega)$ be a sequence of standard split W^*-inclusions with $\mathfrak{A}_1 \subset \mathfrak{A}_n$ for all n. If $\|\varphi_{\Lambda_n} - \omega_0\| \to 0$ for $n \to \infty$ the universal localizing maps ψ_{Λ_n} converge pointwise strongly to the identity.*

Proof. By (2.4) $\|\varphi_{\Lambda_n} - \omega_0\| \to 0$ implies $\eta_{\Lambda_n} \to \Omega$, hence from the definition of U_{Λ_n} and from $\mathfrak{A}_1 \subset \mathfrak{A}_n$ for all n

$$U_{\Lambda_n}^{-1}(A\Omega \otimes \Omega) \to A\Omega, \quad A \in \mathfrak{A}_1.$$

Ω is cyclic for $\mathfrak{A}_1$, hence for all $\Phi \in \mathscr{H}$,

$$U_{\Lambda_n}^{-1}(\Phi \otimes \Omega) \to \Phi.$$

In particular, for $T \in \mathscr{B}(\mathscr{H})$, $A \in \mathfrak{A}_1$,

$$U_{A_n}^{-1}(TA\Omega \otimes \Omega) \to TA\Omega\,.$$

But from the definition of ψ_{A_n} (2.3) this means

$$\psi_{A_n}(T)A\eta_{A_n} \to TA\Omega\,.$$

η_{A_n} tends to Ω, Ω is cyclic for $\mathfrak{A}_1$ and ψ_{A_n} is uniformly bounded, hence $\psi_{A_n}(T)$ converges strongly to T. q.e.d.

2.2. Corollary. *Under the same assumptions, the transpose of the maps ψ_{A_n} of Theorem 2.1 converge strongly on the predual of $\mathscr{B}(\mathscr{H})$.*

Proof. By a 3ε argument it suffices to consider vector states $\omega = (\Phi, \cdot\, \Phi)$. By the relations

$$(\omega \circ \psi_n - \omega)(T) = (U_n\Phi, T \otimes 1, U_n\Phi) - (\Phi \otimes \Omega, T \otimes 1, \Phi \otimes \Omega)\,;\; U_n\Phi \to \Phi \otimes \Omega\,,$$

the assertion follows as in the Proof of Theorem 2.1. q.e.d.

As anticipated in the Introduction, a very restrictive notion of convergence on operators follows immediately from Theorem 3.1.

2.3. Corollary. *Under the same assumptions, for each selfadjoint (possibly unbounded) operator A on $\mathscr{H}$, the selfadjoint operators $A_n = \psi_{A_n}(A)$ have the same basic measure class as A and, for each bounded Borel function f on $\mathbb{R}$, $f(A_n)$ converges strongly to $f(A)$ as $n \to \infty$.*

The generalized cluster property (1.7) holds under very general circumstances. The following result which we reproduce for the convenience of the reader follows from a generalized Powers argument [14–16].

2.4. Proposition. *Let $A_n = (\mathfrak{A}, \mathfrak{B}_n, \Omega)$ be a sequence of standard split W^*-inclusions with an increasing sequence $\mathfrak{B}_n$. Then*

$$\|\varphi_{A_n} - \omega_0\| \to 0\,,$$

if and only if $\mathfrak{B} = \bigcup \mathfrak{B}_n$ is irreducible.

Proof. Let f_n be the ultraweakly continuous linear functional $\varphi_{A_n} - \omega_0$ on the von Neumann algebra $\mathscr{R}_n = \mathfrak{A} \vee \mathfrak{B}_n'$. By the standard split property, $\mathscr{R}_n$ is spatially isomorphic to $\mathfrak{A} \otimes \mathfrak{B}_n'$ (see e.g. [17]). Thus if $\mathfrak{B}$ is irreducible we have

$$\bigcap \mathscr{R}_n = \mathfrak{A}\,.$$

Now $f_n = f_1 \restriction \mathscr{R}_n$ and $f_n \restriction \mathfrak{A} = 0$. Then $\|f_n\| \to 0$. Otherwise $|f_{n_k}(T_k)| \geqq \delta$ for some $\delta > 0$ and for a sequence $T_k \in \mathscr{R}_{n_k}$, $\|T_k\| = 1$, $n_k \to \infty$. The sequence T_k has a weak limit point T. T belongs to $\mathfrak{A}$ and $|f_1(T)| \geqq \delta$, in contradiction with the definition of f_1. Conversely, if $\|\varphi_{A_n} - \omega_0\| \to 0$, then $\varphi_{A_n} - \omega_0 \restriction \mathfrak{A} \cup \mathfrak{B}' = 0$, i.e. $\omega_0(AC) = \omega_0(A)\omega_0(C)$, $A \in \mathfrak{A}$, $C \in \mathfrak{B}'$. Since Ω is cyclic for $\mathfrak{A}$ and separating for $\mathfrak{B}' \subset \mathfrak{B}_n'$ we get $C\Omega = \omega_0(C)\Omega$ and $C = \omega_0(C)1$. q.e.d.

The application of the abstract results to the field theoretical problem is easy. Let $\mathscr{F}(\mathscr{O})$ for bounded open regions $\mathscr{O}$ be the von Neumann algebras of local field operators acting in some Hilbert space, and assume isotony,

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2) \tag{2.5}$$

and irreducibility

$$\{\bigcup \mathcal{F}(\mathcal{O})\}' = \{\lambda 1 \mid \lambda \in \mathbb{C}\}. \tag{2.6}$$

Furthermore, assume that for each $\mathcal{O}$ there is an $\hat{\mathcal{O}} \supset \mathcal{O}$ such that $\Lambda_{\mathcal{O},\hat{\mathcal{O}}} = (\mathcal{F}(\mathcal{O}), \mathcal{F}(\hat{\mathcal{O}}), \Omega)$ is a standard split W^*-inclusion. Then we find immediately the following result:

2.5. Corollary. *Let $\mathcal{O}_n$ be a nondecreasing sequence of regions. Then there is a sequence of regions $\hat{\mathcal{O}}_n \supset \mathcal{O}_n$ such that*

$$\|\varphi_{\Lambda_{\mathcal{O}_n,\hat{\mathcal{O}}_n}} - \omega_0\| \to 0$$

and $\psi_{\Lambda_{\mathcal{O}_n,\hat{\mathcal{O}}_n}} \to \mathrm{id}$ pointwise strongly.

The results discussed so far yield the desired convergence of local charges.

2.6. Theorem. *Let $\mathcal{O}_n$ be a nondecreasing sequence of double cones. Under the above assumptions there is a sequence of double cones $\hat{\mathcal{O}}_n \supset \mathcal{O}_n$ such that, for each generator $u \in \mathcal{G}$*

$$J_u^{\mathcal{O}_n,\hat{\mathcal{O}}_n} \to J_u$$

in the sense that the basic measure classes are the same and for each L^∞ function f,

$$f(J_u^{\mathcal{O}_n,\hat{\mathcal{O}}_n}) \to f(J_u) \quad \textit{strongly}.$$

More generally (even if the global gauge group is not a Lie group), given any global charge operator Q (i.e. Q belongs to the center of the von Neumann algebra generated by the global internal symmetry transformations), the corresponding local charge operators, $Q^{\mathcal{O}_n,\hat{\mathcal{O}}_n} = \psi_{\Lambda_{\mathcal{O}_n,\hat{\mathcal{O}}_n}}(Q)$ [2] converge strongly to Q.

A further consequence of the preceding discussion is the convergence of "globalized partial states." Let ω be any normal state on $\mathcal{B}(\mathcal{H})$. The globalization of the partial state $\omega \restriction \mathcal{F}(\hat{\mathcal{O}}_n)$ is $\omega \circ \psi_{\Lambda_{\mathcal{O}_n,\hat{\mathcal{O}}_n}} \equiv \omega_n$. By Corollary 2.2 (ω_n) converges in norm to ω as $n \to \infty$.

A purely algebraic argument as above cannot say how fast $\hat{\mathcal{O}}_n$ has to grow with respect to $\mathcal{O}_n$ (cf. Sects. 3 and 4).

For instance in a dilation covariant theory there are automorphisms δ_λ of $\mathcal{F} = \mathcal{C}^*\left(\bigcup_{\mathcal{O}} \mathcal{F}(\mathcal{O})\right)$, $\lambda > 0$ such that $\omega_0 \circ \delta_\lambda = \omega_0$, $\delta_\lambda(\mathcal{F}(\mathcal{O}_r)) = \mathcal{F}(\mathcal{O}_{\lambda r})$, where $\mathcal{O}_r$ is the double cone of radius r centered at the origin. Then for each $\lambda > 0$, $\|\varphi_{\mathcal{O}_r,\mathcal{O}_R} - \omega_0\| = \|\varphi_{\mathcal{O}_{\lambda r},\mathcal{O}_{\lambda R}} - \omega_0\|$, and by Proposition 2.4 we have that $\|\varphi_{\mathcal{O}_r,\mathcal{O}_R} - \omega_0\| \to 0$ as $r \to \infty$ if and only if $R/r \to \infty$.

We conclude noting that similar convergence properties hold if we keep $\hat{\mathcal{O}}$ constant and let $\mathcal{O}_n$ shrink to a point, using instead of the irreducibility assumption (2.6) the following property

$$\bigcap_n \mathcal{F}(\mathcal{O}_n) = \mathbb{C} \cdot 1 \quad \text{if} \quad \bigcap_n \mathcal{O}_n = \{\text{point}\},$$

which follows from the general structure of quantum field theory [18].

We turn now to a brief discussion of general continuity properties of standard split W^*-inclusions related to the continuity of local currents at finite $\mathcal{O}, \hat{\mathcal{O}}$.

If $\mathfrak{A}_n$ is an increasing (respectively decreasing) sequence of von Neumann algebras such that $\bigvee \mathfrak{A}_n = \mathfrak{A}$ (respectively $\bigwedge \mathfrak{A}_n = \mathfrak{A}$) we shall write $\mathfrak{A}_n \nearrow \mathfrak{A}$ (respectively $\mathfrak{A}_n \searrow \mathfrak{A}$).

Lemma 2.6. *Let $\mathfrak{A}_n$ be a monotone sequence of von Neumann algebras with $\mathfrak{A}_n \nearrow \mathfrak{A}$ or $\mathfrak{A}_n \searrow \mathfrak{A}$ and Ω, η cyclic and separating vectors for $\mathfrak{A}$ and $\mathfrak{A}_n$ for all n. Let J_n, J and Δ_n, Δ denote the relative modular conjugation and the relative modular operators of $\mathfrak{A}_n$, $\mathfrak{A}$, respectively, with respect to the pair Ω, η. Then $J_n \to J$ in the strong operator topology, and $\Delta_n \to \Delta$ in the strong resolvent sense.*

Proof. In the case $\mathfrak{A}_n \nearrow \mathfrak{A}$ the statement follows by an application of Lemma A.1 (Appendix) to the closure of the antilinear operator $A\Omega \to A^*\eta$, $A \in \mathfrak{A}_n$. The case $\mathfrak{A}_n \searrow \mathfrak{A}$ may be obtained from the first case by looking at the commutants. q.e.d.

2.7. Proposition. *Let $\mathfrak{A}_n$ be a sequence of von Neumann algebras such that $\mathfrak{A}_n \nearrow \mathfrak{A}$ or $\mathfrak{A}_n \searrow \mathfrak{A}$ and let Ω be a cyclic and separating vector for all $\mathfrak{A}_n$ and for $\mathfrak{A}$. Let φ be a faithful normal state of $\bigvee \mathfrak{A}_n$ and η_n, η the vector representatives of $\varphi \restriction \mathfrak{A}_n$ and $\varphi \restriction \mathfrak{A}$ in $P^\natural_\Omega(\mathfrak{A}_n)$ and $P^\natural_\Omega(\mathfrak{A})$, respectively. Then $\|\eta_n - \eta\| \to 0$.*

Proof. For simplicity we shall prove the proposition only for the following particular cases which suffice for our applications:

(i) $\quad \mathfrak{A}_n \searrow \mathfrak{A}$ and η_1 is cyclic for $\mathfrak{A}$.

(ii) $\quad \mathfrak{A}_n \nearrow \mathfrak{A}$ and η is cyclic for $\mathfrak{A}$.

The general case can be obtained similarly.

(i) Since η_1 and η_n induce the same state on $\mathfrak{A}_n$, the formula ($\mathfrak{A}_\infty = \mathfrak{A}$, $\eta_\infty = \eta$)

$$V_n A\eta_1 = A\eta_n, \qquad A \in \mathfrak{A}_n, \qquad n \in \mathbb{N} \cup \{\infty\},$$

determines a unitary $V_n \in \mathfrak{A}_n'$. Since V_n maps $P^\natural_{\eta_1}(\mathfrak{A}_n)$ onto $P^\natural_{\eta_n}(\mathfrak{A}_n)$, V_n is nothing but the standard implementation (cf. e.g. [5, Appendix]) of the identity on $\mathfrak{A}_n$ with respect to the cones $P^\natural_{\eta_1}(\mathfrak{A}_n)$ and $P^\natural_{\eta_n}(\mathfrak{A}_n) = P^\natural_\Omega(\mathfrak{A}_n)$, and therefore

$$V_n = J^{(n)}_\Omega J^{(n)}_{\Omega, \eta_1},$$

where $J^{(n)}_\Omega$ and $J^{(n)}_{\Omega,\eta_1}$ are the modular conjugation and the relative modular conjugation of $\mathfrak{A}_n$, $n \in \mathbb{N} \cup \{\infty\}$. By Proposition 2.7 we have $J^{(n)}_\Omega \to J^{(\infty)}_\Omega$ and $J^{(n)}_{\Omega,\eta_1} \to J^{(\infty)}_{\Omega,\eta_1}$ in the strong operator topology, and hence

$$\eta_n = J^{(n)}_\Omega J^{(n)}_{\Omega,\eta_1} \eta_1 \to J^{(\infty)}_\Omega J^{(\infty)}_{\Omega,\eta_1} \eta_1 = \eta_\infty = \eta .$$

(ii) In this case η_n and η induce the same state on $\mathfrak{A}_n$, thus one defines a unitary $V_n \in \mathfrak{A}_n'$ by

$$V_n A\eta = A\eta_n, \qquad A \in \mathfrak{A}_n .$$

One finds

$$V_n = J^{(n)}_\Omega J^{(n)}_{\Omega,\eta}$$

and proceeds as in the first case. q.e.d.

2.8. Theorem. *Let $(\mathfrak{A}_n, \mathfrak{B}_n, \Omega)$ and $(\mathfrak{A}, \mathfrak{B}, \Omega)$ be standard split W^*-inclusions acting on $\mathscr{H}$ such that $\mathfrak{A}_n \nearrow \mathfrak{A}$ and $\mathfrak{B}_n \searrow \mathfrak{B}$ (or $\mathfrak{A}_n \searrow \mathfrak{A}$ and $\mathfrak{B}_n \nearrow \mathfrak{B}$). Then the canonical product vectors converge in norm and the universal localizing maps converge pointwise strongly.*

Proof. Due to the split property $\mathfrak{A}_n \vee \mathfrak{B}'_n \nearrow \mathfrak{A} \vee \mathfrak{B}'$ (respectively $\mathfrak{A}_n \vee \mathfrak{B}'_n \searrow \mathfrak{A} \vee \mathfrak{B}'$). Thus the first assertion follows immediately from Proposition 2.7 applied to the canonical product state. The convergence of the universal localizing maps follows then as in Theorem 2.1. q.e.d.

The last theorem tells us that the natural interpolating type I factor $\mathscr{N}_n = \psi_n(\mathscr{B}(\mathscr{H}))$, ψ_n denoting the universal localizing map associated to $(\mathfrak{A}_n, \mathfrak{B}_n, \Omega)$, converges in a precise sense to $\mathscr{N} = \psi(\mathscr{B}(\mathscr{H}))$. A similar continuity property holds for semistandard pseudonormal W^*-inclusions (as defined in [5]).

In Quantum Field Theory, Theorem 2.8 applies immediately to yield the continuity of the local generator $J_u^{\mathcal{O},\hat{\mathcal{O}}}$ provided the correspondence $\mathcal{O} \to \mathscr{F}(\mathcal{O})$ has certain continuity properties. Let us write $\mathcal{O}_n \searrow \mathcal{O}$ (respectively $\mathcal{O}_n \nearrow \mathcal{O}$) when $\mathcal{O}_n$ is a nonincreasing (respectively nondecreasing) sequence of double cones with the double cone $\mathcal{O}$ as intersection (respectively as closure of the union). We call the net $\mathcal{O} \to \mathscr{F}(\mathcal{O})$ continuous if

$$\mathscr{F}(\mathcal{O}_n) \searrow \mathscr{F}(\mathcal{O}) \quad \text{if} \quad \mathcal{O}_n \searrow \mathcal{O}, \tag{2.7}$$

$$\mathscr{F}(\mathcal{O}_n) \nearrow \mathscr{F}(\mathcal{O}) \quad \text{if} \quad \mathcal{O}_n \nearrow \mathcal{O}. \tag{2.8}$$

Note that (2.7) would follow from twisted duality [12, 1].

Now let $\mathcal{O} \to \mathscr{F}(\mathcal{O})$ be continuous and let $\mathcal{O}_n \nearrow \mathcal{O}$, $\hat{\mathcal{O}}_n \searrow \hat{\mathcal{O}}$ or $\mathcal{O}_n \searrow \mathcal{O}$, $\hat{\mathcal{O}}_n \nearrow \hat{\mathcal{O}}$. Then

$$\psi_{\mathcal{O}_n, \hat{\mathcal{O}}_n} \to \psi \tag{2.9}$$

pointwise strongly, and the local generators $J_u^{\mathcal{O}_n, \hat{\mathcal{O}}_n}$ converge as above to $J_u^{\mathcal{O},\hat{\mathcal{O}}}$.

3. The Rate of Clustering and the Buchholz-Wichmann Partition Function

Buchholz and Wichmann [7] have shown that the split property of local algebras can be derived from a so-called nuclearity condition. This condition expresses in a certain sense the expectation of old quantum mechanics that locally there are only finitely many states with finite energy. It is stronger than the similar Haag-Swieca compactness criterion [19] and is thought to imply the existence of temperature states for all positive temperatures. It is based on the set of operators

$$T_{\beta,\mathcal{O}}: \mathscr{F}(\mathcal{O}) \to \mathscr{H}, \qquad F \to e^{-\beta H}(F - \omega_0(F))\Omega, \qquad \beta > 0,$$

where H is the Hamiltonian[2]. $T_{\beta,\mathcal{O}}$ can be represented as a pointwise converging sum of rank 1 operators

$$t_{\varphi,\Phi}: \mathscr{F}(\mathcal{O}) \to \mathscr{H}, \qquad t_{\varphi,\Phi}(F) = \varphi(F)\Phi, \tag{3.1}$$

where φ is a bounded linear functional on $\mathscr{F}(\mathcal{O})$ and $\Phi \in \mathscr{H}$. The Buchholz-Wichmann partition function $Z_{\mathrm{BW}}(\beta, \mathcal{O})$ is now defined by

$$Z_{\mathrm{BW}}(\beta, \mathcal{O}) = 1 + \mathscr{N}(\beta, \mathcal{O}), \tag{3.2}$$

[2] Ω is the ground state vector of H, unique up to a phase, and ω_0 is the state induced by Ω

where $\mathcal{N}(\beta, \mathcal{O})$ is the nuclearity index of the set $T_{\beta,\mathcal{O}}(\mathcal{F}(\mathcal{O})_1)$, $\mathcal{F}(\mathcal{O})_1$ denoting the unit ball of $\mathcal{F}(\mathcal{O})$,

$$\mathcal{N}(\beta, \mathcal{O}) = \inf \left\{ \sum_i \|t_i\|,\ t_i : \mathcal{F}(\mathcal{O}) \to \mathcal{H} \text{ rank 1 for all } i \text{ and } T_{\beta,\mathcal{O}} = \sum_i t_i \right\}. \quad (3.3)$$

We assume that Z_{BW} satisfies the bound

$$Z_{\mathrm{BW}}(\beta, \mathcal{O}) \leqq e^{v(\mathcal{O})\beta p(\beta)}, \quad (3.4)$$

where in analogy to thermodynamics $v(\mathcal{O}) > 0$ may be interpreted as the (spatial) volume of the system and $p(\beta)$ as the pressure, β denoting the inverse temperature.

Buchholz and Wichmann have shown that for the free massive scalar field one gets

$$v(\mathcal{O}_r) \leqq cr^3, \qquad r \geqq \frac{1}{m}$$

and

$$p(\beta) \leqq \beta^{-4} e^{-m\beta}. \quad (3.5)$$

One may therefore assume in (3.4)

$$p(\beta) \leqq \beta^{-(n+1)} e^{-m\beta} \quad (3.6)$$

for some $n > 0$. According to Buchholz and Wichmann the bound (3.6) on the pressure implies the "distal split property," i.e. for each $\mathcal{O}$ there is an $\hat{\mathcal{O}} \supset \mathcal{O}$ such that $(\mathcal{F}(\mathcal{O}), \mathcal{F}(\mathcal{O}), \Omega)$ is a standard split W^*-inclusion [7].

The proof of this fact relies on an estimate of the norm distance between the product state $\varphi_{A\mathcal{O},\hat{\mathcal{O}}}$ and the vacuum. Thus we can use the methods of [7] for our problem.

Let $\mathcal{O} \subset \mathcal{O}_d$ be double cones with $\mathcal{O} + te \subset \mathcal{O}_d$ for $|t| < d$, where e denotes the unit vector in time direction. Let $A_i \in \mathcal{F}(\mathcal{O})$, $B'_i \in \mathcal{F}(\mathcal{O}_d)'$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, such that $\|\sum A_i B'_i\| \leqq 1$, and consider the function

$$f(z) = \begin{cases} \sum_i \{(\Omega, A_i e^{izH} B'_i \Omega) - (\Omega, A_i \Omega)(\Omega, B'_i \Omega)\}, & \operatorname{Im} z > 0, \\ \sum_i \{(\Omega, B'_i e^{-izH} A_i \Omega) - (\Omega, A_i \Omega)(\Omega, B'_i \Omega)\}, & \operatorname{Im} z < 0 \quad \text{or} \quad |\operatorname{Re} z| < d. \end{cases} \quad (3.7)$$

Due to the spectrum condition, f is analytic in the cut plane

$$\mathbb{C}_d = \{\operatorname{Im} z \neq 0 \text{ or } |\operatorname{Re} z| < d\}. \quad (3.8)$$

Moreover, f is bounded by $\sum \|A_i\|\ \|B'_i\|$ and satisfies according to (3.2) the estimate

$$|f(z)| \leqq \mathcal{N}(|\operatorname{Im} z|), \quad (3.9)$$

where the r.h.s. is independent of A_i, B'_i.

Buchholz and Wichmann have already shown that this implies that $|f(0)|$ is bounded by a constant $c(\mathcal{N}, d)$ with $c(\mathcal{N}, d) \to 0$ for $d \to \infty$; thus

$$\|\varphi_{\mathcal{O},\mathcal{O}_d} - \omega_0\| = \sup_f |f(0)| \leqq c(\mathcal{N}, d) \quad (3.10)$$

tends to zero for $d \to \infty$.

We want to give a more explicit estimate. For this purpose we use the following lemma whose proof is deferred to the Appendix (Corollary B.2).

3.1. Lemma. *Let f be analytic and bounded in the cut plane $\mathbb{C}_d$, and assume that f fulfills for* $\operatorname{Im} z \neq 0$ *the inequality*

$$|f(z)| \leqq N(|\operatorname{Im} z|), \tag{3.11}$$

where N is a positive, unbounded, twice differentiable, monotonically decreasing and strictly logarithmically convex function on $\mathbb{R}_+$. Then

$$|f(0)| \leqq N(\beta_0) e^{-\beta_0 (\log N)'(\beta_0)}, \tag{3.12}$$

where β_0 satisfies the condition

$$d \geqq \frac{2}{\pi} \int_0^{\beta_0} d\beta \left\{ \log 2 - \beta \frac{(\log N)''(\beta)}{(\log N)'(\beta)} \right\}. \tag{3.13}$$

3.2. Theorem. *If the nuclearity condition* (3.4), (3.6) *is fulfilled, there are constants $c_1, c_2 > 0$ such that for $d > c_1/m$,*

$$\|\varphi_{\mathcal{O}, \mathcal{O}_d} - \omega_0\| \leqq \exp\{c_2 v(\mathcal{O}) m^n e^{-\sqrt{\pi m d}}\} - 1. \tag{3.14}$$

Proof. For all $\alpha \in \mathbb{R}$ the function f in (3.7) satisfies the bound

$$|1 + e^{i\alpha} f(z)| \leqq Z_{\mathrm{BW}}(|\operatorname{Im} z|, \mathcal{O}), \tag{3.15}$$

$N(\beta) = Z_{\mathrm{BW}}(\beta, \mathcal{O}) = \exp\{v(\mathcal{O}) \beta^{-n} e^{-m\beta}\}$ fulfills the conditions of Lemma 3.1, thus

$$1 + |f(0)| = \sup_{\alpha \in \mathbb{R}} |1 + e^{i\alpha} f(z)| \leqq N(\beta_0) e^{-\beta_0 (\log N)'(\beta_0)}. \tag{3.16}$$

To determine β_0 we evaluate (3.13). We obtain

$$d \geqq \frac{2}{\pi} \beta_0 (\log 2 + n) + \frac{m}{\pi} \beta_0^2 + \frac{2}{\pi} \frac{n}{m} \log\left(1 + \frac{m}{n} \beta_0\right). \tag{3.17}$$

Thus $\beta_0 \leqq \frac{1}{m}(\sqrt{\pi m d} - c')$, where $c' > 0$ depends only on n. Inserting this upper bound for β_0 into (3.16) we obtain the estimate

$$|f(0)| \leqq \exp\{c_2 v(\mathcal{O}) m^n e^{-\sqrt{\pi m d}}\} - 1 \tag{3.18}$$

for $d > c_1/m$, where $c_1, c_2 > 0$ depend only on n. The statement on the norm difference $\|\varphi_{\mathcal{O}, \mathcal{O}_d} - \omega_0\|$ follows now as in (3.10). q.e.d.

The function $v(\mathcal{O}_r)$ is expected to be bounded by the volume in general [7], i.e. for large r

$$v(\mathcal{O}_r) \leqq \mathrm{const}\, r^3. \tag{3.19}$$

In theories fulfilling (3.19), by the discussion in the previous section, Theorem 3.2 yields the convergence of the localizing maps $\psi_{\mathcal{O}_r, \mathcal{O}_{r+d}}$ and of the associated local charges or currents provided that (r is bounded below and)

$$d/(\log r)^2 \to \infty; \tag{3.20}$$

when $r \to \infty$ it suffices that $md \geqq \frac{c}{\pi}(\log mr)^2$ for a constant $c > n^2$, where

$$v(\mathcal{O}_r) \leqq \mathrm{const}\, r^n, \quad r \geqq 1/m.$$

As discussed in the next section, explicit bounds on $\|\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}-\omega_0\|$ in the case of a free massive field [which could be extended to generalized free fields with a sequence of masses compatible with assumptions (3.4)–(3.6)] yield a condition weaker than (3.20) [cf. (4.14)]. It is an open problem whether by a different method condition (3.20) can be improved in a general theory.

4. The Rate of Convergence in the Free Massive Case; Models with Distal Splitting

For the free massive scalar field the estimates of Buchholz and Wichmann of the nuclearity index [7] together with the results of the preceding section lead to an estimate of the rate of convergence of local charge operators. These estimates can be improved by a different method.

According to Sect. 2, it is sufficient to estimate the norm distance $\|\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}-\omega_0\|$. In the present case such an estimate can be obtained in the following way. Using the method of [17] one finds two vectors $\Omega\otimes\Omega$ and ξ in the duplicated theory inducing the states ω_0 and $\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}$ respectively. The norm difference of the states can then be estimated in terms of the norm difference of these vectors.

4.1. Theorem. *In the vacuum Hilbert space of the free massive scalar field there is a vector Φ inducing the product state $\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}$ such that*

$$\|\Phi-\Omega\|^2=2|1-(\Phi,\Omega)|\leqq \mathrm{const}\left(m\left(r+\frac{d}{2}\right)^6\right)(md)^2e^{-md}. \tag{4.1}$$

Proof. We choose Φ as that vector representative of $\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}$ having the smallest distance from Ω,

$$\|\Phi-\Omega\|^2=\inf\{\|\Phi'-\Omega\|^2,\ \omega_{\Phi'}=\varphi_{\mathcal{O}_r,\mathcal{O}_{t+d}}\}\equiv d(\omega_0,\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}), \tag{4.2}$$

where d denotes the Bures distance [20, 12][3]. Clearly $(\Phi,\Omega)\geqq 0$, thus

$$\|\Phi-\Omega\|^2=2|1-(\Phi,\Omega)|. \tag{4.3}$$

To estimate the Bures distance we construct a vector representative of $\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}$ in the vacuum Hilbert space of the duplicated theory $\mathcal{O}\to\mathcal{F}(\mathcal{O})\otimes\mathcal{F}(\mathcal{O})$. The duplicated theory is the theory of two hermitian fields $\phi_1=\phi\otimes 1$ and $\phi_2=1\otimes\phi$, where ϕ denotes the free field in the original theory. There is a global gauge symmetry $\phi_1+i\phi_2\to e^{i\alpha}(\phi_1+i\phi_2)$ with a corresponding conserved current

$$j_\mu(x)=\phi_1(x)\partial_\mu\phi_2(x)-\phi_2(x)\partial_\mu\phi_1(x). \tag{4.4}$$

For a suitable test function f,

$$e^{ij_0(f)}\phi_1(x)e^{-ij_0(f)}=\begin{cases}\phi_2(x), & x\in\mathcal{O}_r,\\ \phi_1(x), & x\in\mathcal{O}'_{r+d}.\end{cases} \tag{4.5}$$

[3] The Bures distance of states is equivalent to the distance given by taking vector representatives in the natural cone [13, 18]

Hence $\xi_f = e^{ij_0(f)}\Omega\otimes\Omega$ induces the product state $\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}$,

$$(\xi_f, (A\otimes 1)\xi_f) = \varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}(A), \qquad A\in\mathscr{F}(\mathcal{O}_r)\vee\mathscr{F}(\mathcal{O}_{r+d})', \tag{4.6}$$

and $\Omega\otimes\Omega$ induces the vacuum state. We have

$$\|\xi_f - (\Omega\otimes\Omega)\| = \|(e^{ij_0(f)}-1)(\Omega\otimes\Omega)\| \leqq \|j_0(f)(\Omega\otimes\Omega)\|, \tag{4.7}$$

where the last inequality comes from the functional calculus since $|e^{it}-1|\leqq|t|$. A standard computation yields

$$\|j_0(f)(\Omega\otimes\Omega)\|^2 = \int_{4m^2}^{\infty} d\varrho(\kappa^2)\int\frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2+\kappa^2}}|\tilde{f}(\sqrt{\mathbf{p}^2+\kappa^2},\mathbf{p})|^2|\mathbf{p}|^2, \tag{4.8}$$

where $d\varrho(\kappa^2) = \text{const}\left(1-\frac{4m^2}{\kappa^2}\right)^{1/2} d\kappa^2$.

It remains to find a suitable function f. Adopting an idea of Requardt [6] we use a variable time smearing and define

$$f(x^0,\mathbf{x}) = \tfrac{\pi}{2}\chi(\mathbf{x})\frac{2}{d}g\left(\frac{2x^0}{d}\right), \tag{4.9}$$

where χ is the characteristic function of the 3-ball of radius $r+\frac{d}{2}$ and $g\in\mathscr{D}(\mathbb{R})$ with $\int g=1$ and $\operatorname{supp} g\subset(-1,1)$. f has the desired properties (cf. [17]). Using the estimates

$$c \equiv \int_{4m^2}^{\infty} d\varrho(\kappa^2)\int\frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2+\kappa^2}}|\mathbf{p}|^2(|\mathbf{p}|^2+\kappa^2)^{-3}<\infty \tag{4.10}$$

and

$$|\tilde{\chi}(\mathbf{p})| \leqq \frac{4}{3}\pi\left(r+\frac{d}{2}\right)^3, \tag{4.11}$$

we find

$$\|j_0(f)(\Omega\otimes\Omega)\|^2 \leqq c\left(\frac{4}{3}\pi\right)^2\left(r+\frac{d}{2}\right)^6 \sup_{\omega\geqq md}|\omega^3\tilde{g}(\omega)|^2\left(\frac{2}{d}\right)^{-6}. \tag{4.12}$$

Proceeding as in the proof of exponential clustering by Haag and Swieca [19] we now take the infimum over all admissible g on the right-hand side. In the Appendix (Lemma C.1) we show for $a>0$

$$\inf_{\substack{g\in\mathscr{D}(-1,1)\\ \int g=1}} \sup_{\omega\geqq a}|\omega^3\tilde{g}(\omega)| \leqq \text{const}\, a^{3+\frac{1}{2}}e^{-\frac{a}{2}}. \tag{4.13}$$

Inserting (4.13) into (4.12) and (4.12) into (4.7) gives the desired upper bound on the Bures distance. q.e.d.

4.2. Corollary. *For the free massive scalar neutral field one has the estimate*

$$\|\omega_0-\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}\| \leqq \text{const}\left(m\left(r+\frac{d}{2}\right)\right)^3 mde^{-md/2}.$$

Proof. If the vectors Φ and Ω induce the states $\varphi_{\mathcal{O}_r, \mathcal{O}_{r+d}}$ and ω_0, respectively, one has the bound

$$\|\omega_0 - \varphi_{\mathcal{O}_r, \mathcal{O}_{r+d}}\|^2 \leqq 4(1 - |(\phi, \Omega)|^2).$$

The statement follows now from Theorem 4.1. q.e.d.

Therefore we have the desired convergence of local currents whenever (r is bounded below and)

$$d/\log r \to \infty; \tag{4.14}$$

when $r \to \infty$ is suffices that $md \geqq 2c \log mr$ for a constant c larger than the space time dimension.

The estimate obtained in Theorem 4.1 is good enough to construct examples of theories with a nonzero splitting distance, i.e. there is a "splitting distance" $d(r) > 0$ such that the inclusion $\mathcal{F}(\mathcal{O}_r) \subset \mathcal{F}(\mathcal{O}_{r+d})$ is split for $d > d(r)$ and is not split for $d < d(r)$.

The idea is to consider generalized free scalar fields ϕ_μ with Källen-Lehmann measure

$$d\mu(\kappa^2) = \sum_{n=1}^{\infty} C_n \delta(\kappa^2 - m_n^2) d\kappa^2 \tag{4.15}$$

such that $\sum C_n e^{am_n} < \infty$ for some $a > 0$. In [5] it has been shown that the algebras $\mathcal{F}(\mathcal{O})$ generated by ϕ_μ are isomorphic to infinite tensor products of the algebras $\mathcal{F}_{m_n}(\mathcal{O})$ of the free field with mass m_n with respect to the product state $\omega_0 = \omega_0^{(m_1)} \times \ldots \times \omega_0^{(m_n)} \times \ldots$, where $\omega_0^{(m)}$ denotes the vacuum state for the free field of mass m. These theories have (for a suitable sequence of masses m_n) a maximal temperature $T_{\max}$ [9]. Here we show that (under essentially the same conditions) they also have a nonzero splitting distance $d \sim T_{\max}^{-1}$.

4.3. Theorem. *Given any $d_0 > 0$ there is a generalized free field with splitting distance $d(r)$ such that*

$$d_0 \leqq d(r) \leqq 2d_0$$

for all $r > 0$.

Proof. According to Theorem 4.1 there is for each $m > 0$ a vector $\Phi^{(m)}$ in the Hilbert space $\mathcal{H}^{(m)}$ of the free field with mass m which induces the product state $\varphi^{(m)}_{\mathcal{O}_r, \mathcal{O}_{r+d}}$ and satisfies the bound

$$|1 - (\Omega^{(m)}, \Phi^{(m)})| \leqq k(r, d) m^8 e^{-md} \tag{4.16}$$

for some constant $K(r, d)$. Let $m_n = \dfrac{1}{2d_0} \log(n+1)$. Then for $d > 2d_0$

$$\sum_{n=1}^{\infty} |1 - (\Omega^{(m)}, \Phi^{(m)})| \leqq k(r, d) \left(\frac{1}{2d_0}\right)^8 \sum_{n=1}^{\infty} (\log(n+1))^8 (n+1)^{-\frac{d}{2d_0}} < \infty, \tag{4.17}$$

thus $\bigotimes \Phi^{(m_n)}$ is an element of the incomplete tensor product $(\bigotimes \mathcal{H}^{(m_n)}, \bigotimes \Omega^{(m_n)})$. Since it induces a product state the inclusion $\mathcal{F}(\mathcal{O}_r) \subset \mathcal{F}(\mathcal{O}_{r+d})$ is split, hence $d(r) \leqq 2d_0$ for all r (cf. [5]). Below we shall establish the lower bound

$$\|\omega_0^{(m)} - \varphi^{(m)}_{\mathcal{O}_r, \mathcal{O}_{r+d}}\| \geqq C(r, d) m^{1/2} e^{-md} \tag{4.18}$$

for $mr>1$ and some constant $C(r,d)>0$. Thus for $d<d_0$,

$$\sum_{n=1}^{\infty} \|\omega_0^{(m_n)}-\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}^{(m_n)}\|^2 \geqq \sum_{\substack{n=1\\ m_n r>1}}^{\infty} C(r,d)^2 \frac{1}{2d} \log(n+1)(n+1)^{-\frac{d}{d_0}} = \infty\,, \tag{4.19}$$

hence from [5] the inclusion $\mathscr{F}(\mathcal{O}_r)\subset\mathscr{F}(\mathcal{O}_{r+d})$ is not split for any r, i.e. $d(r)\geqq d_0$.

It remains to prove the bound (4.18). Let f, g be real test functions with $\operatorname{supp} f\subset\mathcal{O}_r$ and $\operatorname{supp} g\subset\mathcal{O}'_{r+d}$. The Weyl operator $e^{i\phi(f+g)}$ is a unitary in $\mathscr{F}(\mathcal{O}_r)\vee\mathscr{F}(\mathcal{O}_{r+d})'$ hence

$$\begin{aligned}\|\omega_0^{(m)}-\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}^{(m)}\| &\geqq |(\omega_0^{(m)}-\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}})(e^{i\phi(f+g)})|\\ &= e^{-\frac{1}{2}(\|f\|^2+\|g\|^2)}|e^{-(f,g)}-1|\end{aligned} \tag{4.20}$$

with $(f,g)=(\phi(f)\Omega,\phi(g)\Omega)$ and $\|f\|^2=(f,f)$. Note that (f,g) is real by locality. Replacing f and g in (4.20) by λf and λg, respectively, with $\|f\|=\|g\|=1$ and $\lambda>0$ and maximizing with respect to $e^{-\lambda^2}$ we get

$$\|\omega_0^{(m)}-\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}^{(m)}\| \geqq b(1+b)^{-\left(\frac{1}{b}+1\right)} \geqq \tfrac{1}{4}b \tag{4.21}$$

with $b=|(f,g)|$, where the second inequality holds since $0\leqq b\leqq 1$.

A bound which though not optimal is sufficient for our purpose can be obtained by choosing f, g as appropriate translates of the same function. Let χ be the characteristic function of $B_\varepsilon=\{\mathbf{x}\in\mathbb{R}^3\,|\,|\mathbf{x}|\leqq\varepsilon\}$, $0<\varepsilon<r$ and $h(x)=\chi(\mathbf{x})\delta(x^0)$. Then choose $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ such that $|\mathbf{a}|+\varepsilon<r$, $|\mathbf{b}|-\varepsilon>r+d$ and $|\mathbf{a}-\mathbf{b}|\leqq d+3\varepsilon$. Then inserting $f=h_{\mathbf{a}}\|h\|^{-1}$, $g=h_{\mathbf{b}}\|h\|^{-1}$ into (4.21) gives

$$\|\omega_0^{(m)}-\varphi_{\mathcal{O}_r,\mathcal{O}_{r+d}}^{(m)}\| \geqq \frac{(h_{\mathbf{a}},h_{\mathbf{b}})}{4\|h\|^2}\,. \tag{4.22}$$

One has

$$(h_{\mathbf{a}},h_{\mathbf{b}}) = \int_{\substack{|\mathbf{x}|\leqq\varepsilon\\ |\mathbf{y}|\leqq\varepsilon}} d^3\mathbf{x}\,d^3\mathbf{y}\,\Delta_+(0,\mathbf{b}-\mathbf{a}+\mathbf{y}-\mathbf{x},m^2) \geqq (\tfrac{4}{3}\pi\varepsilon^3)^2 \inf_{|\mathbf{c}|\leqq d+5\varepsilon} \Delta_+(0,\mathbf{c},m^2) \tag{4.23}$$

with

$$\Delta_+(x,m^2)=(2\pi)^{-3}\int \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2+m^2}}\, e^{i\sqrt{\mathbf{p}^2+m^2}x^0-i\mathbf{p}\cdot\mathbf{x}}$$

and

$$\|h\|^2=\int\frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2+m^2}}|\tilde{\chi}(\mathbf{p})|^2 \leqq \frac{1}{2m}\,\frac{4}{3}\pi\varepsilon^3\,. \tag{4.24}$$

Using the lower bound (Lemma D.1)

$$\Delta_+(0,\mathbf{c},m^2) \geqq (2\pi)^{-3/2}(m|\mathbf{c}|)^{-3/2}\frac{m^2}{2}e^{-m|\mathbf{c}|}, \tag{4.25}$$

we find (4.18) by setting $\varepsilon=\dfrac{1}{m}$ which is admissible if $rm>1$. q.e.d.

Appendix

A. Monotone Continuity of the Polar Decomposition

A.1. Lemma. *Let A be a closed densely defined linear (or antilinear) operator on a Hilbert space $\mathscr{H}$ with domain $D(A)$. Let $D_n \subset D(A)$ be an increasing sequence of dense vector spaces such that $D = \bigcup_{n=1}^{\infty} D_n$ is a core for A. Denote by A_n the closure of $A \upharpoonright D_n$ and put*

$$A = VH, \qquad A_n = V_n H_n \tag{A.1}$$

for the polar decomposition. Then

$$V_n \to V \quad \textit{strongly} \tag{A.2}$$

and

$$H_n \to H \quad \textit{in the strong resolvent sense}. \tag{A.3}$$

Proof. We follow [21] (see also [22]). Let $\mathscr{H}_A$ be the Hilbert space $D(A)$ with the graph norm

$$\|x\|_A^2 = \|x\|^2 + \|Ax\|^2, \qquad x \in D(A). \tag{A.4}$$

Let Γ denote the identification map from $\mathscr{H}_A$ onto $D(A)$ as a subset of $\mathscr{H}$. Clearly $\|\Gamma\| \leqq 1$. For $x, y \in D(H^2)$ we have

$$\begin{aligned}(\Gamma^{-1}x, \Gamma^{-1}y)_A &= (x, y) + (Ax, Ay) = ((1 + A^*A)x, y)\\ &= ((1 + H^2)x, y) = ((1 + H^2)^{1/2}x, (1 + H^2)^{1/2}y),\end{aligned} \tag{A.5}$$

and since $D(H^2)$ is a core for H, i.e. $\Gamma^{-1}D(H^2)$ is dense in $\mathscr{H}_A$, $W = \Gamma^{-1}(1 + H^2)^{-1/2}$ is a unitary operator from $\mathscr{H}$ onto $\mathscr{H}_A$; analogously $W_n = \Gamma^{-1}(1 + H_n^2)^{-1/2}$ is an isometry from $\mathscr{H}$ onto the closure of $\Gamma^{-1}D_n$ in $\mathscr{H}_A$. Since D is a core for A, the projections $W_n W_n^*$ converge strongly to 1 in $\mathscr{H}_A$. By the formulas (A.5) we also have $\Gamma\Gamma^* = (1 + H^2)^{-1}$, $\Gamma W_n W_n^* \Gamma^* = (1 + H_n^2)^{-1}$, hence H_n converges to H in the strong resolvent sense.

It follows that for $y \in D((\Gamma\Gamma^*)^{-1}) = D(H^2)$, and $x \in \mathscr{H}$

$$(\Gamma^{-1}y, (W_n - W)x) = ((\Gamma\Gamma^*)^{-1}y, [(1 + H_n^2)^{-1/2} - (1 + H^2)^{-1/2}]x) \to 0. \tag{A.6}$$

As $\Gamma^{-1}D(H^2)$ is dense in $\mathscr{H}_A$, this proves that the isometries W_n converge weakly (hence strongly) to the isometry W. Furthermore, $\|A_n\Gamma\| \leqq 1$ and $A_n\Gamma \to A\Gamma$ strongly, hence

$$A_n \Gamma W_n \to A\Gamma W \quad \text{strongly}. \tag{A.7}$$

But $A_n\Gamma W_n = V_n H_n (1 + H_n^2)^{-1/2}$, $A\Gamma W = VH(1 + H^2)^{-1/2}$, thus

$$\begin{aligned}(V_n - V)H(1 + H^2)^{-1/2} &= A_n\Gamma W_n - A\Gamma W + V_n(H(1 + H^2)^{-1/2} - H_n(1 + H_n^2)^{-1/2})\\ &\to 0 \quad \text{strongly},\end{aligned} \tag{A.8}$$

where we used (A.7), the strong resolvent convergence $H_n \to H$ and the uniform boundedness of V_n. We conclude that $V_n \to V$ strongly on $(\ker H)^\perp = (\ker V)^\perp$ hence everywhere, since $\ker V_n \subset \ker V$. q.e.d.

B. An Inequality of the Phragmen-Lindelöf Type

In this appendix we prove Lemma 3.1.

First we derive a class of estimates which are related to Jensen's inequality by suitable conformal mappings.

B.1. Proposition. *Let f be an analytic bounded function in the cut plane* $\mathbb{C}_d = \{z \in \mathbb{C}, |\mathrm{Re}\, z| < d$ or $\mathrm{Im}\, z \neq 0\}$, *and assume that f fulfills for* $\mathrm{Im}\, z \neq 0$, *the inequality*

$$|f(z)| \leqq N(|\mathrm{Im}\, z|), \tag{B.1}$$

where N is a positive continuous function on $\mathbb{R}_+$. *Then one has the following estimate*

$$|f(0)| \leqq \exp\left\{\frac{2}{\pi} \int_0^\infty \frac{ds}{\cosh s} \log N(h(s))\right\}, \tag{B.2}$$

where h is an arbitrary positive continuous function on $\mathbb{R}_+$ *with*

$$\frac{2}{\pi} \int_0^\infty ds h(s) \leqq d. \tag{B.3}$$

Proof. Let $f \not\equiv 0$ (otherwise the statement above is trivially satisfied). Let g be analytic in the unit disc $\{z \in \mathbb{C} \,|\, |z| < 1\}$ with $g(0) = 0$ and $g(z) \in \mathbb{C}_d$. Then $f \circ g$ is analytic in the unit disc, and according to Jensen's inequality one has for $0 < r < 1$,

$$|f(0)| = |f \circ g(0)| \leqq \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} d\theta \log |f \circ g(re^{i\theta})|\right\}. \tag{B.4}$$

To perform the limit $r \to 1$ in (B.4) we consider the sequence of nonnegative functions

$$\varphi_n(\theta) = -\log \left| f \circ g\left(\frac{n}{n+1} e^{i\theta}\right)\right| + c \tag{B.5}$$

with $c = \log \sup\limits_{z \in \mathbb{C}_d} |f(z)|$. Fatou's lemma yields

$$\liminf_n \int_0^{2\pi} d\theta \varphi_n(\theta) \geqq \int_0^{2\pi} d\theta \liminf_n \varphi_n(\theta), \tag{B.6}$$

hence

$$\limsup_n \int_0^{2\pi} d\theta \log \left| f \circ g\left(\frac{n}{n+1} e^{i\theta}\right)\right| \leqq \int_0^{2\pi} d\theta \limsup_n \log \left| f \circ g\left(\frac{n}{n+1} e^{i\theta}\right)\right|. \tag{B.7}$$

According to the assumptions one has the estimate

$$\limsup_n \log \left| f \circ g\left(\frac{n}{n+1} e^{i\theta}\right)\right| \leqq \limsup_n \log N\left(\left|\mathrm{Im}\, g\left(\frac{n}{n+1} e^{i\theta}\right)\right|\right) \tag{B.8}$$

[we set $N(0) = \infty$]. If $\mathrm{Im}\, g(re^{i\theta})$ converges almost everywhere for r tending to 1 to $h_1(\theta) \neq 0$, the limit on the right-hand side of (B.8) is equal to

$$\log N(|h_1(\theta)|) \tag{B.9}$$

almost everywhere. Thus we find under these conditions the estimate

$$|f(0)| \leqq \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} d\theta \log N(|h_1(\theta)|)\right\}. \tag{B.10}$$

Now we want to use the function h of the theorem for the construction of an analytic function g with the desired properties. Let $h \in L^1(\mathbb{R}_+)$ be a positive and continuous function with $\frac{2}{\pi}\|h\|_1 \leqq d$. We set

$$h_1(\theta) = ah(-\log\tan\theta/2), \qquad 0<\theta<\tfrac{\pi}{2}, \tag{B.11}$$

where $a>0$ will be fixed later on, and extend h_1 to a periodic function of θ with period 2π by

$$\begin{aligned} &\text{(i)} && h_1(\theta) = -h_1(-\theta), \\ &\text{(ii)} && h_1(\pi-\theta) = h_1(\theta), \\ &\text{(iii)} && h_1(0) = 0. \end{aligned} \tag{B.12}$$

We then define g by the formula ($|z|<1$)

$$g(z) = \frac{i}{2\pi}\int_0^{2\pi} d\theta h_1(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}. \tag{B.13}$$

g is the unique analytic function in the unit disc with $g(0)=0$ and $\lim_{r\to 1}\operatorname{Im} g(re^{i\theta}) = h_1(\theta)$ for $\theta \neq 0, \pi$. Moreover, $g(z) \in \mathbb{C}_d$ for a suitable choice of a. In fact, for $\operatorname{Im} z \neq 0$ one has

$$\begin{aligned} \operatorname{Im} g(z) &= \frac{1}{4\pi}\int_0^{2\pi} d\theta h_1(\theta)\left\{\frac{e^{i\theta}+z}{e^{i\theta}-z} + \frac{e^{-i\theta}+\bar{z}}{e^{-i\theta}-\bar{z}}\right\} \\ &= \frac{1}{2\pi}\int_0^{\pi} d\theta h_1(\theta)(1-|z|^2) \\ &\quad \times \{[1+||z|^2 - 2|z|\cos(\arg z - \theta)]^{-1} - [1+|z|^2-2|z|\cos(\arg z+\theta)]^{-1}\} \end{aligned} \tag{B.14}$$

from the antisymmetry of h_1, thus from the positivity of h_1 in the interval $(0,\pi)$

$$\operatorname{Im} g(z) \lesseqgtr 0 \quad \text{for} \quad \begin{cases} \pi < \arg z < 2\pi, \\ 0 < \arg z < \pi. \end{cases} \tag{B.15}$$

If on the other hand $\operatorname{Im} z = 0$, one finds from (B.12),

$$\begin{aligned} |\operatorname{Re} g(z)| &= \frac{1}{4\pi}\left|\int_0^{2\pi} d\theta h_1(\theta)\left\{\frac{e^{i\theta}+z}{e^{i\theta}-z} - \frac{e^{-i\theta}+\bar{z}}{e^{-i\theta}-\bar{z}}\right\}\right| \\ &= \frac{2}{\pi}\int_0^{\pi/2} d\theta h_1(\theta)\sin\theta\left\{\frac{|z|}{1+z^2-2z\cos\theta} + \frac{|z|}{1+z^2+2z\cos\theta}\right\}. \end{aligned} \tag{B.16}$$

The expression in the curly brackets is always smaller than $(\sin\theta)^{-2}$ for $z \in \mathbb{R}$, $|z|<1$. Thus by the substitution $\theta = 2\arctan e^{-s}$, $ds = \frac{d\theta}{\sin\theta}$ one obtains

$$|\operatorname{Re} g(z)| < \frac{2a}{\pi}\int_0^{\infty} ds h(s) \leqq ad \tag{B.17}$$

and for $a=1$, g has all properties required before.

Finally we make the substitution $\theta = 2\arctan e^{-s}$ also in (B.10) and arrive at the theorem. q.e.d.

To apply Proposition B.1 one has to find appropriate functions h. If for example $N(\beta) = e^{-m\beta}$, $m > 0$, the infimum of the right-hand side of (B.2) is obtained by choosing a δ-sequence of positive functions k_n, $\int k_n = 1$, and setting $h_n = \frac{\pi}{2} dk_n$. Then

$$\lim_{n\to\infty} \frac{2}{\pi} \int_0^\infty \frac{ds}{\cosh s}(-m\tfrac{\pi}{2}dk_n(s)) = -md,$$

thus $|f(0)| \leqq e^{-md}$, which is the bound obtained in [26] by a direct application of Jensen's inequality.

The main interest on Proposition B.1 is its applicability to unbounded functions N. This will be treated in the following

B.2. Corollary (Lemma 3.1). *Let f and N be as in Proposition B.1 and assume that N is unbounded, twice differentiable, monotonically decreasing and strictly logarithmically convex. Then a function h with the desired properties for which the right-hand side of* (B.2) *is finite exists if and only if* $\log|(\log N)'|$ *is locally integrable. In this case the best bound for $|f(0)|$ is obtained by choosing*

$$h(s) = ((\log N)')^{-1}(-\lambda \cosh s), \tag{B.18}$$

where $\lambda > 0$ is determined by the condition

$$\frac{2}{\pi}\int_0^\infty ds h(s) = d. \tag{B.19}$$

A more explicit (but not optimal) bound is

$$|f(0)| \leqq N(\beta_0) e^{\beta_0 |(\log N)'(\beta_0)|}, \tag{B.20}$$

where $\beta_0 > 0$ satisfies the condition

$$\frac{2}{\pi}\int_0^{\beta_0} d\beta[\log 2 - \beta(\log N)''(\beta)/(\log N)'(\beta)] \leqq d. \tag{B.21}$$

Proof. A minimum of the functional

$$F(h) = \frac{2}{\pi}\int_0^\infty \frac{ds}{\cosh s}(\log N)(h(s)) \tag{B.22}$$

with the constraint (B.19) must satisfy the equation

$$\frac{(\log N)'(h(s))}{\cosh s} + \lambda = 0, \tag{B.23}$$

hence

$$h(s) = ((\log N)')^{-1}(-\lambda \cosh s) \equiv h_\lambda(s), \tag{B.24}$$

where the Lagrange multiplier $\lambda > 0$ has to be determined by the constraint.

Now let $\lambda > 0$ be arbitrary and assume that there is a positive continuous function k with $\int dsk(s) < \infty$ and $F(k) < \infty$. We consider the convex combinations

$$k_\mu(s) = \mu k(s) + (1-\mu)h_\lambda(s), \quad 0 \leqq \mu \leqq 1. \tag{B.25}$$

We have for $h_\lambda(s) \neq k(s)$,

$$\frac{d}{d\mu}\left[\frac{(\log N)(k_\mu(s))}{\cosh s} + \lambda k_\mu(s)\right]$$
$$= [(\log N)'(h_\lambda(s) + \mu(k - h_\lambda)(s)) - (\log N)'(h_\lambda(s))](k - h_\lambda)(s)(\cosh s)^{-1} > 0 \tag{B.26}$$

due to the strict logarithmic convexity of N, hence

$$F(h_\lambda) + \lambda \frac{2}{\pi} \int_0^\infty ds h_\lambda(s) < F(k) + \lambda \frac{2}{\pi} \int_0^\infty ds k(s) \tag{B.27}$$

if $h_\lambda \neq k$. Thus provided some admissible function k exists for which F is finite, $\int h_\lambda$ and $F(h_\lambda)$ are finite for all $\lambda > 0$. Moreover, $\lambda \to \int h_\lambda$ is continuous and strictly monotonically decreasing, and $\int h_\lambda \to 0$ for $\lambda \to \infty$ and $\int h_\lambda \to \infty$ for $\lambda \to 0$, hence there is a unique $\lambda \equiv \lambda(d)$ with $\frac{2}{\pi} \int h_\lambda = d$. Inequality (B.27) then implies that $h_{\lambda(d)}$ is the unique solution of the minimum problem.

We now want to investigate under which conditions the functional F is finite for some admissible k. According to the preceding discussion this amounts to check whether h_λ is integrable and $F(h_\lambda)$ is finite for some (hence all) $\lambda > 0$.

Actually $F(h_\lambda)$ is always finite. This may be seen as follows. We substitute $s = \cosh^{-1}\left(\frac{1}{\cos\alpha}\right)$, thus $\frac{ds}{\cosh s} = d\alpha$ and

$$F(h_\lambda) = \frac{2}{\pi} \int_0^{\pi/2} d\alpha (\log N)(\beta(\alpha)), \tag{B.28}$$

where $\beta(\alpha) = ((\log N)')^{-1}\left(-\frac{\lambda}{\cos\alpha}\right)$. Then $\alpha = \alpha(\beta) = \cos^{-1}\left(-\frac{\lambda}{(\log N)'(\beta))}\right)$ and $(\beta_0 = \beta(0))$

$$F(h_\lambda) = -\frac{2}{\pi} \int_0^{\beta_0} d\beta (\log N)(\beta) \frac{d\alpha}{d\beta}. \tag{B.29}$$

By partial integration of the right-hand side of (B.29) we obtain

$$F(h_\lambda) = \lim_{\beta_1 \to 0} \frac{2}{\pi}\left[(\log N)(\beta_1)\alpha(\beta_1) - (\log N)(\beta_0)\alpha(\beta_0) + \int_{\beta_1}^{\beta_0} d\beta (\log N)'(\beta)\alpha(\beta)\right]. \tag{B.30}$$

Using $\alpha(\beta_0) = 0$, $(\log N)'(\beta_0) < 0$ for all $\beta < 0$ and the inequality

$$\tfrac{\pi}{2}(1 - x) \leqq \cos^{-1} x \leqq \tfrac{\pi}{2}, \tag{B.31}$$

which holds for $0 \leqq x \leqq 1$, we find the desired bound

$$F(h_\lambda) \leqq (\log N)(\beta_0) - \beta_0 (\log N)'(\beta_0) \tag{B.32}$$

with $\beta_0 = ((\log N)')^{-1}(-\lambda)$. Equation (B.20) now follows by inserting (B.32) into (B.20) and applying Proposition B.1.

In contrast to $F(h_\lambda)$, $\int h_\lambda$ is not always finite. By the substitution $\beta=\beta(s) =((\log N)')^{-1}(-\lambda\cosh s)$, we find $(\beta_0=\beta(0))$,

$$\int_0^\infty ds h_\lambda(s) = -\int_0^{\beta_0} d\beta\beta \frac{ds}{d\beta} = -\beta_0 s(\beta_0) + \lim_{\beta_1\to 0}\left[\beta_1 s(\beta_1) + \int_{\beta_1}^{\beta_0} d\beta s(\beta)\right], \quad \text{(B.33)}$$

where $s(\beta)=\cosh^{-1}(-(\log N)'(\beta)/\lambda)$ and $s(\beta_0)=0$. Using the inequalities

$$\log x \leqq \cosh^{-1} x \leqq \log 2x, \quad \text{(B.34)}$$

which hold for $x\geqq 1$, we conclude that $\int_0^{\beta_0} d\beta s(\beta)<\infty$ if and only if $\log|(\log N)'(\beta)|$ is locally integrable. Since s is monotonically decreasing, we have in this case

$$0\leqq \beta_1 s(\beta_1) \leqq \int_0^{\beta_1} d\beta s(\beta)\to 0, \quad \beta_1\to 0. \quad \text{(B.35)}$$

Hence

$$\int_0^{\beta_0} d\beta \log(-(\log N)'(\beta)/\lambda) \leqq \int_0^\infty ds h_\lambda(s) \leqq \int_0^{\beta_0} d\beta \log(-2(\log N)'(\beta)/\lambda), \quad \text{(B.36)}$$

thus h_λ is integrable if and only if $\log|(\log N)'|$ is locally integrable. Equation (B.21) follows now from the right inequality in (B.36) by partial integration and by $\lambda=-(\log N)'(\beta_0)$. q.e.d.

C. Inequalities in $\mathscr{D}$-Spaces

C.1. Lemma. *Let $k\in\mathbb{Z}_+$ and $m\geqq\max(2k,2)$. Then*

$$\inf \sup_{\omega\geqq m} |\omega^k \tilde{g}(\omega)| \leqq m^{k+\frac{1}{2}} e^{-\frac{m}{2}} \frac{\sqrt{2}}{4} e^{1/4},$$

where the infimum is taken over all $g\in\mathscr{D}(-1,1)$ with $\int g=1$.

Proof. We have for all $n\in\mathbb{N}$, $n\geqq k$

$$\sup_{\omega\geqq m} |\omega^k\tilde{g}(\omega)| \leqq m^{k-n} \sup|\omega^n \tilde{g}(\omega)| \leqq m^{k-n} \frac{1}{\sqrt{2\pi}} \int dt |g^{(n)}(t)|. \quad \text{(C.1)}$$

In Lemma C.2 below we show

$$\inf_g \int dt |g^{(n)}(t)| = \tfrac{1}{2} 2^n n!. \quad \text{(C.2)}$$

Inserting this formula with $n=\left[\frac{m}{2}\right]$, [ ] denoting the integer part of a real number, we find for $m\geqq\max(2k,2)$

$$\inf_g \sup_{\omega\geqq m} |\omega^k\tilde{g}(\omega)| \leqq m^k \frac{1}{\sqrt{2\pi}} \frac{1}{2} n^{-n} n! \left(\frac{m}{2n}\right)^{-n}. \quad \text{(C.3)}$$

Using Stirling's formula

$$n! \leqq n^{n+1/2} e^{-n} \sqrt{2\pi} e^{1/4n}, \quad \text{(C.4)}$$

and the estimate

$$\left(1+\frac{\varepsilon}{n}\right)^{-n} e^{\varepsilon} \leqq \left(1+\frac{\varepsilon}{n}\right)^{\frac{\varepsilon}{2}}$$

for $\varepsilon = \frac{m}{2} - n < 1$, we obtain

$$\inf_{g} \sup_{\omega \geqq m} |\omega^k \tilde{g}(\omega)| \leqq m^{k+\frac{1}{2}} e^{-\frac{m}{2}} \frac{\sqrt{2}}{4} e^{1/4}. \quad \text{q.e.d.} \tag{C.5}$$

C.2. Lemma. $\inf_{\substack{g \in \mathscr{D}(-1,1) \\ \int g = 1}} \|g^{(n)}\|_1 = \frac{1}{2} 2^n n!,\ n \in \mathbb{N}.$

Proof. First we show that for all $g \in \mathscr{D}(-1,1)$ and $n \in \mathbb{N}$,

$$\|g^{(n)}\|_1 \geqq 2^{n-1} n! \int g. \tag{C.6}$$

By multiple partial integration

$$\int g(x) dx = \frac{(-1)^n}{n!} \int dx x^n g^{(n)}(x), \tag{C.7}$$

and for $0 \leqq k < n$, $k \in \mathbb{Z}$,

$$\int dx x^k g^{(n)}(x) = 0. \tag{C.8}$$

Thus

$$|\int g| \leqq \frac{1}{n!} \inf_{p \in P_n} \max_{x \in [-1,1]} |(px)| \int |g^{(n)}|, \tag{C.9}$$

where P_n is the set of normalized polynomials of degree n. The normalized polynomial of degree n with the smallest maximum modulus in the interval $[-1,1]$ is $2^{1-n} T_n$, where $T_n(x) = \cos(n \arccos x)$ is the n-th Tschebyscheff polynomial [23]. Thus (C.6) follows from $\max|T_n| = 1$.

We now want to show that the bound (C.6) is optimal. Equation (C.9) becomes an equality if $g^{(n)}$ has support only at the extrema of T_n, i.e. if it is of the form

$$g^{(n)}(x) = \sum_{k=0}^{n} \lambda_k \delta(x - x_k), \qquad x_k = \cos\frac{k\pi}{n} \tag{C.10}$$

with $\lambda_k \in \mathbb{C}$, $k = 0, \ldots, n$. The coefficients λ_k are fixed by the condition that $g^{(n)}$ is the n-th derivative of a function with compact support with integral 1. From (C.8) and (C.9) this is equivalent to the system of linear equations

$$\sum_{k=0}^{n} \lambda_k T_m(x_k) = \begin{cases} 0, & m = 0, \quad n-1, \\ 2^{n-1} n!, & m = n \end{cases} \tag{C.11}$$

which has the unique solution $\lambda_0 = C$, $\lambda_k = (-1)^k 2C$, $k = 1, \ldots, n-1$, $\lambda_n = (-1)^n C$, $C = 2^{n-1} n!/(2n)$.

The function g so found does not belong to $\mathscr{D}(-1,1)$, but can be approximated by functions in $\mathscr{D}(-1,1)$. Let $0 < \varepsilon < 1$ and $\varphi \in \mathscr{D}(-\varepsilon, \varepsilon)$ with $\int \varphi = 1$ and $\varphi \geqq 0$. Set $g_\varepsilon(x) = \frac{1}{1-\varepsilon} g\left(\frac{x}{1-\varepsilon}\right)$. The convolution $f_\varepsilon = g_\varepsilon * \varphi$ belongs to $\mathscr{D}(-1,1)$ and has integral 1. The n-th derivative is $f_\varepsilon^{(n)} = g_\varepsilon^{(n)} * \varphi = \left(\frac{1}{1-\varepsilon}\right)^n \sum_{k=0}^{n} \lambda_k \varphi(x - (1-\varepsilon)^2 x_k)$, hence

$$\inf_{0<\varepsilon<1} \|f_\varepsilon^{(n)}\|_1 = \sum_{k=0}^{n} |\lambda_k| = 2^{n-1} n! \quad \text{q.e.d.} \tag{C.12}$$

D. A Lower Bound on $\Delta_+(0, \mathbf{a}; m)$

For the paper to be selfcontained, we add here a proof of the lower bound on the 2-point function of the free scalar massive field which has been used in Sect. 4.

D.1. Lemma. *Let* $\Delta_+(0, \mathbf{a}; m) = (2\pi)^{-3} \int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2 + m^2}} e^{i\mathbf{k}\cdot\mathbf{a}}$, $\mathbf{a} \neq 0$. *Then*

$$\Delta_+(0, \mathbf{a}; m) \geqq (2\pi)^{-3/2} \frac{m^2}{2} (m|\mathbf{a}|)^{-3/2} e^{-m|\mathbf{a}|}. \tag{D.1}$$

Proof. Let $S(x)$ denote the euclidean 2-point function (Schwinger function),

$$S(x) = (2\pi)^{-4} \int \frac{d^4k}{k^2 + m^2} e^{ikx}, \qquad k = (k_0, \mathbf{k}), \qquad k^2 = k_0^2 + \mathbf{k}^2. \tag{D.2}$$

We have $\Delta_+(0, \mathbf{a}; m) = S(0, \mathbf{a})$, and since S is rotation invariant, $S(0, \mathbf{a}) = S(|\mathbf{a}|, 0)$, thus after integration over k_0, we find

$$\Delta_+(0, \mathbf{a}; m) = (2\pi)^{-3} \int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2 + m^2}} e^{-\sqrt{\mathbf{k}^2 + m^2}|\mathbf{a}|} \tag{D.3}$$

(cf. e.g. [24]). Now by integration over the angles and by the substitution $\omega = \sqrt{\mathbf{k}^2 + m^2}$, $d\omega = \frac{|\mathbf{k}|}{\omega} d|\mathbf{k}|$, we obtain

$$\Delta_+(0, \mathbf{a}; m) = (2\pi)^{-2} \int_m^\infty d\omega \sqrt{\omega^2 - m^2} e^{-\omega|\mathbf{a}|} \equiv m^2 (m|\mathbf{a}|)^{-1} K_1(m|\mathbf{a}|), \tag{D.4}$$

where K_1 denotes the modified Bessel function of the third kind of order 1 [25]. We now set $\omega = m\left(1 + \frac{x^2}{2}\right)$, $d\omega = mx\,dx$, $\sqrt{\omega^2 - m^2} = mx\sqrt{1 + \frac{x^2}{4}}$, and using $\sqrt{1 + \frac{x^2}{4}} \geqq 1$, we find

$$\begin{aligned} \Delta_+(0, \mathbf{a}; m) &\geqq (2\pi)^{-2} m^2 e^{-m|\mathbf{a}|} \int_0^\infty dx\, x^2 e^{-m|\mathbf{a}|x^2/2} \\ &= (2\pi)^{-3/2} \frac{m^2}{2} (m|\mathbf{a}|)^{-3/2} e^{-m|\mathbf{a}|}. \quad \text{q.e.d.} \end{aligned} \tag{D.5}$$

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