ON SELBERG'S ZETA FUNCTION FOR COMPACT RIEMANN SURFACES

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Received 31 December 1986

We present several novel relations for Selberg's zeta function for compact Riemann surfaces. The results find direct application in calculating determinants entering multiloop amplitudes in string theory, and in determining the energy spectrum of chaotic motion on a surface of constant negative curvature.

In recent years the geometry of the Laplace operator on compact Riemann surfaces with two or more handles [1-3] and the corresponding Selberg zeta function [4] play an important role in a variety of physical applications. We mention only two examples:

(i) The first example is the string theory path integral [5,6] for the closed bosonic Polyakov string in the critical dimension $d=26$. It turns out that the loop expansion for on-shell amplitudes is an expansion in the number of handles of the two-dimensional surface, the world-sheet of the string [7]. Since the Polyakov action is quadratic in the space–time coordinates, the path integration can be carried out and one is left with integrals over Teichmüller (or moduli) space. The integrals of the multiloop amplitudes contain the determinant of the laplacian acting on functions and vectors on compact Riemann surfaces. Recently it has been realized [8] that the relevant determinants can be expressed in terms of the Selberg zeta function. In the mathematical literature similar results have been discussed [9,10].

(ii) In his work on quantum chaos, Gutzwiller [11] has studied the spectrum of enery levels in a beautiful chaotic hamiltonian system, the free motion (quantum billiard) on a surface of constant negative curvature. Applying his trace formula [12] (derived from a semiclassical approximation to the Feynman path integral) he was able to express the trace of the Green's function by a sum over the classical periodic orbits on the given Riemann surface with genus $g \geqslant 2$. This "closed orbit sum" is nothing but a special case of Selberg's trace formula [4] as applied to compact Riemann surfaces, and the final result can again be formulated in terms of the Selberg zeta function.

It is obvious from the above examples that a detailed knowledge of the Selberg zeta function is most desirable. In this note we shall present several new relations for the function in question. Detailed proofs and applications will be given elsewhere.

Let M be a compact Riemann surface of genus $g \geqslant 2$. M can be identified with H/Γ, the action of a fuchsian group Γ on the upper half-plane $H = \{z = x + iy \colon y > 0\}$ endowed with the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. This is the classical model for hyperbolic geometry of constant negative curvature, $R = -1$. Γ is a discrete subgroup of $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$, the group of Möbius transformations. From the Gauss–Bonnet theorem we infer $RA = 2\pi\chi = 4\pi(1-g)$, where A denotes the area of M and χ its Euler characteristic, i.e. $A = 4\pi(g-1)$. In the Poincaré metric the laplacian on M (Laplace–Beltrami operator) is given by $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, and we are interested in the eigenvalue problem $-\Delta u = \lambda u$. The spectrum of Δ on M is discrete and real [‡1], $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ with # (eigenvalues λ_n with $\lambda_n \leqslant \lambda$) $\sim (A/4\pi)\lambda$ asymptotically (Weyl's law) [13–15].

Since the elements $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ are hyperbolic, i.e. $|\mathrm{Tr}\,\gamma| = |a+d| > 2$, they are conjugate with Γ to a Möb-

[‡1] For quantum billiards on Riemann surfaces of constant negative curvature R, the energy spectrum is given by $E_n = (\hbar^2/2mR^2)\lambda_n$.

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(North-Holland Physics Publishing Division)

ius transformation of the form $z \to N(\gamma)z$, $1 < N(\gamma) < \infty$, where $N(\gamma)$ is called the norm of γ. (γ_1, $\gamma_2 \in \Gamma$ are conjugate within Γ if there exists a $\gamma_3 \in \Gamma$ such that $\gamma_1 = \gamma_3 \gamma_2 \gamma_3^{-1}$.) The class of all elements in Γ which are conjugate to a given γ is called the conjugacy class of γ in Γ and is denoted by $\{\gamma\}$. The number $N(\gamma)$ is, of course, the same within a conjugacy class and measure the "magnification". $N(\gamma)$ has, however, another striking geometrical interpretation, since there exists a unique relationship between the conjugacy classes in Γ and the homotopy classes of closed paths on the surface M. In each class one defines a length $l(\gamma)$ by the length of the shortest closed path measured by means of the Poincaré distance. One then obtains $N(\gamma) = \exp[l(\gamma)]$, $l(\gamma) > 0$. Thus the conjugacy classes in Γ can be uniquely parametrized by their length spectrum $\{l(\gamma)\}$. Given any $\gamma \in \Gamma$ there is a unique γ_0 such that $\gamma = \gamma_0^n$, $n \in \mathbb{N}$; γ_0 is called primitive element of Γ, since it cannot be expressed as a power of any other element of Γ. The corresponding closed orbit with length $l(\gamma_0)$ is called a prime geodesic on M. Obviously $l(\gamma) = l(\gamma_0^n) = nl(\gamma_0)$, since in this case the prime geodesic is traversed n times. For the length spectrum of M one has Huber's law [16] $\nu(x) \sim e^x/x$ for $x \to \infty$, where $\nu(x)$ is the number of inconjugate primitive γ's with $l(\gamma) \leqslant x$.

Our starting is the *Selberg trace formula* [4] which can be considered as a generalization and non-commutative analogue of the classical Poisson summation formula

$$\sum_{n=0}^{\infty} h(r_n) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}r\, r \tanh \pi r\, h(r) + \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} \frac{l(\gamma)}{\sinh \frac{1}{2} nl(\gamma)} g(nl(\gamma)) . \qquad (1)$$

Here all series and the integral converge absolutely under the following conditions on the function $h(r)$: (i) $h(-r) = h(r)$, (ii) $h(r)$ is holomorphic in a strip $|\mathrm{Im}\, r| \leqslant \frac{1}{2} + \epsilon$, $\epsilon > 0$, (iii) $|h(r)| \leqslant a(1 + |r|^2)^{-1-\epsilon}$, $a > 0$. The function $g(u)$ is the Fourier transform of $h(r)$

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}r \exp(-\mathrm{i}ur) h(r) . \qquad (2)$$

On the left-hand side of (1) the sum runs over the eigenvalues of Δ parametrized in the form $\lambda_n = \frac{1}{4} + r_n^2$, i.e. over the pairs $(r_n, -r_n)$, $r_n \in \mathbb{C}$; $r = 0$ has to be counted twice if $\frac{1}{4}$ happens to be an eigenvalue. On the right-hand side the sum is taken over all primitive conjugacy classes in Γ, denoted by $\{\gamma\}_p$. The trace formula (1) establishes a relation between the eigenvalues of Δ on M and the lengths of the closed geodesics on M, i.e. a very striking duality between the quantum mechanical energy spectrum and the lengths of the classical closed periodic orbits.

In order to calculate the trace of the resolvent of Δ, $\mathrm{Tr}(-\Delta + z)^{-1}$, one is lead to substitute $h(r) = (r^2 + z)^{-1}$ in the trace formula. This function violates, however, the growth condition for $|r| \to \infty$. The reason is that the resolvent operator is not of trace class, since the eigenvalues behave as $\lambda_n \sim (4\pi/A)n$ for $n \to \infty$ as follows from Weyl's law. Thus the resolvent has to be regularized properly. A very convenient regularization is given by the following choice ($\mathrm{Re}\, s$, $\mathrm{Re}\, \sigma > 1$):

$$h(r) = \frac{1}{r^2 + (s - \frac{1}{2})^2} - \frac{1}{r^2 + (\sigma - \frac{1}{2})^2} , \qquad (3)$$

which fulfills all the conditions in the trace formula. Here σ plays the role of a regulator. With (3) we find from (1) the "sum rule" (see also refs. [14,15])

$$\sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n + s(s-1)} - \frac{1}{\lambda_n + \sigma(\sigma - 1)} \right)$$

$$= -2(g-1)[\psi(s) - \psi(\sigma)] + \frac{1}{2s-1} \frac{Z'(s)}{Z(s)} - \frac{1}{2\sigma - 1} \frac{Z'(\sigma)}{Z(\sigma)} , \qquad (4)$$

where $\psi(s) \equiv \Gamma'(s)/\Gamma(s)$ is the digamma function and $Z(s)$ denotes the *Selberg zeta function on M* defined by

$$Z(s) \equiv \prod_{\{\gamma\}_p} \prod_{n=0}^{\infty} \{1-\exp[-(s+n)l(\gamma)]\} . \tag{5}$$

Huber's law assures the convergence of (5) if Re $s>1$. Notice that on the left-hand side of (4) the term in brackets cannot be broken up, otherwise convergence is lost. From (5) follows $Z(s)$, $Z'(s)>0$ for s real and >1, since the "neck" of M, the length of the shortest path, is larger than zero, $|\mathrm{Tr}\,\gamma| = 2\cosh \frac{1}{2}l(\gamma) > 2$ for all $\gamma \in \Gamma$.

To get rid of the σ terms in (4), we make use of the following observation. Isolating the zero mode on the left-hand side of (4), we can take the limit $\sigma \to 1+$, since the following limit exists:

$$\lim_{\sigma\to 1+}\left(\frac{1}{2\sigma-1}\frac{Z'(\sigma)}{Z(\sigma)}-\frac{1}{\sigma(\sigma-1)}\right)=\frac{1}{2}\frac{Z''(1)}{Z'(1)}-1\equiv B . \tag{6}$$

We thus obtain our *fundamental sum rule for the logarithmic derivative of Z(s)* [‡2]

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)}=B+\frac{1}{s(s-1)}+1(g-1)[\psi(s)-\psi(1)]+\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n+s(s-1)}-\frac{1}{\lambda_n}\right) . \tag{7}$$

[$\psi(1)=-\gamma$, γ = Euler's constant] [‡3]. It is obvious that the sum rule (7) extends meromorphically to all $s \in \mathbb{C}$. In fact, the Selberg zeta function is an entire function of s of order 2 whose "trivial" zeros can easily be read of from (7): $s=1$ is a simple zero, $s=0$ is a zero of multiplicity $2g-1$, and $s=-k$, $k \in \mathbb{N}$ are zeros with multiplicity $2(g-1)(2k+1)$. Apart from a finite number of zeros on the real line between 0 and 1 (corresponding to eigenvalues $\lambda_n \leqslant \frac{1}{4}$), the "non-trivial" zeros are located at $s=\frac{1}{2}\pm i r_n$, $r_n \in \mathbb{R}^+$, i.e. they lie on the critical line Re $s=\frac{1}{2}$.

From (7) we deduce the following *Laurent expansion near s = 1*:

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)}=\frac{1}{s-1}+(B-1)+\sum_{n=1}^{\infty}a_n(s-1)^n ,$$

$$a_n=(-1)^{n+1}\left[1+2(g-1)\zeta(n+1)+\sum_{l=0}^{[n/2]}(-1)^{l+1}\binom{n-l}{l}\zeta_\Delta(n+1-l)\right], \quad n\in\mathbb{N} . \tag{8}$$

Here $\zeta(s)$ is the Riemann zeta function, and $\zeta_\Delta(s)$ denotes the *zeta function of Minakshisundaram–Pleijel* (MP) [17] for the laplacian on M

$$\zeta_\Delta(s) \equiv \mathrm{Tr}'(-\Delta)^{-s}=\sum_{n=1}^{\infty}\frac{1}{\lambda_n^s} . \tag{9}$$

(Notice that the zero mode has been omitted. For a definition of $(-\Delta)^{-s}$ see ref. [18]). The Dirichlet series (9) converges for Re $s>1$ due to Weyl's law. By expanding $Z(s)$ on the left-hand side of (8) in a Taylor series around $s=1$, we obtain explicit relations for the derivatives of $Z(s)$ at $s=1$

$$\frac{n}{n+1}\rho_{n+1}=(B+1)\rho_n+\sum_{k=1}^{n-1}\binom{n}{k}k!(a_k+2a_{k-1})\rho_{n-k} ,$$

$$\rho_n \equiv Z^{(n)}(1)/Z'(1) . \tag{10}$$

The first two non-trivial relations are

$$\rho_3=Z'''(1)/Z'(1)=3B(B+4)+\pi^2(g-1)-3\zeta_\Delta(2) ,$$

‡2 A similar but simpler relation was obtained in ref. [13], which is, however, incorrect. See also ref. [15].

‡3 From (7) one derives immediately the well-known functional relation for $Z(s)$ [4].

$$\rho_4 = Z^{(4)}(1)/Z'(1) = 4B(B^2+9B+6)+4\pi^2(g-1)[B+\tfrac{7}{3}-(4/\pi^2)\zeta(3)]-12(B+3)\zeta_\Delta(2)+8\zeta_\Delta(3)\,. \tag{11}$$

To obtain a meromorphic continuation of the zeta function (9) it is convenient to introduce the *trace of a generalized resolvent of* Δ *on* M defined by

$$k_\Delta(s;\sigma) \equiv \mathrm{Tr}(-\Delta+\sigma(\sigma-1))^{-s} = \sum_{n=0}^{\infty} \frac{1}{[\lambda_n+\sigma(\sigma-1)]^s}\,, \quad \mathrm{Re}\, s, \mathrm{Re}\,\sigma > 1\,. \tag{12}$$

(Notice that in this case the zero mode has been included.) The zeta function (10) can then be obtained by the following limit:

$$\zeta_\Delta(s) = \lim_{\sigma\to 1+}\left(k_\Delta(s;\sigma) - \frac{1}{[\sigma(\sigma-1)]^s}\right). \tag{13}$$

The trace (12) is given by a Mellin transform of the *trace of the heat kernel* (partition function)

$$k_\Delta(s;\sigma) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}\exp[-\sigma(\sigma-1)t]\,\theta_\Delta(t), \quad \theta_\Delta(t) \equiv \mathrm{Tr}\exp(\Delta t) = \sum_{n=0}^{\infty}\exp(-\lambda_n t)\,. \tag{14a,b}$$

The trace of the heat kernel is obtained from the Selberg trace formula (1) by substituting $h(r) = \exp[-(r^2+\frac{1}{4})t]$, which for $t>0$ fulfills the required conditions. One then finds [13]

$$\theta_\Delta(t) = \theta_\Delta^{(1)}(t) + \theta_\Delta^{(2)}(t)\,, \tag{15a}$$

$$\theta_\Delta^{(1)}(t) = \frac{A\exp(-t/4)}{(4\pi t)^{3/2}}\int_0^\infty \mathrm{d}b\, \frac{b\exp(-b^2/4t)}{\sinh\frac{1}{2}b}\,, \tag{15b}$$

$$\theta_\Delta^{(2)}(t) = \frac{\exp(-t/4)}{2(4\pi t)^{1/2}}\sum_{\{\gamma\}_p}\sum_{n=1}^{\infty}\frac{l(\gamma)\exp\{-[nl(\gamma)]^2/4t\}}{\sinh\frac{1}{2}nl(\gamma)}\,. \tag{15c}$$

Since $\theta_\Delta^{(2)}$ vanishes exponentially for $t\to 0+$, the small $-t$ behaviour of the heat kernel is completely determined by $\theta_\Delta^{(1)}$, and is explicitly given by the *asymptotic expansion*

$$\theta_\Delta(t) = \frac{g-1}{t}\sum_{n=0}^{N} b_n t^n + \mathrm{O}(t^N)\,,$$

$$b_0 = 1, \quad b_n = \frac{(-1)^n}{2^{2n}n!}\left[1+2\sum_{k=1}^{n}\binom{n}{k}(2^{2k-1}-1)\,|B_{2k}|\right], \quad n\in\mathbb{N}\,, \tag{16}$$

where B_{2k} are the Bernoulli numbers. ($b_1 = -\frac{1}{3}$, $b_2 = \frac{1}{15}$). (Weyl's law follows immediately from the leading term in (16).) From (16) and the large $-t$ behaviour of θ_Δ we can conclude that (14a) can be meromorphically continued to all $s\in\mathbb{C}$ with a simple pole at $s=1$ if $\mathrm{Re}\,\sigma>1$. We get

$$k_\Delta(s;\sigma) = \frac{g-1}{s-1}\,\pi\int_0^\infty \mathrm{d}r\,\frac{[r^2+(\sigma-\frac{1}{2})^2]^{1-s}}{\cosh^2\pi r} + \frac{I(s;\sigma)}{\Gamma(s)}\,, \tag{17a}$$

$$I(s;\sigma) \equiv \frac{(2\sigma-1)^{1/2-s}}{\sqrt{4\pi}}\sum_{\{\gamma\}_p} l(\gamma)\sum_{n=1}^{\infty}\frac{[nl(\gamma)]^{s-1/2}K_{s-1/2}(\frac{1}{2}\,nl(\gamma)(2\sigma-1))}{\sinh\frac{1}{2}\,nl(\gamma)}\,, \tag{17b}$$

where $K_\nu(z)$ is the modified Bessel function. (For $\mathrm{Re}\,s<0$, $\sigma=1$, a similar result has been derived by Randol [19]). Inserting (17a), (17b) in (13) we obtain for the MP zeta function

$$\zeta_\Delta(s)=\zeta_\Delta^{(1)}(s)+\zeta_\Delta^{(2)}(s)\,, \tag{18a}$$

$$\zeta_\Delta^{(1)}(s)=\frac{g-1}{s-1}\pi\int_0^\infty \mathrm{d}r\,\frac{(r^2+\frac{1}{4})^{1-s}}{\cosh^2\pi r}=\frac{g-1}{s-1}+\mathrm{FP}\zeta_\Delta^{(1)}(1)+\sum_{n=1}^\infty c_n(s-1)^n\,, \tag{18b}$$

$$c_n=(g-1)\frac{(-1)^{n+1}}{(n+1)!}\pi\int_0^\infty \mathrm{d}r\,\frac{[\ln(r^2+\frac{1}{4})]^{n+1}}{\cosh^2\pi r}\,,$$

$$\zeta_\Delta^{(2)}(s)=\lim_{\sigma\to 1+}\left(\frac{I(s;\sigma)}{\Gamma(s)}-\frac{1}{[\sigma(\sigma-1)]^s}\right),\quad \zeta_\Delta(0)=-\tfrac{1}{3}(g-1)\,. \tag{18c,d}$$

In (18b) FP denotes the finite part

$$\mathrm{FP}\zeta_\Delta^{(1)}(1)=2(g-1)\gamma\,. \tag{18e}$$

For $s\in\mathbb{Z}^+$ the above relations can be expressed in terms of Riemann's and Selberg's zeta function, respectively,

$$\zeta_\Delta^{(1)}(N)=2(g-1)\sum_{n=0}^{N-2}\frac{(N-2+n)!}{n!}(N-1-n)\zeta(N-n),\quad N=2,3,\ldots\,, \tag{19a}$$

$$I(N;\sigma)=(-1)^{N-1}\delta_\sigma^N\ln Z(\sigma),\quad N\in\mathbb{N}_0\,, \tag{19b}$$

$$\zeta_\Delta^{(2)}(N)=\frac{(-1)^{N-1}}{(N-1)!}\lim_{\sigma\to 1+}\delta_\sigma^N\ln[Z(\sigma)/\sigma(\sigma-1)],\quad N\in\mathbb{N}\,. \tag{19c}$$

Here δ_σ denotes the differential operator $(2\sigma-1)^{-1}\mathrm{d}/\mathrm{d}\sigma$. Let us mention that $\zeta_\Delta(s)$ can also be calculated for $s\in\mathbb{Z}^-$.) Eqs. (18e) and (19c) combined with (6) lead to the following explicit expression for the finite part of $\zeta_\Delta(s)$ at $s=1$

$$\mathrm{FP}\zeta_\Delta(1)=2(g-1)\gamma+B\equiv\gamma_\Delta\,, \tag{20}$$

where we have introduced the "*generalized Euler constant*" γ_Δ of the laplacian on M. (See in this connection ref. [20].)

If $Z(\sigma)$ in (19c) is expanded in a Taylor series around $\sigma=1$ one obtains in combination with (19a) explicit formulae for the MP zeta function at $s=2, 3, \ldots$, which involve only the Riemann zeta function and the derivatives of the Selberg zeta function at $s=1$. (These are just the inverse relations of (10).) For $s=2, 3$ they read

$$\zeta_\Delta(2)=\tfrac{1}{3}\pi^2(g-1)+B(B+4)-\tfrac{1}{3}Z'''(1)/Z'(1)\,,$$

$$\zeta_\Delta(3)=\tfrac{1}{3}\pi^2[1+(6/\pi^2)\zeta(3)](g-1)+B^2(B+6B+15)-\tfrac{1}{3}(B+3)Z'''(1)/Z'(1)+\tfrac{1}{8}Z^{(4)}(1)/Z'(1)\,. \tag{21}$$

The above relations can be used to derive bounds for the derivatives of $Z(s)$ and for the smallest non-vanishing eigenvalue λ_1. We give two examples. Since $\zeta_\Delta(s)>0$ for $s>1$ we obtain from (21)

$$Z'''(1)<[\pi^2(g-1)+3B(B+4)]Z'(1)\,. \tag{22}$$

From the "sum rule method" [21] recently introduced for the calculation of small eigenvalues we get [‡4]

$$[\zeta_\Delta(2)]^{-1/2}<\lambda_1<\zeta_\Delta(2)/\zeta_\Delta(3)\,, \tag{23}$$

[‡4] For simplicity we assume that λ_1 is nondegenerate.

which can be optimized with respect to g. A detailed discussion of these bounds will be given elsewhere.

Below we shall require the derivative of $\zeta_\Delta(s)$ at $s=0$. From (18b) one derives [‡5]

$$\frac{d}{ds}\zeta_\Delta^{(1)}(0)=2(g-1)C\,,$$

$$C\equiv-\frac{\pi}{2}\int_0^\infty dr\,\frac{(r^2+\frac{1}{4})}{\cosh^2\pi r}[1-\ln(r^2+\tfrac{1}{4})]=\tfrac{1}{4}-\tfrac{1}{2}\ln 2\pi-2\zeta'(-1)\simeq-0.338\,, \tag{24a}$$

while (18c) and (19b) yield

$$\frac{d}{ds}\zeta_\Delta^{(2)}(0)=\lim_{\sigma\to 1+}[I(0;\sigma)+\ln\sigma(\sigma-1)]=-\lim_{\sigma\to 1+}\ln[Z(\sigma)/\sigma(\sigma-1)]=-\ln Z'(1)\,. \tag{24b}$$

Thus we obtain

$$\zeta_\Delta'(0)=2(g-1)C-\ln Z'(1)\,. \tag{25}$$

At this point it is appropriate to introduce the *functional determinant of* Δ *on* M

$$D_\Delta(z)\equiv\det{}'(-\Delta+z),\quad z\in\mathbb{C}\,, \tag{26}$$

where the prime indicates that the zero mode has been omitted. In differential geometry [9] and quantum field theory [23] the following definition (zeta function regularization) has proven useful:

$$D_\Delta(0)\equiv\exp[-\zeta_\Delta'(0)]\,. \tag{27}$$

We thus obtain with (25)

$$D_\Delta(0)=Z'(1)\exp[-2(g-1)C]\,. \tag{28}$$

For $z\neq 0$ we define the determinant (16) by the following infinite product over the zeros at $z=-\lambda_n$:

$$D_\Delta(z)=D_\Delta(0)\exp(\gamma_\Delta z)\prod_{n=1}^\infty[(1+z/\lambda_n)\exp(-z/\lambda_n)]\,. \tag{29}$$

(The product converges as a consequence of Weyl's law). With (29) we get for the determinant our final result

$$D_\Delta(z)=Z'(1)\exp[\gamma_\Delta z-2(g-1)C]\prod_{n=1}^\infty[(1+z/\lambda_n)\exp(-z/\lambda_n)]\,, \tag{30}$$

which shows again the fundamental role played by the two quantities $Z'(1)$ and γ_Δ. For small z the determinant can be completely expressed in terms of the MP zeta function

$$D_\Delta(z)=\exp\left(-\sum_{n=0}^\infty(-1)^N d_N z^N\right),\quad |z|<\lambda_1\,,$$

$$d_0=\zeta_\Delta'(0),\quad d_1=\mathrm{FP}\zeta_\Delta(1),\quad d_N=\zeta_\Delta(N)/N,\quad N=2,3,\ldots\,. \tag{31}$$

From the above relations we can derive several bounds. It is easy to see that $\zeta_\Delta'(0)$ must be negative, which yields $D_\Delta(0)>1$ and the important lower bound

$$Z'(1)>\exp[2(g-1)C]\cong[0.509]^{g-1}\,, \tag{32}$$

[‡5] $\zeta'(-1)$ has been calculated from the asymptotic series [22] $-\frac{1}{6}(1-\frac{1}{120}+\frac{1}{840})=-\frac{139}{840}\cong-0.1655$. Our value for C agrees with a numerical evaluation of the integral (24a) reported in ref. [10], where $K\equiv 2C$.

while $D'_\Delta(0)>0$ yields the bounds

$$\gamma_\Delta>0,\quad B>-2(g-1)\gamma,\quad Z''(1)>-2[2(g-1)\gamma-1]Z'(1)\,. \tag{33}$$

Finally, we would like to establish a closed formula for the Selberg zeta function itself. If we multiply our sum rule (7) by $(2s-1)$ and then integrate over s from $\sigma>1$ to $s>\sigma$, we obtain

$$Z(s)=\frac{Z(\sigma)}{\sigma(\sigma-1)}s(s-1)\frac{V(s)}{V(\sigma)}\,, \tag{34}$$

where the function $V(s)$ satisfies $V(1)=1$. In the derivation of (34) all integrations are straightforward except the integral over the digamma function. The latter can be evaluated by means of Alexeiewsky's theorem [24,25]

$$\int_0^z \mathrm{d}t\,\ln\Gamma(t+1)=\tfrac{1}{2}z\ln 2\pi-\tfrac{1}{2}z(z+1)+z\ln\Gamma(z+1)-\ln G(z+1)\,, \tag{35}$$

where G denotes the Barnes G-function defined by

$$G(z+1)=(2\pi)^{z/2}\exp[-\tfrac{1}{2}z-\tfrac{1}{2}(1+\gamma)z^2]\prod_{n=1}^{\infty}[(1+z/n)^n\exp(-z+z^2/2n)]$$

$$=(2\pi)^{z/2}\exp[-\tfrac{1}{2}z-\tfrac{1}{2}(1+\gamma)z^2]\prod_{n=1}^{\infty}\left(\frac{\Gamma(n)}{\Gamma(z+n)}\exp[z\psi(n)+\tfrac{1}{2}z^2\psi'(n)]\right). \tag{36}$$

The function $G(z+1)$ is an entire function with zeros at the points $z=-n$, $n\in\mathbb{N}$, the zero at $z=-n$ being of order n. It is a natural extension of the gamma function, and possesses many properties analogous to those of the latter [25]. We only give the following ones:

$$G(z+1)=\Gamma(z)G(z),\quad G(1)=1\,,$$

$$\frac{G'(z+1)}{G(z+1)}=\tfrac{1}{2}\ln 2\pi+\tfrac{1}{2}+z[\psi(z)-1],\quad \ln\frac{G(1-z)}{G(1+z)}=\int_0^z \mathrm{d}t\,\pi t\cot\pi t-z\ln 2\pi\,. \tag{37}$$

In eq. (34) the regulator σ can now be removed by taking the llimit $\sigma\to 1+$ (see (24b)), and we arrive at our *fundamental relation for the Selberg zeta function*

$$Z(s)=Z'(1)\,s(s-1)\,V(s)\,, \tag{38a}$$

$$V(s)\equiv\exp[\gamma_\Delta s(s-1)][(2\pi)^{1-s}\exp[s(s-1)]G(s)G(s+1)]^{A/2\pi}\prod_{n=1}^{\infty}\{[1+s(s-1)/\lambda_n]\exp[-s(s-1)/\lambda_n]\}\,. \tag{38b}$$

Relation (38a) gives a factorization of the entire function $Z(s)$ in terms of canonical products formed with its zeros. Indeed, the last product in (38b) reproduces all the "non-trivial" zeros of $Z(s)$, while the G-functions (together with the factor $s(s-1)$) generate all the "trivial" zeros (see the discussion after (7)). Eqs. (38a), (38b) constitute a duality relation between the length spectrum of the closed periodic geodesics on M and the spectrum of the laplacian on M. The relation depends only on the three fundamental "constants" A, γ_Δ and $Z'(1)$.

Comparison of (38b) with our expression for the determinant allow us to rewrite (38a), (38b) as

$$Z(s)=s(s-1)D_\Delta(s(s-1))\{(2\pi)^{1-s}\exp[C+s(s-1)]G(s)G(s+1)\}^{2(g-1)}\,. \tag{39}$$

Substituting for the constant C our explicit result (24a), we arrive at an expression recently given by Voros [26]. The specific dependence of the determinant (30) on $Z'(1)$ and γ_Δ has, however, not been determined in ref. [26].

To summarize, we have presented several novel relations for the Selberg zeta function for compact Riemann surfaces. The results are useful not only for further mathematical investigations along the lines of refs. [14–19], but they are also given in a form to be used directly in applications of current physical interest [8,11].

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