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# The Representation Theory of the Symmetry Group of Lattice Fermions as a Basis for Kinematics in Lattice QCD

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**Abstract.** The symmetry group of staggered lattice fermions is discussed as a discrete subgroup of the symmetry group of the Dirac-Kähler equation. For the representation theory of this group, G. Mackey's generalization of E.P. Wigner's procedure for the construction of unitary representations of groups with normal subgroups is used. A complete classification of these irreducible representations by "momentum stars", "flavour orbits" and "reduced spins" is given.

## 1. Introduction

The most symmetric lattice approximation of Dirac fields is given by staggered fermions [1] together with their geometric interpretation by Dirac-Kähler forms [2, 3]. In this description of Dirac fields their lattice degeneracy [4] is controlled by the Susskind flavour symmetry group [5]. Furthermore, in the massless case a one parameter continuous chiral symmetry group survives the lattice approximation, which is spontaneously broken in the strong coupling approximation [7]. Finally, the symmetry group of staggered fermions (called the 'lattice fermion group' (LFG) in this paper) is in the Dirac-Kähler description a geometric restriction of the corresponding continuum symmetry group [6].

The LFG is also a symmetry group of staggered fermions with gauge interaction, i.e. of Dirac-Kähler fermions with Susskind coupling. It follows that it is a symmetry group of Green's functions calculated in lattice approximation of Euclidean QCD. Therefore it is suited for the classification of particle states, a fact which should be used more systematically in strong coupling calculations [7] and in numerical calculations [8]. For all these reasons it is worthwhile to base the kinematics of lattice fermions on a systematic representation theory of the LFG, a procedure which is very familiar from the treatment of the space-time symmetry of the Poincare group [12, 9].

The aim of this paper is the construction of all irreducible representations of the LFG [10]. However, the LFG shows many of the complexities of the crystallographic groups. In a way we may consider the LFG as an extension of a special crystallographic group of four dimensional space [11] by flavour transformations. Since this extension leads to a non-symmorphous space group, we have to use a generalization of Wigner's well known procedure [12] for the construction of the irreducible representations of groups which are semidirect products with Abelian normal subgroups. Such a procedure is due to Mackey [13]. It is well suited for our purpose.

This is the content of our paper. In Sect. 2 we give a detailed description of the LFG, its subgroups, and its relation to the symmetry group of the continuous Dirac-Kähler equation. In the following Sect. 3 we describe the complete construction of all the irreducible (unitary) representations of the LFG with the help of the methods by Wigner and Mackey. Finally in Sect. 4 we give a short outlook on applications. There is other work related to ours [14], however it is less systematic and less complete than our treatment.

## 2 The Symmetry Group of Lattice Fermions

### 2.1 Basic Concepts

It is well known that staggered fermions are the result of a systematic lattice approximation of the Dirac-Kähler equation (DKE) [2]:

$$(d-\delta+m)\,\Phi=0. \tag{1}$$

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Here $d$ denotes exterior differentiation, $\delta$ the codifferential operator, and $\Phi$ an inhomogeneous differential form

$$\Phi=\sum_{H=1,\mu_i}^{di}\frac{1}{h!}\phi_{\mu_1\dots\mu_h}(x)\,dx^{\mu_1}\wedge\dots\wedge dx^{\mu_h}\equiv\sum_H\phi(x,H)\,dx^H. \tag{2}$$

$\phi(x,H)$ are the independent complex components of $\Phi$ in a multi-index notation $H=(\mu_1,\dots\mu_h)$, $\mu_1<\mu_2<\dots<\mu_h$, $h=1,\dots,di$, $di$=dimension of space-time.

In Euclidean space $R^{di}$, '$di$' even, we may decompose $\Phi$ into $2^{di/2}$ independent Dirac fields $\phi^b(x)$, in such a way that solutions of the DKE get decomposed into solutions of the Dirac equation:

$$(\gamma^\mu\partial_\mu+m)\,\phi^b(x)=0,\qquad b=1,\dots 2^{di/2}. \tag{3}$$

The main tool for proving the equivalence between (1) and (3) is the introduction of an associative Clifford product for differential forms [15] which is defined by the relation $dx^\mu\vee dx^\nu=dx^\mu\wedge dx^\nu+\delta^{\mu\nu}$ for the 'Cartesian' basis elements $dx^\mu$. In the 'Dirac' basis $Z=(Z_a^b)$:

$$Z=2^{-d/2}\sum_H(\gamma^H)^T\mathscr{B}\,dx^H, \tag{4}$$

with

$$dx^H\equiv dx^{\mu_1}\wedge\dots\wedge dx^{\mu_h},$$

and

$$\mathscr{B}\,dx^H=(-1)^{h(h-1)/2}\,dx^H,$$

this Clifford product is represented as a matrix algebra:

$$dx^\mu\vee Z=\gamma^{\mu T}Z,\qquad Z\vee dx^\mu=Z\gamma^{\mu T},\qquad Z_a^b\vee Z_c^d=Z_a^d\,\delta_c^b. \tag{5}$$

With the help of the Clifford product, the Dirac-Kähler operator $d-\delta$ can be brought into the simple form

$$(d-\delta)\,\Phi=dx^\mu\vee\partial_\mu\Phi. \tag{6}$$

Beginning with this expression, a direct calculation using (4), (5) demonstrates that the Dirac components $\phi^b(x)=(\phi_a^b(x))$ of a solution of the DKE, $\Phi=\sum_{a,b}\phi_a^b(x)\,Z_a^b$, satisfy the Dirac equation (3).

The lattice approximation of the DKE is based on an imitation of the well known [16] mapping of DeRham complexes on simplicial complexes. Let $\Gamma$ be an infinite cubic lattice with points $x=b(n^1,\dots,n^{di})$, lattice unit vectors $e_\mu$, $h$-dimensional lattice cubes $[x,H]$ spanned at $x$ by $e_\mu$, $\mu\in H$ embedded in Euclidean space, (see Fig. 1). Then, corresponding to the differential forms $\Phi$, we get cochains on the lattice ('lattice fields'):

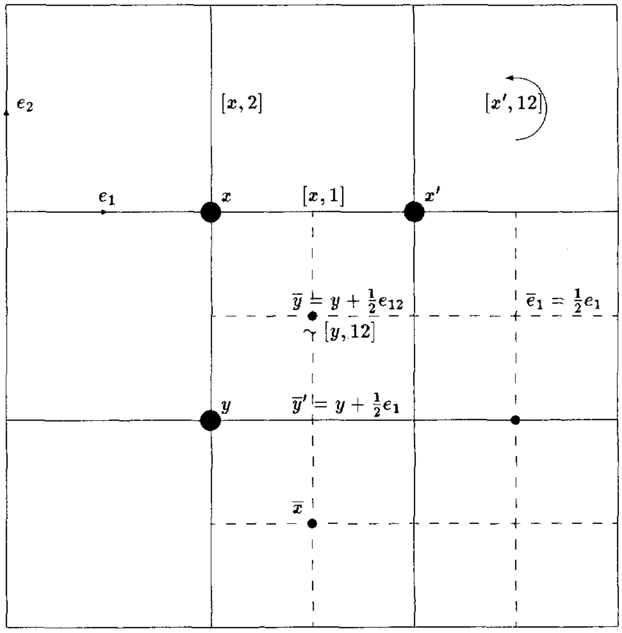


Fig. 1. Illustration of the lattice notions

$$\Phi(\mathscr{C})=\int_{\mathscr{C}}\Phi, \tag{7}$$

where $\mathscr{C}$ is a sum of lattice cubes. $\Phi(\mathscr{C})$ is a linear functional on linear combinations of cubes. It can be expressed in a basis $\{d^{x,H}\}$:

$$\Phi=\sum_{x,h}\phi(x,H)\,d^{x,H},\qquad d^{x,H}([x',H'])=\delta_{x'}^x\,\delta_{H'}^H. \tag{8}$$

This is the lattice analogue of (2). Because of Stokes' theorem, the mapping (7) implies

$$d\to\check{\Delta},\quad \textit{with}\ \check{\Delta}\Phi(\mathscr{C})=\Phi(\Delta\mathscr{C}),$$
$$\delta\to\check{\nabla},\quad \textit{with}\ \check{\nabla}\Phi(\mathscr{C})=\Phi(\nabla\mathscr{C}).$$

Here $\Delta$ and $\nabla$ are the boundary and the coboundary operator applied to lattice cells. Thus the DKE on the lattice becomes

$$(\check{\Delta}-\check{\nabla}+m)\,\Phi=0. \tag{9}$$

The lattice Dirac-Kähler field $\phi(x,H)$ becomes a staggered fermion field $\chi(y)$ if we make the identification

$$\phi(x,H)=\phi(y,H(y))=\chi(y),\qquad y=x+\tfrac{1}{2}e_H,$$
$$e_H=\sum_{\mu\in H}e_\mu. \tag{10}$$

By this mapping the $r$-cochains of the 'coarse' lattice $\Gamma$: $\phi(x, H)$ get identified with lattice fields defined at the points of the 'fine' lattice $y \in \bar{\Gamma}$ which are central points of the cells $[x, H]$ (see Fig. 1). For the proof of this proposition one shows that a Dirac field $\psi_a(y) = \gamma_{ai}^{H(y)} \phi(y, H(y))$, '$i$' arbitrarily fixed, satisfies the naive Dirac equation, iff $\phi(x, H)$ satisfies the DKE. For a more detailed discussion of the formalism and of the results we mentioned here, we refer to the literature [2].

### 2.2 The Lattice Symmetry Group

Now we approach our first problem, the determination of the symmetry group of the DKE on the lattice, and its geometric description as the lattice restriction of the symmetry group of the continuum.

The free DKE in the continuum is equivalent to simultaneous Dirac equations of four degenerate Dirac fields. Therefore infinitesimal spinorial Euclidean transformations, and $SU(4)$-'flavour' transformations

$$\begin{aligned}(M_{\mu\nu}\Phi)_a^b(x) &= (x_\mu \partial_\nu - x_\nu \partial_\mu)\,\Phi_a^b(x) \\ &\quad + \tfrac{1}{2}(\gamma_\mu \gamma_\nu)_{ac}\,\Phi_a^b(x), \qquad \mu \neq \nu, \\ (P_\mu \Phi)_a^b(x) &= \partial_\mu\, \Phi_a^b(x), \\ (T^i \Phi)_a^b(x) &= \tfrac{1}{2}\lambda_c^{ib}\,\Phi_a^c(x), \qquad i = 1, \ldots 15. \end{aligned} \tag{11}$$

generate a symmetry group $\mathcal{SE} \times SU(4)$. Here $\lambda^i$ denote the 15 Gell-Mann matrices for $SU(4)$ [17]; we use the summation convention. There are additional symmetries like space reflection:

$$(\Pi_s \Phi)_a^b(x) = \gamma_{ac}^4\, \Phi_c^b(\Pi_s x),$$
$$(\Pi_s x) = (-x^1, -x^2, -x^3, x^4),$$

charge conjugation, general phase transformations, and chiral transformations in the special case of mass $m=0$. At the moment we restrict our considerations to the symmetry group $\mathcal{G}$ generated by the spinorial Euclidean group $\mathcal{SE}'$ including space reflections, and by the $SU(4)$ flavour group. The general finite element of this group $\mathcal{G} = \{(f, a, s)\}$ is composed by a flavour transformation $(f) \equiv (f, 0, 1)$ a translation $(a) \equiv (1, a, 1)$, and an extended spin transformation $(s) \equiv (1, 0, s)$, $s = s'$ or $s'\Pi_s$, $s' \in \mathcal{SP} \equiv$ 'spin group'. On $\Phi_a^b(x)$ the transformation $(-1, 0, -1)$ acts trivially. Thus we should consider $\mathcal{SE} \times SU(4)/Z_2$ as the proper symmetry group. This implies restrictions on the representations which we regard later, (see Sect. 3.3 (2)).

The lattice restriciton $\mathcal{T}_L$ of the translation group $\mathcal{T}$ is found trivially

$$\mathcal{T} \supset \mathcal{T}_L = \{a \mid a = b(n^1, n^2, n^3, n^4),\, n^i \in Z\} = \{[a]\}.$$

The lattice constant '$b$' of the coarse lattice $\Gamma$ we set most of the time equal to 1.

In order to understand the lattice restriction of $\mathcal{SP}(4) \times SU(4)$ from a geometric point of view, we have to consider the relation between the transformations of the Dirac fields and the transformations of the differential forms. It follows from (5) that flavour transformations correspond to Clifford right multiplication:

$$(\Phi \vee c(u))_a^b = \sum_d \psi_a^d u_d^b$$
$$\text{for } c(u) = \sum_H u(H)\, dx^H = \sum_{a,b} u_a^b Z_a^b. \tag{12}$$

The spinor rotations of the Dirac components can be expressed directly as operations on the forms:

$$\delta_{\mu\nu}\Phi = (x_\mu \partial_\nu - x_\nu \partial_\mu)\,\Phi + \tfrac{1}{2} S_{\mu\nu} \vee \Phi,$$
$$S_{\mu\nu} = dx_\mu \wedge dx_\nu. \tag{13}$$

On the other hand, if the Cartesian components transform as $O(4)$ tensors, we get the transformation law

$$\delta_{\mu\nu}^G \Phi = (x_\mu \partial_\nu - x_\nu \partial_\mu)\,\Phi + \tfrac{1}{2}(S_{\mu\nu} \vee \Phi - \Phi \vee S_{\mu\nu}). \tag{14}$$

These 'geometric' rotations differ from spinor rotations by flavour transformations defined above, (12). The lattice restriction of the geometric rotations follows by geometric intuition from the correspondence (7). The cubic lattice allows only rotations belonging to the symmetry group $W_4$ of the 4-dimensional cube. $W_4$ is a group with 384 elements which is generated by rotations in the $(\mu, \nu)$-plane by $\frac{\pi}{2}$ and by a space reflection $\Pi_s$. These rotations map $r$-cubes, $r = 1, \ldots, 4$, onto $r$-cubes, and they commute with the boundary and coboundary operation. Therefore the DKE is invariant under the lattice restriction of the 'geometric Euclidean group' $\mathcal{GE} \supset \mathcal{GE}_L \simeq \mathcal{T}_L \otimes W_4$.

For a similar geometric approach to the discussion of the lattice restriction of the flavour transformations, we need, according to (12), a definition of a $\vee$-product on the lattice. Such a product was defined on the basis of the cup product and cap product of algebraic topology in [2, 18]. This distributive product is defined by the $\vee$-multiplication of the basis elements $d^{x,H}$ defined in (8):

$$d^{x,H} \vee d^{y,K} = \check{\rho}_{H,K}\, \delta^{x+e_H, y}\, d^{x+e_A, H\Delta K},$$
$$A = H \cap K, \quad H\Delta K = H \cup K - A. \tag{15}$$

The sign $\check{\rho}_{H,K}$ is the same as in the Clifford product of forms: $dx^H \vee dx^H = \check{\rho}_{H,K}\, dx^{H\Delta K}$. This product is not associative, but satisfies

$$(d^{x - e_{H \cap L}, H} \vee d^{y,K}) \vee d^{z - e_{H \cap K}, L} = d^{x,H} \vee (d^{y,K} \vee d^{z,L}). \tag{16}$$

The DK-operator can be written with help of $d^H \equiv \sum_x d^{x,H}$:

$$\breve{\Delta} - \breve{\nabla} = d^\mu \vee \partial_\mu^-, \quad (\partial_\mu^- f)(x) = f(x) - f(x - e_\mu). \tag{17}$$

It follows immediately with help of (16) that $\varepsilon d^K$, $\varepsilon = \pm 1$

$$\varepsilon d^K \Phi: \Phi \to \varepsilon \Phi \vee (d^K)^{-1},$$

is a symmetry transformation of the DKE. A direct calculation leads to $(d^K)^2 = [-e_K]$, i.e. the flavour transformations generate translations. The group $\mathscr{F}\mathscr{T}_L$ generated by the $\varepsilon d^K$ contains $\mathscr{T}_L$ as a normal subgroup. The factor group $\mathscr{F}_L = \mathscr{F}\mathscr{T}_L / \mathscr{T}_L \simeq \mathscr{K}_4$ is isomorphic to the multiplicative group $\mathscr{K}_4$ of the Dirac matrices $\{\pm\gamma^K\}$, which has order 32 $(=2^{di+1})$. However $\mathscr{F}\mathscr{T}_L$ is not a semidirect product $\mathscr{T}_L \otimes \mathscr{K}_4$, but what is called in a cristallographic language a non-symmorphous extension of the lattice translation group.

Adding up these considerations, we can formulate a proposition on the lattice restriction of the symmetry group $\mathscr{G}$ of the DKE and its action on staggered fermion fields. For this we compose a general element $(f, a, s)$ of $\mathscr{G}$ by a flavour transformation $\bar{f}$, a translation $[a]$, and a geometric rotation $R(s)$

$$g = (\bar{f}) \circ (s, a, s) = (\bar{f}s, a, s) = [\bar{f}, a, R(s)]. \tag{18}$$

In these two equivalent forms the group multiplication is:

$$\begin{aligned} &(f, a, s) \circ (f', a', s') = (ff', R(s)\, a' + a, ss'), \\ &[f, a, R(s)] \circ [f', a', R(s')] \\ &\quad = [fsf's^{-1}, R(s)\, a' + a, R(s)\, R(s')]. \end{aligned} \tag{19}$$

With this notation we have

**Proposition 2.1.** *The lattice restriction of* $\mathscr{G}$ *is*

$$\mathscr{G}_L = \{[\varepsilon d^K, -\tfrac{1}{2} e_K + a, R] \mid a \in \mathscr{T}_L, R \in W_4\}$$

*with the composition law*

$$\begin{aligned} &[\varepsilon d^K, -\tfrac{1}{2} e_K + a, R] \circ [\varepsilon' d^L, -\tfrac{1}{2} e_L + a', R'] \\ &\quad = [\varepsilon\varepsilon' \rho(R, R \circ L)\, \breve{\rho}_{K, R \circ L}\, d^{K \Delta R \circ L}, \\ &\quad -\tfrac{1}{2}(e_K + R e_L) + R a' + a, RR']. \end{aligned} \tag{20}$$

*It is a symmetry group of the free* DKE *if it acts on staggered fermion fields according to*

$$\begin{aligned} &([a]\,\chi)(y, H(y)) = \chi(y - a^\mu e_\mu, H(y)), \\ &H(y - a^\mu e_\mu) = H(y), \\ &([R]\,\chi)(y, H(y)) = \rho(R, H(y))\, \chi(R^{-1} y, H(R^{-1} y)), \\ &R \in W_4, \\ &(\varepsilon d^K \chi)(y, H(y)) = \varepsilon \breve{\rho}_{H(y), K}\, \chi(y + \tfrac{1}{2} e_K,\ H(y + \tfrac{1}{2} e_K)), \end{aligned} \tag{21}$$

*where the sign* $\rho(R, H)$ *is the same as in the transformation of the basis differentials of the continuum:* $R\, dx^H = R^{-1} \vee dx^H \vee R = \rho(R, H)\, dx^{R^{-1} \circ H}$.

### *2.3 Subgroup Structure*

The structure of the LFG described by Proposition 2.1 is rather involved. In Fig. 2 we illustrate the relations of different continuum and lattice subgroups. These are: The continuous symmetry group $\mathscr{G}$, the spinorial Euclidean group $\mathscr{S}\mathscr{E}$, the geometric Euclidean group $G\mathscr{E}$, the translation group $\mathscr{T}$, the flavour symmetry group $\mathscr{F} \simeq SU(4)$, the spin group $\mathscr{S}\mathscr{P}'(4)$ including the space reflections, the geometric rotation group $O(4)$, the lattice symmetry group LFG $\simeq \mathscr{G}_L$ as the lattice restriction of $\mathscr{G}$, the group $\mathscr{F}\mathscr{T}_L$ generated by the lattice flavour transformations, the lattice space group $G\mathscr{E}_L$ as lattice restriction of $G\mathscr{E}$, the symmetry group $W_4$ of the 4-cube. Of course, there are many more subgroups, several of which have to be used in the representation theory of $\mathscr{G}$ and $\mathscr{G}_L$. The groups in Fig. 2 are only the most important ones for the general understanding of our mathematical and physical considerations. We draw attention to two points in Fig. 2 which are responsible for the somewhat unfamiliar features of the lattice symmetry of fermions and its relation to continuum symmetry. One point is the different 'splitting' of $\mathscr{F} \times \mathscr{S}\mathscr{P}'(4)$ into $\mathscr{F}$ and $\mathscr{S}\mathscr{P}'(4)$ on the one hand, and into $\mathscr{F}$ and $O(4)$ on the other, in a way as described in (18). The other point is the fact that because of the non-symmorphous extension of $\mathscr{T}$ to $\mathscr{T}\mathscr{F}_L$, the lattice flavour group $\mathscr{K}_4 \simeq \mathscr{F}\mathscr{T}_L / \mathscr{T}_L \subset SU(4)$, and the lattice spinor group $S_L \simeq \mathscr{G}_L / \mathscr{F}\mathscr{T}_L \subset \mathscr{S}\mathscr{P}'(4)$ are not subgroups of $\mathscr{G}_L$. The subgroup structure of $\mathscr{G}_L$ is relevant for the rather involved representation theory of the LFG.

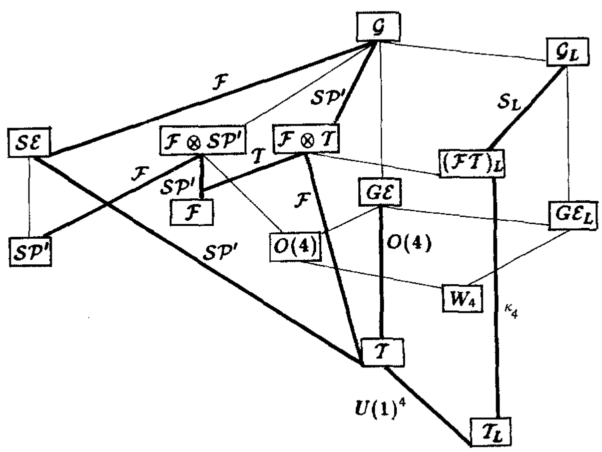


**Fig. 2.** Subgroup structure of the symmetry group of Dirac-Kähler fields. The symbols are explained in Sect. 2.3. Symbols in boxes denote subgroups. Thick lines connect groups with normal subgroups. Symbols beside thick lines denote factor groups

## 3 The Construction of the Representations of the Lattice Fermion Group

### 3.1 The Wigner-Mackey construction

The main topic of this paper is the construction of the irreducible, unitary representations ('irreps') of the LFG. Its subgroup structure, represented in Fig. 2, shows that the LFG has normal subgroups. Therefore the induction procedure may simplify our task. However, the normal subgroups are not always embedded in the LFG in the manner of semidirect products, nor are these always Abelian subgroups. Therefore the induction procedure well known to physicists, which is due to Wigner [12], is not general enough. We have to use a generalization due to Mackey [13]. Because we have to apply repeatedly this method, we shall first give a short description of this scheme, omitting all unnecessary mathematical sophistication.

Consider a group $G$ with a normal subgroup $N$ and the factor group $F$. The aim is to construct the irreps of $G$ with help of the irreps of $N$ and the irreps of some subgroups of $F$. The building blocks of the procedure are:

(1) *G-Orbits in* $\hat{N}$. Let $\{L^p\}=\hat{N}$ be a set representing all inequivalent irreps of $N$. Because $N$ is a normal subgroup of $G$, $L^p(g^{-1}ng)$ is equivalent to some $L^{p'}(n)$:

$$L^p(g^{-1}ng)\simeq L^{gp}, \quad n\in N, \quad g\in G, \quad p\in\hat{N}. \tag{22}$$

This means that $G$ acts as a transformation group of the dual $\hat{N}$ of $N$: $p\xrightarrow{g}gp$. Hence $\hat{N}$ can be decomposed in $G$-orbits: $\theta^j=\{g\bar{p}_j|g\in G\}$, where $\bar{p}_j$ denotes a reference point for each orbit. The group $\bar{S}_j$ leaving $\bar{p}_j$ invariant: $\bar{S}_j=\{s|s\bar{p}_j=\bar{p}_j,\ s\in G\}$ is called the little group of first kind. $N$ is a normal subgroup of $\bar{S}_j$. The little group of second kind is defined as the factor group $S_j=\bar{S}_j/N$. The orbit $\Theta^j$ can be described by the right cosets $G/\bar{S}_j$: if $g\bar{p}_j=g'\bar{p}_j$ then $g=g's$, $s\in\bar{S}_j$. We fix this description by choosing a definite set of 'boost transformations' $\{g(p)\}$ which represent these cosets $G/\bar{S}_j$, the points $p\in\Theta^j$:

$$g(p)\,\bar{p}_j=p, \qquad g(\bar{p}_j)=e. \tag{23}$$

For the explicit construction of the irreps of $G$ we standardize the matrices $L^p_{mm'}(n)$ by suitable equivalence transformations in such a way that the equivalence in (22) becomes an equality for $g(p)$:

$$L^p_{mm'}(n)=L^{\bar{p}_j}_{mm'}(g^{-1}(p)\,n\,g(p)). \tag{24}$$

(2) *Extension of* $L^{\bar{p}_j}(n)$ *to a projective irrep* $\bar{L}^{\bar{p}_j}(n)$ *of* $\bar{S}_j$. Let $\{s\}$ be a set of representatives of the cosets $\bar{S}_j/N$, and thus of the elements of $S_j$. Because of (22) and $s\bar{p}_j=\bar{p}_j$, there is a linear transformation $A(s)$ of the representation space $\mathscr{H}$ of $L^{\bar{p}_j}$ satisfying

$$L^{\bar{p}_j}(s^{-1}ns)=A^{-1}(s)\,L^{\bar{p}_j}(n)\,A(s). \tag{25}$$

Because of Schur's Lemma, $A(s)$ is determined up to a factor. We take the unit element $e$ as representative of $N$, and we set $A(e)=1$. Now writing an arbitrary element of $\bar{s}\in\bar{S}_j$ in the form $nsn'$, we may define (omitting the index $\bar{p}_j$):

$$\bar{L}(\bar{s})=\bar{L}(nsn')=L(n)\,A(s)\,L(n'). \tag{26}$$

Equation (26) determines a consistent mapping $\bar{s}\to\bar{L}(\bar{s})$, $\bar{s}\in\bar{S}_j$. We have to show, that if $nsn'=\tilde{n}s'\tilde{n}'$, then $\bar{L}(\tilde{n}s\tilde{n}')=\bar{L}(nsn')$. For this we make the following simple consideration. First $s=s'$ because $N$ is a normal subgroup of $\bar{S}_j$. Then from $\tilde{n}'=s^{-1}\tilde{n}^{-1}nsn'$ follows

$$\begin{aligned}\bar{L}(\tilde{n}s\tilde{n}')&=L(\tilde{n})\,A(s)\,L(\tilde{n}')=L(\tilde{n})\,A(s)\,L(s^{-1}\tilde{n}^{-1}nsn')\\&=L(\tilde{n})\,A(s)\,A(s^{-1})\,L(\tilde{n}^{-1}n)\,A(s)L(n')=\bar{L}(nsn').\end{aligned}$$

Now we shall show that $\bar{L}(\bar{s})$ is a projective representation of $\bar{S}_j$. First we remark that

$$\bar{L}^{-1}(\bar{s})\,L(n)\,\bar{L}(\bar{s})=L(\bar{s}^{-1}n\bar{s}) \tag{27}$$

follows from a simple calculation. With help of this formula we see that

$$\begin{aligned}\bar{L}^{-1}(\bar{s})\,\bar{L}^{-1}(\bar{t})\,L(n)\,\bar{L}(\bar{t})\,\bar{L}(\bar{s})&=L(\bar{s}^{-1}\bar{t}^{-1}n\bar{t}\bar{s})\\&=\bar{L}^{-1}(\bar{t}\bar{s})\,L(n)\,\bar{L}(\bar{t}\bar{s}).\end{aligned}$$

This means that $\bar{L}(\bar{t})\,\bar{L}(\bar{s})\,\bar{L}^{-1}(\bar{t}\bar{s})$ commutes with the irreducible representation $L(n)$ of $N$, and hence it follows from Schur's Lemma that

$$\bar{L}(\bar{t})\,\bar{L}(\bar{s})=\sigma(\bar{t},\bar{s})\,\bar{L}(\bar{t}\bar{s}), \tag{28}$$

where $\sigma(\bar{t},\ \bar{s})$ is a numerical factor. It follows from (26) and the normalization $A(e)=1$ that for the restriction of this projective representation of $\bar{S}_j$ to $N$ we get $\bar{L}(n)=L(n)$, $n\in N$. The multiplier $\sigma(\bar{t},\bar{s})$ satisfies as usual

$$\sigma(\bar{u},\bar{s})\,\sigma(\overline{us},\bar{t})=\sigma(\bar{u},\overline{ss})\,\sigma(\bar{s},\bar{t})$$

and is normalized: $\sigma(e,\ e)=1$. One sees easily that $\sigma(\bar{u},\ \bar{s})$ depends only on the $N$-cosets of $\bar{u}$ and $\bar{s}$, and may therefore be considered as a multiplier of the little group of second kind $S_j$. A possible change by a factor $\mu(\bar{u})$ of $A(\bar{u})$ defined by (25) would lead to an equivalence transformation of the multiplier:

$$\sigma'(s,s')=\frac{\mu(s)\,\mu(s')}{\mu(ss')}\,\sigma(s,s'), \qquad s,s'\in S_j. \tag{29}$$

Adding up the considerations of this part, we have proven the following Lemma (Mackey): The representation of $N$: $n \to L^{\bar{p}_j}$ related to the reference point $\bar{p}_j$ of the $G$-orbit $\Theta^j$ in $\hat{N}$ has an extension to a projective representation of the little group of first kind $\bar{S}_j$. This extension is uniquely determined up to equivalence transformations of the multiplier. The multiplier $\sigma(s,t)$ is also a multiplier of the little group of the second kind.

(3) *The Projective Representations of the Little Groups of Second Kind.* For the construction of the irreps of $G$ in the Wigner-Mackey scheme one needs the projective irreps $D^\chi(s)$ of $\bar{S}_j/N \subset G/N$ with the muliplier $\sigma^{-1}(s,t)$, where $\sigma$ is the multiplier defined by (28). In our case we get these irreps of the little groups of second kind by applying the Wigner-Mackey scheme once more. The little groups appearing then have well known representations.

(4) *The Main Theorem (Mackey).* All irreducible, unitary representations of a group $G$ with normal subgroup $N$ are characterized by the $G$-orbits $\Theta^j$ in $\hat{N}$, and the irreducible projective representations of the related little groups of second kind $S_j \ni s \to D^\chi(s)$ with multiplier of the equivalence class $\sigma_j^{-1}(s,s')$.

The basic ideas of the proof of these statements follow from the construction of the irreps of $G$ with help of $\Theta^j$ and $D^\chi(S_j)$.

(5) *Explicit Construction.* Given an irrep $g \to U(g)$ in the representation space $\mathscr{H}$, we can find an orthonormal base $\Phi^{j,\chi}$ of $\mathscr{H}$

$$\Phi^{j,\chi} = \left| \begin{matrix} j & & \chi \\ p, & m, & r \end{matrix} \right\rangle$$

in which the restriction $U(g)|_N$ decomposes into irreps of $N$:

$$U(n)\left| \begin{matrix} j, & & \chi \\ p, & m, & r \end{matrix} \right\rangle = \left| \begin{matrix} j, & & \chi \\ p, & m', & r \end{matrix} \right\rangle L^p_{m'm}(n). \tag{30}$$

The boost transformations $g(p)$, (23), transform the different irreducible subspaces into each other. Because of (24) we can set, in agreement with (30),

$$U(g(p))\left| \begin{matrix} j, & & \chi \\ \bar{p}_j, & m, & r \end{matrix} \right\rangle = \left| \begin{matrix} j, & & \chi \\ p, & m, & r \end{matrix} \right\rangle. \tag{31}$$

This follows from the simple calculation:

$$U(n)\,U(g(p))\left| \begin{matrix} j, & & \chi \\ \bar{p}_j, & m, & r \end{matrix} \right\rangle$$

$$= U(g(p))\,U(g^{-1}(p)\,n\,g(p))\left| \begin{matrix} j, & & \chi \\ \bar{p}_j, & m, & r \end{matrix} \right\rangle$$

$$= U(g(p))\left| \begin{matrix} j, & & \chi \\ \bar{p}_j, & \bar{m}, & r \end{matrix} \right\rangle L^{\bar{p}_j}_{\bar{m}m}(g^{-1}(p)\,n\,g(p))$$

$$= \left| \begin{matrix} j, & & \chi \\ p, & m', & r \end{matrix} \right\rangle L^p_{m'm}(n). \tag{32}$$

The subspace associated with the irrep $L^{\bar{p}_j}(n)$ is invariant under the transformations of the little group of the first kind $\bar{S}_j$. The irrep of $\bar{S}_j$ in that subspace can be represented as a direct product of the extension $\bar{L}^{\bar{p}_j}(\bar{s})$ of $L^{\bar{p}_j}(n)$ and an unfaithful projective irrep of $\bar{S}_j$: $D(\bar{s}) \equiv D^\chi(s)$ which is an irrep of the factor group $S_j \simeq \bar{S}_j/N$ with inverse multiplier:

$$U(\bar{s})\left| \begin{matrix} j, & & \chi \\ \bar{p}_j, & m, & r \end{matrix} \right\rangle = \left| \begin{matrix} j, & & \chi \\ \bar{p}_j, & m', & r' \end{matrix} \right\rangle \bar{L}^{\bar{p}_j}_{m'm}(\bar{s})\, D^\chi_{r'r}(\bar{s}). \tag{33}$$

$\bar{L}^{\bar{p}_j}(\bar{s})$ was discussed under Point (2), $D^\chi(s)$ under Point (3).

We can write an arbitrary $g \in G$ as a product of boost transformations and an element of $\bar{S}_j$. From

$$g = g(gp)\,\bar{s}(g,p)\,g^{-1}(p), \qquad \bar{s}(g,p) = g^{-1}(gp)\,g\,g(p) \tag{34}$$

we get with help of (23)

$$\bar{s}(g,p)\,\bar{p}_j = \bar{p}_j \Rightarrow \bar{s}(g,p) \in \bar{S}_j. \tag{35}$$

Therefore the transformation of the basis $\Phi^{j,\chi}$ under $U(g)$ is determined explicitly by (31) and (33):

$$U(g)\left| \begin{matrix} j, & & \chi \\ p, & m, & r \end{matrix} \right\rangle$$

$$= U(g(gp))\,U(\bar{s}(g,p))\,U(g^{-1}(p))\left| \begin{matrix} j, & & \chi \\ p, & m, & r \end{matrix} \right\rangle$$

$$= \left| \begin{matrix} j, & & \chi \\ gp, & m', & r' \end{matrix} \right\rangle \bar{L}^{\bar{p}_j}_{m'm}(\bar{s}(g,p))\, D^\chi_{r'r}(\bar{s}(g,p)). \tag{36}$$

One may verify by direct calculation that (36) defines a representation of $G$, which is irreducible if $p$ is restricted to a single $G$-orbit and if $D^\chi(s)$ is irreducible.

(6) *Remarks on Equivalence.* In the construction above, the representation of an $G$-orbit $\Theta^j$ by a reference point $\bar{p}_j$ and boost transformations, (23), introduces some arbitrariness. One convinces oneself easily that a different choice of reference points and boost transformations leads to equivalent representations of $G$. Similarly, equivalence transformations of $D^\chi(s)$ and $L^{\bar{p}_j}(n)$ lead to equivalent representations of $G$.

The Wigner-Mackey procedure allows the explicit construction of the irreps of a group $G$ with normal subgroup $N$ with help of the irreps of $N$, and with help of the *projective* irreps of the little groups of second kind. It was emphasized by Mackey [13], that, after some obvious modifications, essentially the same procedure allows also the construction of the *projective* irreps of $G$. In this sense, the Wigner-Mackey procedure presents a closed scheme for the construction for the irreps of groups by the irreps of smaller groups.

### *3.2 The structure of the irreps of the LFG*

The iterated application of the Wigner-Mackey procedure leads to a complete classification of the irreps of $\mathscr{G}_L$ by a '*Momentum Star* $St_j$', a '*Flavour Orbit* $\Theta_{j,k}$', and a '*Reduced Spin* $\sigma$'. These are the steps leading to this result:

(1) *Momentum Star.* In a first application of the Wigner-Mackey procedure we consider the translation group $\mathscr{T}_L$ as a normal subgroup of $\mathscr{G}_L$. The 1-dimensional irreps of $\mathscr{T}_L$: $[a]\to e^{ipa}$ are labelled by 'momenta' $p=(p_1, \dots p_4)$ varying in the Brillouin zone: $-\frac{\pi}{b}<p_\mu\leqq\frac{\pi}{b}$ of the coarse lattice. We denote by the star $St_j$ the orbit of the rotations $R\in W_4$ applied to the momenta $p\to Rp$. Depending on the orientation of $p$ there are 17 qualitatively different stars, see Table 1. For each $St_j$ one may choose a reference point $\bar{p}_j\in St_j$, boost operators $\Lambda(p)\,\bar{p}_j=p$, $\Lambda(p)\in W_4$, and determine the stability group $S_j=\{R\,|\,R\bar{p}_j=\bar{p}_j\}$.

The little group of the first kind $S_j^{(1)}$ in this application of the Wigner-Mackey procedure is generated by the translations, flavour transformations, and the rotations of $S_j$. The little group of the second kind $S_j^{(2)}\simeq S_j^{(1)}/\mathscr{T}_L$ is generated by $S_j$ and the elements of $\mathscr{K}_4$. An extension of the representation of $\mathscr{T}_L$: $[a]\to e^{i\bar{p}_j a}$ to a representation of $S_j^{(1)}$ is given trivially by: $[\varepsilon d^K, -\frac{1}{2}e_K+a, R]\to e^{i(\bar{p}_j, a-1/2e_K)}$, $R\in S_j$. The representations of $\mathscr{G}_L$: $[\varepsilon d^K, -\frac{1}{2}e_K+a, R]\to U(\varepsilon d^K, -\frac{1}{2}e_K+a, R)$, corresponding to (31), (34), (36) have then the form

$$U(\varepsilon d^K, -\tfrac{1}{2}e_K+a, R)\left|\begin{matrix} j, & \chi \\ p, & m\end{matrix}\right\rangle = e^{i(Rp, a-\frac{1}{2}e_K)}\sum_{m'}\left|\begin{matrix} j, & \chi \\ Rp, & m'\end{matrix}\right\rangle D^\chi_{m'm}(s(g,p)), \tag{37}$$

with $p\in St_j$ and $s(g,p)\in S_j^1$:

$$\begin{aligned} s(g,p)&=[1,0,\Lambda^{-1}(Rp)]\circ[\varepsilon d^K, -\tfrac{1}{2}e_K+a, R]\circ[1,0,\Lambda(p)]\\ &=[\varepsilon\rho(\Lambda^{-1}(Rp),K')\,d^{K'}, \Lambda^{-1}(Rp)(a-\tfrac{1}{2}e_K), \omega(R,p)]\end{aligned} \tag{38}$$

with $K'=\Lambda^{-1}(Rp)\circ K$, $\rho(\Lambda^{-1}(Rp),K')=\rho(\Lambda(R,p),K)$, and $\omega(R,p)=\Lambda^{-1}(Rp)\,R\Lambda(p)$. $D^\chi(s)$ is an irrep of $S_j^{(2)}$ considered as an unfaithful representation of $S_j^{(1)}$. We have to construct these representations.

(2) *The Irreps of* $\mathscr{K}_4$, *Flavour Orbits.* The group $S_j^{(2)}$ contains $\mathscr{K}_4$ as a normal subgroup. We apply for the construction of the irreps of $S_j^{(2)}$ the Wigner-Mackey procedure for a second time. For this we have to consider first the irreps of $\mathscr{K}_4$. These are the well-known 4-dimensional representation $\varepsilon d^K\to\varepsilon\gamma^K$ which we give the label $L=0$, and the 16 one dimensional representations for the factor group with respect to the centre:

$$\mathscr{K}_4:\ \begin{aligned}&\varepsilon d^K\to\varepsilon\gamma^K, \qquad L=0\\ &e d^K\to e^{i\pi(e_L,e_K)}\equiv\Gamma^L(\varepsilon d^K),\end{aligned}$$

$$e_L=\sum_{\mu\in L}e_\mu, \qquad e_\mu=(0,\dots \overset{\mu}{1},\dots 0). \tag{39}$$

$L\neq 0$ is a multi-index like in (2).

For the construction of the representations of $S_j^{(2)}$, we have to consider further the transformations of the irreps of $\mathscr{K}_4$ under the rotations of $S_j$: $\Gamma^L(R^{-1}(\varepsilon d^K)R)\simeq\Gamma^{R\circ L}$. In the case $L=0$, this is an equivalence transformation for all $R$, therefore $R\circ(L=0)=(L=0)$. The set of 1-dimensional representations decomposes under the rotations of $S_j$ in 'flavour orbits' $\Theta_{j,k}$, $k=1,\dots N_j$. For a flavour orbit we again may fix a reference point $\bar{L}_k$, choose boost operators $f(L)\,\bar{L}_k=L$, $f(L)\in S_j$, and determine the stability group $S_{j,k}=\{R\,|\,R\bar{L}_k=\bar{L}_k;\ R\in S_j\}\subset S_j$. In Table 1 we list the reference points $\bar{L}_k$ of the flavour orbits $\Theta_{j,k}$ and the corresponding stability groups $S_{j,k}$. The total flavour orbits can be easily constructed from $\bar{L}_k$. This is illustrated in the example below. Because of their relation to half integer and integer spin representations, we refer to the orbit $\Gamma^0$ as an odd orbit, and to the others as even orbits.

(3) *The Irreps of* $S_j^{(2)}$, *Reduced Spin.* The group $S_j^{(2)}$ is a semidirect product of $S_j$ with $\mathscr{K}_4$ as normal subgroup: $S_j^{(2)}\simeq\mathscr{K}_4\circledS S_j$. The Wigner-Mackey construction of the irreps $D^\chi(s)$ of $S_j^{(2)}$ uses the (flavour-) orbits $\Theta_{j,k}$ of the irreps of $\mathscr{K}_4$ under $S_j$ transformations. These are described above. With the definition given there, we can state that the little group of the first kind $S_{j,k}^{(1)}$ is a semidirect product of $S_{j,k}$ with $\mathscr{K}_4$ as normal subgroup. The little group of the second kind is then $S_{j,k}^{(1)}/\mathscr{K}_4\simeq S_{j,k}$.

For the Wigner-Mackey construction of the irreps of the group $S_j^{(2)}$ we need further the extension of the representation of $\mathscr{K}_4$: $\varepsilon d^K\to\Gamma^{\bar{L}_k}(e d^K)$ to a projective representation of $S_{j,k}^{(1)}\ni\xi\to\tilde{\Gamma}^{\bar{L}_k}(\xi)$. For $L=0$,

**Table 1.** Classification of the irreps of the Lattice Fermion Group. The headings have the following meaining: $j$ numbering of the momentum stars; $S_j$ stability group of the momentum star; $\bar{p}_j$ reference point of the momentum star, ($p_1 \neq p_2$ etc.); $N(S_j)$ order of $S_j$; $S_{j,k}$ reduced spin group, (we distinguish between a $Z_2$-group of reflections and a $\mathrm{Sym}_2$ group of permutations, $\mathrm{Sym}_2^2 \simeq \mathrm{Sym}_2 \times \mathrm{Sym}_2$ etc.); $\bar{L}_k$ reference point of flavour orbits; $\theta_{j,\cdot k}$ flavour orbit; $N(\theta_{j\cdot k})$ number of points on the flavour orbit; $N(\sigma)$ number of different reduced spins; $N(\mathscr{D})$ number of irreps of the group $S_j^{(2)} \simeq \mathscr{K}_4 \otimes S_j$

| j | $S_j$ | $\bar{p}_j$ | $N(S_j)$ | $S_{j,k}$ | $\bar{L}_k$ | $N(\Theta_{j,k})$ | $N(\sigma)$ | $N(\mathscr{D})$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $W_4$ | $(0, 0, 0, 0)$ | 384 | $W_4$ | (0, 0, 0, 0), (1, 1, 1, 1) | 1 | 20 | 40 |
| | | $(\pi, \pi, \pi, \pi)$ | | $W_3 \times Z_2$ | (0, 0, 0, 1), (1, 1, 1, 0) | 4 | 20 | 40 |
| | | | | $D_4 \times D_4$ | (1, 1, 0, 0) | 6 | 25 | 25 |
| | | | | ${}^dW_4$ | $\Gamma^0$ | 1 | 5 | 5 |
| 2 | $W_3 \times Z_2$ | $(0, 0, 0, \pi)$ | 96 | $W_3 \times Z_2$ | (1, 1, 1, 1), (1, 1, 1, 0) | 1 | 20 | 80 |
| | | $(\pi, \pi, \pi, 0)$ | | | (0, 0, 0, 1), (0, 0, 0, 0) | | | |
| | | | | $D_4 \times Z_2^2$ | (1, 1, 0, 0), (0, 0, 1, 0) | 3 | 20 | 80 |
| | | | | | (1, 1, 0, 1), (0, 0, 1, 1) | | | |
| | | | | ${}^d(W_3 \times Z_2)$ | $\Gamma^0$ | 1 | 3 | 3 |
| 3 | $D_4 \times D_4$ | $(0, 0, \pi, \pi)$ | 64 | $D_4 \times D_4$ | (0, 0, 0, 0), (0, 0, 1, 1) | 1 | 25 | 100 |
| | | | | | (1, 1, 0, 0), (1, 1, 1, 1) | | | |
| | | | | $D_4 \times Z_2^2$ | (0, 0, 1, 0), (1, 1, 1, 0) | 2 | 40 | 80 |
| | | | | | (1, 0, 0, 0), (1, 0, 1, 1) | | | |
| | | | | $Z_2^4$ | (1, 0, 1, 0) | 4 | 16 | 16 |
| | | | | ${}^d(D_4 \times D_4)$ | $\Gamma^0$ | 1 | 4 | 4 |
| 4 | $W_3$ | $(0, 0, 0, p)$ | 48 | $W_3$ | (0, 0, 0, 0), (0, 0, 0, 1) | 1 | 10 | 40 |
| | | $(\pi, \pi, \pi, p)$ | | | (1, 1, 1, 0), (1, 1, 1, 1) | | | |
| | | | | $D_4 \times Z_2$ | (0, 0, 1, 0), (1, 1, 0, 0) | 3 | 10 | 40 |
| | | | | | (0, 0, 1, 1), (1, 1, 0, 1) | | | |
| | | | | ${}^dW_3$ | $\Gamma^0$ | 1 | 6 | 6 |
| 5 | $\mathrm{Sym}_4$ | $(p, p, p, p)$ | 24 | $\mathrm{Sym}_4$ | (0, 0, 0, 0), (1, 1, 1, 1) | 1, | 5 | 10 |
| | | | | $\mathrm{Sym}_3$ | (0, 0, 0, 1), (1, 1, 1, 0) | 4 | 3 | 6 |
| | | | | $\mathrm{Sym}_2^2$ | (0, 0, 1, 1) | 6 | 4 | 4 |
| | | | | ${}^d(\mathrm{Sym}_4)$ | $\Gamma^0$ | 1 | 3 | 3 |
| 6 | $D_4 \times Z_2$ | $(0, 0, \pi, p)$ | 16 | $D_4 \times Z_2$ | (0, 0, 0, 0), (1, 1, 1, 1) | 1 | 10 | 80 |
| | | $(\pi, \pi, 0, p)$ | | | (1, 1, 1, 0), (0, 0, 1, 0) | | | |
| | | | | | (1, 1, 0, 1), (0, 0, 0, 1) | | | |
| | | | | | (1, 1, 0, 0), (0, 0, 1, 1) | | | |
| | | | | $Z_2^3$ | (0, 1, 0, 0), (0, 1, 0, 1) | 2 | 8 | 32 |
| | | | | | (0, 1, 1, 0), (0, 1, 1, 1) | | | |
| | | | | ${}^d(D_4 \times Z_2)$ | $\Gamma^0$ | 1 | 4 | 4 |
| 7 | $D_4 \times \mathrm{Sym}_2$ | $(0, 0, p, p)$ | 16 | $D_4 \times \mathrm{Sym}_2$ | (0, 0, 0, 0), (0, 0, 1, 1) | 1 | 10 | 40 |
| | | $(\pi, \pi, p, p)$ | | | (1, 1, 0, 0), (1, 1, 1, 1) | | | |
| | | | | $Z_2^2 \times \mathrm{Sym}_2$ | (0, 1, 0, 0), (0, 1, 1, 1) | 2 | 8 | 16 |
| | | | | $D_4$ | (0, 0, 0, 1), (1, 1, 0, 1) | 2 | 5 | 10 |
| | | | | $Z_2^2$ | (0, 1, 0, 1) | 4 | 4 | 4 |
| | | | | ${}^d(D_4 \times \mathrm{Sym}_2)$ | $\Gamma^0$ | 1 | 4 | 4 |
| 8 | $\mathrm{Sym}_3 \times Z_2$ | $(p, p, p, 0)$ | 12 | $\mathrm{Sym}_3 \times Z_2$ | (0, 0, 0, 0), (1, 1, 1, 0) | 1 | 6 | 24 |
| | | $(p, p, p, \pi)$ | | | (0, 0, 0, 1), (1, 1, 1, 1) | | | |
| | | | | $Z_2 \times \mathrm{Sym}_2$ | (0, 0, 1, 0), (0, 0, 1, 1) | 3 | 4 | 16 |
| | | | | | (1, 1, 0, 0), (1, 1, 0, 1) | | | |
| | | | | ${}^d(\mathrm{Sym}_3 \times Z_2)$ | $\Gamma^0$ | 1 | 2 | 2 |
| 9 | $D_4$ | $(0, 0, p_3, p_4)$ | 8 | $D_4$ | (0, 0, 0, 0), (0, 0, 0, 1) | 1 | 5 | 40 |
| | | $(\pi, \pi, p_3, p_4)$ | | | (0, 0, 1, 0), (0, 0, 1, 1) | | | |
| | | | | | (1, 1, 0, 0), (1, 1, 0, 1) | | | |
| | | | | | (1, 1, 1, 0), (1, 1, 1, 1) | | | |
| | | | | $Z_2^2$ | (0, 1, 0, 0), (0, 1, 0, 1) | 2 | 4 | 16 |
| | | | | | (0, 1, 1, 0), (0, 1, 1, 1) | | | |
| | | | | ${}^dD_4$ | $\Gamma^0$ | 1 | 2 | 2 |

**Table 1** (Continued)

| $j$ | $S_j$ | $\bar{p}_j$ | $N(S_j)$ | $S_{j,k}$ | $\bar{L}_k$ | $N(\Theta_{j,k})$ | $N(\sigma)$ | $N(\mathcal{D})$ |
|---|---|---|---|---|---|---|---|---|
| 10 | $Z_2^2\times \mathrm{Sym}_2$ | $(0, \pi, p, p)$ | 8 | $Z_2^2\times \mathrm{Sym}_2$ | (0, 0, 0, 0), (0, 1, 0, 0) | 1 | 8 | 64 |
| | | | | | (1, 0, 0, 0), (1, 1, 0, 0) | | | |
| | | | | | (0, 0, 1, 1), (0, 1, 1, 1) | | | |
| | | | | $Z_2^2$ | (0, 0, 0, 1), (0, 1, 0, 1) | 2 | 4 | 16 |
| | | | | | (1, 0, 0, 1), (1, 1, 0, 1) | | | |
| | | | | ${}^dZ_2^3$ | $\Gamma^0$ | 1 | 2 | 2 |
| 11 | $\mathrm{Sym}_3$ | $(p, p, p, p_4)$ | 6 | $\mathrm{Sym}_3$ | (0, 0, 0, 0), (1, 1, 1, 0) | 1 | 3 | 12 |
| | | | | | (0, 0, 0, 1), (1, 1, 1, 1) | | | |
| | | | | $\mathrm{Sym}_2$ | (0, 0, 1, 0), (0, 0, 1, 1) | 3 | 2 | 8 |
| | | | | | (1, 1, 0, 0), (1, 1, 0, 1) | | | |
| | | | | ${}^d\mathrm{Sym}_3$ | $\Gamma^0$ | 1 | 3 | 3 |
| 12 | $Z_2^2$ | $(0, \pi, p_3, p_4)$ | 4 | $Z_2^2$ | (0, 0, 0, 0), ..., (1, 1, 1, 1) | 1 | 4 | 64 |
| | | | | ${}^d(Z_2^2)$ | $\Gamma^0$ | 1 | 1 | 1 |
| 13 | $Z_2\times \mathrm{Sym}_2$ | $(0, p, p, p_4)$ | 4 | $Z_2\times \mathrm{Sym}_2$ | (0, 0, 0, 0), (0, 0, 0, 1) | 1 | 4 | 32 |
| | | $(\pi, p, p, p_4)$ | | | (1, 0, 0, 0), (1, 0, 0, 1) | | | |
| | | | | | (0, 1, 1, 0), (0, 1, 1, 1) | | | |
| | | | | | (1, 1, 1, 0), (1, 1, 1, 1) | | | |
| | | | | $Z_2$ | (0, 0, 1, 0), (0, 0, 1, 1) | 2 | 2 | 8 |
| | | | | | (1, 0, 1, 0), (1, 0, 1, 1) | | | |
| | | | | ${}^d(Z_2\times \mathrm{Sym}_2)$ | $\Gamma^0$ | 1 | 1 | 1 |
| 14 | $\mathrm{Sym}_2^2$ | $(p, p, p_4, p_4)$ | 4 | $\mathrm{Sym}_2^2$ | (0, 0, 0, 0), (0, 0, 1, 1) | 1 | 4 | 16 |
| | | | | | (1, 1, 0, 0), (1, 1, 1, 1) | | | |
| | | | | $Z_2$ | (0, 1, 0, 0), (0, 1, 1, 1) | 2 | 2 | 8 |
| | | | | | (0, 0, 0, 1), (1, 1, 0, 1) | | | |
| | | | | $\mathcal{E}$ | (0, 1, 0, 1) | 4 | 1 | 1 |
| | | | | ${}^d(\mathrm{Sym}_2^2)$ | $\Gamma^0$ | 1 | 1 | 1 |
| 15 | $Z_2$ | $(0, p_2, p_3, p_4)$ | 2 | $Z_2$ | (0, 0, 0, 0), ... | 1 | 2 | 32 |
| | | $(\pi, p_2, p_3, p_4)$ | | | ..., (1, 1, 1, 1) | | | |
| | | | | ${}^d(Z_2)$ | $\Gamma^0$ | 1 | 2 | 2 |
| 16 | $\mathrm{Sym}_2$ | $(p, p, p_3, p_4)$ | 2 | $\mathrm{Sym}_2$ | (0, 0, 0, 0), (0, 0, 0, 1) | 1 | 2 | 16 |
| | | | | | (0, 0, 1, 0), (0, 0, 1, 1) | | | |
| | | | | | (1, 1, 0, 0), (1, 1, 0, 1) | | | |
| | | | | | (1, 1, 1, 0), (1, 1, 1, 1) | | | |
| | | | | $\mathcal{E}$ | (0, 1, 0, 0), (0, 1, 0, 1) | 2 | 1 | 4 |
| | | | | | (0, 1, 1, 0), (0, 1, 1, 1) | | | |
| | | | | ${}^dZ_2$ | $\Gamma^0$ | 1 | 2 | 2 |
| 17 | $\mathcal{E}$ | $(p_1, p_2, p_3, p_4)$ | 1 | $\mathcal{E}$ | (0, 0, 0, 0), ... (1, 1, 1, 1) | 1 | 1 | 16 |
| | | | | ${}^d\mathcal{E}$ | $\Gamma^0$ | 1 | 1 | 1 |

$\tilde{\Gamma}^0_{a',a}(\xi)$, is the extension of the 4-dimensional representation of $\mathcal{K}_4$ to a projective representation of $S^{(1)}_{j,0}$. In this case the representation of the rotations $R\in S_{j,k}$ is given by the projective representation of $W_4$ generated by $R_{\mu\nu}\to\frac{1}{2}(1+\gamma_\mu\gamma_\nu)$, $\Pi_s\to\gamma^4$, restricted to $S_{j,k}$. We consider this projective representation of $W_4$ sometimes as a group ${}^dW_4$ which is central extension of $W_4$. In the extension of the 1-dimensional representations: $\tilde{\Gamma}^{\bar{L}_k}_{\bar{a},a}(\xi)$, $\bar{L}_k\neq 0$, the rotations and reflections of $S^{(1)}_{j,k}$ are represented trivially: $R_{\mu\nu}\to 1$, and $\Pi_s\to 1$.

With these concepts we can construct the irreps of $S^{(2)}_j$ according to (31), (34), (36) with help of the projective irreps $\mathcal{D}^\sigma(s)$ of $S_{j,k}$ with the appropriate multiplier

$$\mathcal{U}^D(\varepsilon d^K, R)\left|\begin{matrix}k, & & \sigma\\ L, & a, & n\end{matrix}\right\rangle = \sum_{a',n'}\left|\begin{matrix}k, & & \sigma\\ R\circ L, & a', & n'\end{matrix}\right\rangle \tilde{\Gamma}^{\bar{L}_k}_{a',a}(\xi(s,L))\,\mathcal{D}^\sigma_{n',n}(\xi(s,L)), \quad (40)$$

with

$$L\in\Theta_{j,k},\qquad s=[\varepsilon d^K, R]\in S^{(2)}_j,$$

$$S^{(1)}_{j,k}\ni\xi(s,L)=[1,f^{-1}(R\circ L)]\circ[\varepsilon d^K, R]\circ[1,f(R\circ L]$$
$$=[\varepsilon\rho(f(R\circ L),K)\,d^{K'}, X(R,L)],$$

$$K'=f^{-1}(R\circ L)\circ K,\qquad X(R,L)=f^{-1}(R\circ L)\,R f(L).$$

Thus the irreps of $S_j^{(2)}$ are characterized by the flavour orbits $\Theta_{j,k}$, and the irreps $\mathscr{D}^\sigma$ of $S_{j,k}$ with appropriate multiplier. The groups $S_{j,k}$, the 'reduced spin groups', are direct products of the symmetric groups $\mathrm{Sym}_n$, $n=2, 3, 4$, and the cubic groups $W_{di}$. Their projective irreps are well-known, or can be constructed easily. The irreps of $S_{j,k} \ni \xi \rightarrow \mathscr{D}^\sigma_{r',r}(\xi)$, which determine the reduced spin $\sigma$, may then be labelled by combinations of the primitive characters of these elementary groups. For completeness we summarize the representations defining the reduced spin in Sect. 3.3.

(4) *Proposition on the Irreps of the* LFG. With these preliminaries we may apply the main theorem of the Wigner-Mackey procedure to the construction of the irreps of the LFG. The result we formulate as:

**Proposition 3.1.** *The irreducible, unitary representations of the symmetry group* LFG *of staggered fermions* (*Proposition* 2.1) *are determined by a 'momentum star', a 'flavour orbit', and the 'reduced spin'.*

This proposition follows from the application of the *W-M* procedure to the LFG with the translation group $\mathscr{T}_L$ as normal subgroup, and combining this with the characterization of the irreps of the little group $S_j^{(2)}$ by flavour orbits and reduced spin.

We could get an explicit construction of the irreps of the LFG by combining (37) and (40). For this we must identify the index $m$ in (37) with the indices ($L$, $a$, $n$) in (40), the character $\chi$ of the representation $D^\chi(s)$ of $S_j^{(2)}$ in (37) with ($k$, $\sigma$) in (40), and finally $D^\chi(s)$ in (37) with $\mathscr{U}^D(s)$ in (40). However, in this manner one gets for the general case expressions which are rather involved. There are simplifications if one considers the special structure of the even and odd flavour orbits, and of the representations of the reduced spin groups related to it. Therefore we postpone the discussion of the explicit construction of the irreps of the LFG to Sect. 3.3.

(5) *Example.* Before we continue the general discussion, we illustrate the concepts introduced above with an example which has already been considered [14] in connection with the calculation of the mass spectrum in lattice QCD.

We consider the momentum star of 8 points

$$St_4\colon (p_\mu)=(0,0,0,\pm E), \quad (0,0,\pm E,0), \\ (0,\pm E,0,0), \quad (\pm E,0,0,0).$$

The stability group of the reference point $\bar{p}_j=(0, 0, 0, E)$, $E>0$, of this star is $S_4 \simeq W_3$. In this case there are three types of flavour orbits:

(*a*) The odd 1-point orbit $\Gamma^0$, $k=0$.

(*b*) The even 1-point orbits $\Theta_{4,k}=\{(e_L)\}$, $k=1, \ldots, 4$ with $S_{4,k} \simeq W_3$:

$\{(0, 0, 0, 0)\}$, $\{(0, 0, 0, 1)\}$,
$\{(1, 1, 1, 0)\}$, $\{(1, 1, 1, 1)\}$.

(*c*) The even 3-point orbits $\Theta_{4,k}$, $k=5, \ldots, 8$ with $S_{4,k} \simeq D_4 \times Z_2$:

$\{(1,0,0,0),(0,1,0,0),\underline{(0,0,1,0)}\}$,
$\{(0,1,1,0),(1,0,1,0),\underline{(1,1,0,0)}\}$,
$\{(1,0,0,1),(0,1,0,1),\underline{(0,0,1,1)}\}$,
$\{(0,1,1,1),(1,0,1,1),\underline{(1,1,0,1)}\}$

with $\bar{L}_k$ underlined.

In this example we have $S_{4,k} \simeq W_3 \simeq \mathrm{Sym}_4 \times Z_2$ for $k=0, \ldots, 4$, and $S_{4,k} \simeq D_4 \times Z_2$ for $k=5, \ldots, 8$. For $k=0$ we have to consider projective representations of $W_3$ with the multipliers of the spinor representations. There are two 2-dimensional and one 4-dimensional of such representations for $\mathrm{Sym}_4$. This means altogether six reduced spin-parity combinations for this class of representations characterized by $St_4$ and $\Theta_{4,0}$. We denote them by $(2)^\pm_{\bar{a}W_3}$, $(2')^\pm_{\bar{a}W_3}$, $(4)^\pm_{\bar{a}W_3}$. (Compare Table 3).

The proper cubic group has two 1-dimensional, one 2-dimensional, and two 3-dimensional representations. Together with the parities $\Pi_s=\pm 1$, this gives 10 spin parity combinations related to the reduced spin group of the orbits $\Theta_{4,k}$, $k=1, \ldots 4$. We denote them by $(1^\pm)_{W_3}$, $(1'^\pm)_{W_3}$, $(2^\pm)_{W_3}$, $(3^\pm)_{W_3}$, $(3'^\pm)_{W_3}$. (In the notation $(\bar{m})\chi(\mathscr{L})$ of Table 2 we have the following correspondence:

$(1^\pm)_{W_3}$, $(1'^\pm)_{W_3}$, $(2^\pm)_{W_3} \simeq (\bar{m})(1)$, $(\bar{m})(1')$, $(\bar{m})(2)$ with $(\bar{m})=(0, 0, 0)$ for $+$parity, $(\bar{m})=(1, 1, 1)$ for $-$parity, and $(3^\pm) \simeq (\bar{m})(1^\mp)$, $(3'^\pm) \simeq (\bar{m})(1^\pm)$ with $(\bar{m})=(1, 1, 0)$ for $+$parity and $(\bar{m})=(0, 0, 1)$ for $-$parity).

The dihedral group $D_4$ has four 1-dimensional representations and one 2-dimensional representation. This gives again 10 spin parity combinations, however of a different type which belong to the flavour orbits $\Theta_{j,k}$, $k=5, \ldots 8$. These are denoted by $(1^\pm)_{D_4}$, $(1'^\pm)_{D_4}$, $(1''^\pm)_{D_4}$, $(1'''^\pm)_{D_4}$, $(2^\pm)_{D_4}$. (In the notation of $(m_1, m_2)\otimes(m_3)\ \chi(\mathscr{L})$ of Table 2 these irreps of $D_4 \times Z_2$ correspond to $(0,0)\otimes(m_3)(1^+)$, $(1,1)\otimes(m_3)(1^-)$, $(0,0)\otimes(m_3)(1^-)$, $(1,1)\otimes(m_3)(1^+)$, $(0,1)\otimes(m_3)(1)$, $m_3=0$ for $+$symmetry, $m_3=1$ for $-$symmetry with respect to reflection of the 3-axis.

This is a complete classification of the 86 irreps of $\mathscr{G}_L$ with momentum star $St_4$.

### *3.3 Complete Classification of the Irreps of the* LFG

According to the Proposition 3.1, the irreps of the LFG are determined by a momentum star, a flavour orbit, and the reduced spin. Therefore, with help of

**Table 2.** Classification of the irreps of the cubic groups. The headings have the following meaning: $W_{di}$ $di$ dimensional cubic group; $\bar{m}$ reference point of the reflection orbit; $N(\bar{m})$ number of points on the reflection orbit; $\mathscr{L}$ stability group of the reflection orbit: $\chi(\mathscr{L})$ representations of $\mathscr{L}$, the notation $(\dim\chi)^i$ is defined in Table 2'; $N(\mathscr{D}_{\mathscr{L}})$ number of irreps of $W_{di}$ for given $\mathscr{L}$

| $W_{di}$ | $\bar{m}$ | $N(\bar{m})$ | $\mathscr{L}$ | $\chi(\mathscr{L})$ | $N(\mathscr{D}_{\mathscr{L}})$ |
|---|---|---|---|---|---|
| $Z_2$ | (0), (1) | 1 | $\mathscr{E}$ | (1) | 2 |
| $D_4$ | (0, 0), (1, 1) | 1 | $\mathrm{Sym}_2$ | $(1^+)$, $(1^-)$ | 4 |
| | (0, 1) | 2 | $\mathscr{E}$ | (1) | 1 |
| $W_3$ | (0, 0, 0), (1, 1, 1) | 1 | $\mathrm{Sym}_3$ | (1), (1'), (2) | 6 |
| | (0, 0, 1), (1, 1, 0) | 3 | $\mathrm{Sym}_2$ | $(1^+)$, $(1^-)$ | 4 |
| $W_4$ | (0, 0, 0, 0), (1, 1, 1, 1) | 1 | $\mathrm{Sym}_4$ | (1), (1'), (2), (3), (3') | 10 |
| | (0, 0, 0, 1), (1, 1, 1, 0) | 4 | $\mathrm{Sym}_3$ | (1), (1'), (2) | 6 |
| | (0, 0, 1, 1) | 6 | $\mathrm{Sym}_2^2$ | $(1^{++})$, $(1^{+-})$, $(1^{-+})$, $(1^{--})$ | 4 |

**Table 2'.** Explicit definition of the irreps of the $\mathrm{Sym}_n$ by the representations of the generating cycles

| | | | |
|---|---|---|---|
| $\mathrm{Sym}_2$: | $(1^\pm)$: | $(12)\to\pm 1$ | |
| $\mathrm{Sym}_3$: | (1): | $(12)\to 1$ | $(123)\to 1$ |
| | (1'): | $(12)\to -1$ | $(123)\to 1$ |
| | (2): | $(12)\to\begin{pmatrix}0&1\\1&0\end{pmatrix}$ | $(12)\to\begin{pmatrix}\exp 2\pi i/3 & 0\\ 0 & \exp -2\pi i/3\end{pmatrix}$ |
| $\mathrm{Sym}_4$: | (1): | $(12)\to 1$ | $(1234)\to 1$ |
| | (1'): | $(12)\to -1$ | $(1234)\to -1$ |
| | (2): | $(12)\to\frac{1}{2}\begin{pmatrix}1&\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}$ | $(1234)\to\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ |
| | (3): | $(12)\to\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}$ | $(1234)\to\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$ |
| | $(3')\simeq(3)\oplus(1')$ | | |

Table 1, we may denote these irreps in a most informative manner by a symbol

$$\mathscr{R}(\bar{p}_j, \bar{L}_k, \sigma(S_{j,k}))$$

where $\bar{p}_j$ denotes a reference point of the momentum star, $\bar{L}_k$ a reference point of the flavour orbit, $\sigma(S_{jk})$ an irreducible representation of the reduced spin group. Table 1 describes the range of $\bar{p}_j$ and $\bar{L}_k$. For a complete classification of the irreps of the LFG we need still a survey of the irreps of the reduced spin groups. We shall describe these representations in the next two subsections for the reduced spin groups of even and odd flavour orbits separately, and collect the important information in Table 2 and 3. As a last point, we explain how one should use Table 1, Table 2, and Table 3 for a survey of the explicit construction of all irreps of the LFG.

(1) *Reduced Spins for even Flavour Orbits.* The reduced spin groups $S_{j,k}$ are subgroups of the hyper-cubical group $W_4$. It can be seen from Table 1, that they are direct products with factors $W_3$ (the symmetry group of a 3 cube), $D_4$ (the symmetry group of a square), $Z_2$ (the cyclic group of order 2), $\mathrm{Sym}_4$, $\mathrm{Sym}_3$ and $\mathrm{Sym}_2\simeq Z_2$ (the permutation groups of 4, 3 and 2 objects, respectively). In the product description of the reduced spin groups in Table 1 the factors are understood in the following way. They are groups which transform $di$-dimensional sub-spaces of $R^4$, and leave the orthogonal sup-spaces untransformed. In this spirit we destinguish the isomorphic $Z_2$-groups of the permutation of two axes from the reflections of one axis. Of course the irreps of these groups are well known [19, 22]. In spite of this, we consider shortly the irreps of these groups under the unifying viewpoint that the $W_{di}$ are semidirect products with Abelian normal subgroups:

$$W_{di}\simeq\overbrace{Z_2\times Z_2\times\,\dots\,\times Z_2}^{di}\otimes\mathrm{Sym}_{di}\equiv Z_2^{di}\otimes\mathrm{Sym}_{di}\equiv\Pi\otimes\mathrm{Sym}_{di}.$$

The point of view is particularly appropriate for showing how the reduced spin groups are embedded in $W_4$. Further, it allows the use of the Wigner-Mackey procedure for the construction of the irreps of the

**Table 3.** Classification of the proper irreps of the reduced spin groups of odd flavour orbits. For details see Sect. 3.3 (2)

| $j$ | ${}^dS_{j,0}$ | $\mathscr{G}\mathscr{R}$ | ${}^d\Pi_{di}$ | $F$ | $\sigma$ |
|---|---|---|---|---|---|
| 1 | ${}^dW_4$ | | $\mathscr{K}_4$ | $\mathrm{Sym}_4$ | (4) (4′) (8) (12) (12′) |
| 2 | ${}^d(W_3\times Z_2)$ | | $\mathscr{K}_4$ | $\mathrm{Sym}_3$ | (4) (4′) (8) |
| 3 | ${}^d(D_4\times D_4)$ | – | $\mathscr{K}_4$ | $\mathrm{Sym}_2\times\mathrm{Sym}_2$ | $(4^{\pm\pm})$ |
| 4 | ${}^dW_3$ | | $\mathscr{K}_3$ | $\mathrm{Sym}_3$ | $(2)^{\pm}\ (2')^{\pm}\ (4)^{\pm}$ |
| 5 | ${}^d\mathrm{Sym}_4$ | | | | (2) (2′) (4) |
| 6 | ${}^d(D_4\times Z_2)$ | | $\mathscr{K}_3$ | $\mathrm{Sym}_2$ | $(2^{\pm})^{\pm}$ |
| 7 | ${}^d(D_4\times\mathrm{Sym}_2)$ | | $\mathscr{K}_2$ | $\mathrm{Sym}_2\times\mathrm{Sym}_2$ | $(2^{\pm\pm})$ |
| 8 | ${}^d(\mathrm{Sym}_3\times Z_2)$ | $\mathscr{K}_3$ | | | $(2)^{\pm}$ |
| 9 | ${}^dD_4$ | | $\mathscr{K}_2$ | $\mathrm{Sym}_2$ | $(2^{\pm})$ |
| 10 | ${}^d(\mathrm{Sym}_2\times Z_2^2)$ | | $\mathscr{K}_2$ | $\mathrm{Sym}_2$ | $(2^{\pm})$ |
| 11 | ${}^d\mathrm{Sym}_3$ | $\simeq\mathrm{Sym}_3$ | | | (1) (1′) (2) |
| 12 | ${}^d(Z_2^2)$ | $\mathscr{K}_2$ | $\mathscr{K}_2$ | $\mathscr{E}$ | (2) |
| 13 | ${}^d(Z_2\times\mathrm{Sym}_2)$ | $\mathscr{K}_2$ | | | (2) |
| 14 | ${}^d(\mathrm{Sym}_2\times\mathrm{Sym}_2)$ | $\mathscr{K}_2$ | | | (2) |
| 15 | ${}^dZ_2$ | $Z_2\times Z_2$ | | | $(1^{\pm})^{-}$ |
| 16 | ${}^d\mathrm{Sym}_2$ | $Z_2\times Z_2$ | | | $(1^{\pm})^{-}$ |
| 17 | ${}^d\mathscr{E}$ | $Z_2$ | | | $(1)^{-}$ |

reduced spin groups. We describe shortly this structure of the 'cubic' groups.

It is geometrically evident that the cubic groups $W_2\simeq D_4$, $W_3$, $W_4$, ... are generated by rotations $R_{\mu\nu}$ in the $(\mu,\nu)$-plane by $\pi/2$, $\mu,\nu=1,\dots,di$, and a reflection, e.g. the reflection of the 1-axis $\Pi_1$. $(R_{\mu\nu})^2$ describes a reflection of the two axes of the $(\mu,\nu)$-plane. The reflections $\Pi_1$ and $(R_{\mu\nu})^2$ generate an Abelian normal subgroup $\Pi\simeq(Z_2)^{di}$ of $W_{di}$, of which the reflections of an axis $\Pi_\mu$, $\mu=1,\dots,di$, form a basis. The factor group $W_{di}/Z_2^{di}$ is the permutation group of the different axes. Since the permutations of the axes form a subgroup of $W_{di}$, it follows that $W_{di}=\{(z_1,\dots,z_{di};h)\}$, is a semidirect product with the multiplication rule

$$(z_1,\dots,z_{di};h)\bullet(z'_1,\dots z'_{di};h')\\=(z_1+h\circ z'_1,\dots,z_{di}+h\circ z'_{di};hh').$$

Here $z_i=0,1 \bmod (2)$ denote the elements of $Z_2$, generated by the reflection $\Pi_i$, written additively. $\mathrm{Sym}_{di}=\{h\}$ is the group of permutations of the axes acting on the $z_i$: $h\circ z_i=z_j$. In these terms, the isomorphism $W_{di}\simeq Z_2^{di}\otimes\mathrm{Sym}_{di}$ is given by the mapping of the generating elements; $R_{\mu\nu}\to(z,h_{\mu\nu})$, $h_{\mu\nu}$ transposition of $\mu$ and $\nu$, $z_i=1$ *for* $i=\nu$, $z_i=0$ *for* $i\neq\nu$.

According to the main theorem of the Wigner Mackey procedure, the irreps of $W_{di}$ are characterized by the orbits of the permutation groups $\mathrm{Sym}_{di}$ in the set of irreps of $Z_2^{di}$, ('reflection orbits'):

$$(z,1)\to e^{i\pi(m,z)},\qquad m=(m_1,\dots,m_{di}),\qquad m_i=0,1 \tag{41}$$

and the irreps of the related little groups of second kind: $\mathscr{L}$. (In the case of a semidirect product with Abelian normal subgroup, we have for the multiplier $\sigma(u,v)\equiv 1$.) These little groups of second kind are either $\mathrm{Sym}_4$, or $\mathrm{Sym}_3$, or $\mathrm{Sym}_2\simeq Z_2$, or the trivial group $\mathscr{E}$. The result of this consideration is summarized in Table 2 for $di=1,2,3,4$. In this Table $\bar{m}$ denotes a reference point of a reflection orbit of size $N(\bar{m})$. We assume that the irreps $\chi(\mathscr{L})$ of the little groups $Z_2$: $(1^+)$, $(1^-)$; $\mathrm{Sym}_3$: (1), (1′), (2), and $\mathrm{Sym}_3$: (1), (1′), (2), (3), (3′) are indeed very familiar [19]. We make this assumption in spite of the fact, that we could have applied the Wigner-Mackey procedure for the construction of these representations once more! In order to fix the notation, we define these representations explicitely in Table 2′. $N(\mathscr{D}_{\mathscr{L}})$ denotes the number of irreps of $W_{di}$ we get by combining the reflection orbits with the irreps of the little groups $\mathscr{L}$. The dimension of these representations is given by $\dim(\mathscr{D}_{\mathscr{L}})=N(\bar{m})\cdot\dim(\chi(\mathscr{L}))$.

For physics, the representation of the space reflection $\Pi_s=\Pi_1\Pi_2\Pi_3$ plays as parity $\pi$ an important role. According to (41), we have: $\Pi_s\to\exp i(m_1+m_2+m_3)\pi$. Thus the parity of the representations of $W_3$ is $+1$ for $\bar{m}=(0,0,0)$, $(1,1,0)$, and '$-1$' for $\bar{m}=(0,0,1)$, $(1,1,1)$. Similar we get the parity for a composite reduced spin group like $D_4\times Z_2\subset S_3$: $\pi=+1$ for $\bar{m}=(0,0)\otimes(0)$, $(1,1)\otimes(0)$, $(0,1)\otimes(1)$, and $\pi=-1$ for $\bar{m}=(0,0)\otimes(1)$, $(1,1)\otimes(1)$, $(0,1)\otimes(0)$.

(2) *Reduced Spins for odd Flavour Orbits.* The projective representations of the reduced spin groups related to the odd flavour orbits have a non-trivial multiplier. As we have seen in Sect. 3.2 (3), the projective extension of the 4-dimensional representation of $\mathscr{K}_4$ to a representation of $S_{j,0}^{(1)}$ leads to a representation of the

reduced spin group $S_{j,0}$: $S_{j,0}^{(1)} \supset S_{j,0} \ni s \to \tilde{\Gamma}^0(s)$ by linear combinations of $\gamma$-matrices. This defines a central extension ${}^dS_{j,0}$ of $S_{j,0}$ which might be considered as a subgroup of ${}^dW_4$. However, the reduced spin groups are subgroups of the geometric rotation group, the ('single valued') $O(4)$. Thus on physical grounds, i.e. because the continuous symmetry group is $\mathscr{S}\mathscr{E} \times SU(4)/Z_2$ rather than $\mathscr{S}\mathscr{E} \times SU(4)$, the multiplier of the irreps defining the reduced spin in the case of the odd flavour orbit must compensate the multiplier ('double valuedness') of $\tilde{\Gamma}^0(s)$. This means that we have to study the ('proper') representations of ${}^dS_{j,0}$ in which the phase factor, i.e. the central element $\varepsilon = -1$, is represented faithfully. The results are summarized in Table 3.

As a guideline for the construction of these representations we use again the description of the cubic groups as semidirect products with an Abelian normal subgroup generated by reflections. In the matrix group ${}^dW_4$, the reflection of the $\mu$-axis is represented as $\Pi_\mu \to i\gamma^5\gamma_\mu$. The transpositions $h_{\mu\nu}$ are represented as $h_{\mu\nu} \to \pm i\gamma^5(\gamma_\mu - \gamma_\nu)/\sqrt{2}$. With the phase convention for $\Pi_\mu$ the space reflection is represented by $\Pi_s = \Pi_1 \Pi_2 \Pi_3 \to i\gamma_4$ which has imaginary eigenvalues $\pm i$. For this we denote the space parity by the eigenvalue of $-i\Pi_s$. The reflections generate the group $\mathscr{K}_{di}$, $di = 1, 2, 3, 4$, which is the central extension of the Abelian normal subgroup $\Pi_{di}$ of $W_{di}$. $\mathscr{K}_{di}$ is normal subgroup of ${}^dW_{di}$ with the factor group ${}^dW_{di}/\mathscr{K}_{di} \simeq \mathrm{Sym}_{di}$. There is a similar situation with respect to the extensions of the composite reduced spin groups. In Table 3 we give the extended reflection groups $\mathscr{K}_{di}$, $di \geqq 2$ contained as normal subgroups in the different ${}^dS_{j,0}$, together with the corresponding factor groups ${}^dS_{j,0}/\mathscr{K}_{di} \simeq F$.

We apply again the Wigner-Mackey procedure, Sect. 3.1, for the construction of the irreps of ${}^dS_{j,0}$, starting from the $\mathscr{K}_{di}$ as normal subgroups. Since we are interested in the irreps with faithful representation of the central element $\varepsilon = -1 \to -1$, we can restrict ourselves to the faithful representations of $\mathscr{K}_{di}$. These are for $\mathscr{K}_4$ the 4-dimensional representation by the $\gamma$-matrices; for $\mathscr{K}_3$ the two inequivalent 2-dimensional representations by the Pauli matrices $\Pi_i \to \sigma_i$ and $\Pi_i \to -\sigma_i$, $i = 1, 2, 3$; for $\mathscr{K}_2$ the 2-dimensional representation by Pauli matrices. We checked that the two inequivalent representations of $\mathscr{K}_3$ are not transformed into each other by transformations of ${}^dS_{j,0}$, $j = 4, 6$ (see Table 3). Thus the irreps of ${}^dS_{j,0}$ with faithful representation of the centre contain 1-point orbits in $\hat{\mathscr{K}}_{di}$. The corresponding little groups of the first kind are the ${}^dS_{j,0}$ itself. The extension of an irrep of $\mathscr{K}_{di}$ to an irrep of ${}^dS_{j,0}$, according to Sect. 3.1 (2), can be easily derived from the defining $\gamma$-matrices in ${}^dW_4 \supset {}^dS_{j,0}$. According to the Wigner-Mackey procedure the irreps of ${}^dS_{j,0}$ are given by the irreps of the little groups of second kind, i.e. of the factor groups ${}^dS_{j,0}/\mathscr{K}_{di}$, (Sect. 3.1 (4)). However, these are the well known representations of the $\mathrm{Sym}_{di}$ which already appear in the colomn '$\chi(\mathscr{L})$' of Table 2. These considerations explain the 'reduced spin' $\sigma({}^dS_{j,0})$ for $j = 1$, 2, 3, 4, 6, 7, 9, 10, 12, the case where ${}^dS_{j,0}$ contains a normal subgroup $\mathscr{K}_{di}$ generated by reflections. In the symbols for $\sigma({}^dS_{j,0})$: $(n^i)^{\pm d}_{S_{j,0}}$, (or $(n^i)^\pm$ as in Table 3), $n$ denotes the dimension of the representation:

$$n = \dim(\mathscr{K}_{di}) \dim(\mathscr{L})$$

with $\dim(\mathscr{K}_{di})$: dimension of the $\mathscr{K}$-representation, $\dim(\mathscr{L})$: dimension of the ${}^dS_{j,0}/\mathscr{K}_{di}$-representation.

The index $i$ is the same as for $\chi(\mathscr{L})$ in Table 2. The remaining sign in $(\,)^\pm$ destinguishes the two inequivalent irreps of $\mathscr{K}_3$. For $j = 3$ and $j = 6$ it coincides with the space parity.

There remain a few other cases. Direct calculations with the expressions for $\Pi_\mu = i\gamma^5\gamma_\mu$ and $h_{\mu\nu} = \pm i\gamma^5(\gamma_\mu - \gamma_\nu)$ show that ${}^dS_{j,0} \simeq \mathscr{G}\mathscr{R} = \mathscr{K}_3$, $\mathscr{K}_2$, $Z_2 \times Z_2$, $Z_2$ for the momentum stars $j = 8$, 12, 13, 14, 15, 16, 17. $\mathscr{G}\mathscr{R}$ is listed in Table 3. The representations of these groups are well known and appeared already several times in our discussion. The interpretation of ${}^dZ_2$ and ${}^d\mathrm{Sym}_2$ as $Z_2 \times Z_2$ is somewhat ambiguous. It depends on our phase conventions for $\Pi_\mu$ and $h_{\mu,\nu}$ introduced above. The fact that the central group should be represented faithfully restricts the representations of $Z_2$ and $Z_2 \times Z_2$. This is the meaning of the index '$-$' in $(1^\sigma)^-$. The extension of ${}^d\mathrm{Sym}_3$ becomes $\mathrm{Sym}_3$ by an equivalence transformation of the multiplier, (29). Thus its irreps are those of $\mathrm{Sym}_3$ with the appropriate phase. Finally we have to discuss ${}^d\mathrm{Sym}_4$, ($j = 5$). The extension ${}^d\mathscr{V}_4$ of the group $\mathscr{V}_4 = \{(1), (12)\,(34), (13)\,(24), (14)\,(23)\} \subset \mathrm{Sym}_4$ is isomorphic to $\mathscr{K}_2$. The factor group is ${}^d\mathrm{Sym}_4/\mathscr{K}_2 \simeq \mathrm{Sym}_3$. The Wigner-Mackey construction leads with help of the irreps of $\mathrm{Sym}_3$ to the irreps of ${}^d\mathrm{Sym}_4$: (2), (2'), (4).

This completes the explanation of Table 3.

(3) *Survey of the Irreps of the* LFG. Table 1 together with Table 2 and Table 3 allows a complete survey of the irreps of the LFG. For this one has to combine the variety of momentum stars $St_j$ characterized by $\bar{p}_j$ with the variety of the flavour orbits $\Theta_{j,k}$ labelled by $\bar{L}_k$, and with the variety of reduced spins $\sigma$ described in Table 2 and Table 3. We don't want to explain explicitly all the details of this procedure. It can be performed easily for each special case. Also our experience shows, that for a definite physical problem one needs to consider only a few cases. However, for orientation we give some further hints. The

number of different reduced spins, i.e. the number of inequivalent irreps of the reduced spin group $S_{j,k}$ can be found in Table 1 in the column headed by $N(\sigma)$. The number of different flavour orbits i.e. different $\bar{L}_k$ in the sixth column, multiplied by $N(\sigma)$ gives the number $N(\mathscr{D})$ of irreps of the LFG for a given momentum star and reduced spin groups. The LFG of an infinite lattice is an infinite group, the total number of its irreps, being infinite, does not have a clear meaning. Instead, the numbers $N(\mathscr{D})$ associated with the irreps of a continuous class of momentum stars give an appropriate description of the multiplicity of the irreps of the LFG.

The dimension of an irrep of the LFG is given by the following factors:

(*a*) $N(St_j)=384/N(S_j)$, the number of points of the momentum star, 384 is the order of $W_4$, $N(S_j)$ the order of the stability group $S_j$ of $St_j$.

(*b*) $N'(\Theta_{j,k})$ the number of points of the flavour orbit, multiplied by 4 in the case of $\Gamma_0$:

$$N'(\Theta_{j,k})=N(\Theta_{j,k}) \text{ for } k\neq 0,$$
$$N'(\Theta_{j,k})=4N(\Theta_{j,k}) \text{ for } k=0.$$

(*c*) $\dim(D^\sigma(S_{j,k}))$, the dimension of the representation $D^\sigma(S_{j,k})$, of the reduced spin group, which is given by the symbols $(n)_{S_{j,k}}$, $n=\dim(D^\sigma(S_{j,k}))$ in Tables 1, 2, 3.

Thus we may calculate the dimension of an irrep $\mathscr{R}(\bar{p}_j, \bar{L}_k, \sigma(S_{j,k}))$ by information given in the Tables:

$$\dim(\mathscr{R}(\bar{p}_j, \bar{L}_k, \sigma(S_{j,k})))=\frac{384}{N(S_j)} N'(\Theta_{j,k}) \dim(D^\sigma(S_{j,k})). \tag{42}$$

The explanation of the symbol $\mathscr{R}(\bar{p}_j, \bar{L}_k, \sigma(S_{j,k}))$ for the irreps of the LFG by the Tables 1–3 is rather involved. The reason for this is partly that we treat even and odd flavour orbits more or less together. It could be simplified by treating the two cases separately. The representations with even flavour orbits ('mesonic' representations) are essentially determined by the eigenvalues of commuting operators, mainly reflection operators: $T(a)$, $(d^\mu)$, $\Pi_\nu$ restricted by some symmetry. For certain purposes a notation based on this fact could be advantageous. We don't want to elaborate this possibility which is implicitly contained in our tables. The representations with odd flavour orbits ('baryonic' representations) are completely determined by the group ${}^dS_j$ of the star $St_j$ and its irreps.

(4) *Remark on the Construction of the Irreps.* The Wigner-Mackey procedure allows an explicit construction of the LFG irreps. For the general case, the explicit formulas become rather involved. We can get some simplifications by treating the cases of even and odd flavour orbits separately.

In the case of even flavour the representation $S_{j,k}^{(1)} \ni \xi \to \tilde{\Gamma}_{a',a}^{L_k}(\xi)$ in (40) is 1-dimensional. We may omit the index $a$. Then a simple calculation, using (39) and the definition of $\xi(s, L)$, results in a simplified version of (40):

$$\mathscr{U}^D(\varepsilon d^K, R)\left|\begin{matrix} k, & \sigma \\ L, & n \end{matrix}\right\rangle = e^{i\pi(e_{R\circ L}, e_K)} \sum_{n'} \left|\begin{matrix} k, & \sigma \\ R\circ L, & n' \end{matrix}\right\rangle \mathscr{D}^\sigma_{n'n}(X(s, L)), \tag{43}$$

with $X(R, L)=f^{-1}(R\circ L)\, R f(L) \in S_{j,k}$. Now we insert this in (37), and get an expression for the transformation of our standard basis

$$U(\varepsilon d^K, -\tfrac{1}{2}e_K+a, R)\left|\begin{matrix} j, & k, & \sigma \\ p, & L, & n \end{matrix}\right\rangle = e^{i(Rp, a-\frac{1}{2}e_K)} e^{i\pi(e_{L'}, e_K)} \cdot \sum_{n'} \left|\begin{matrix} j, & k, & \sigma \\ Rp, & \omega(R,p)\circ L, & n' \end{matrix}\right\rangle D^\sigma_{n'n}(X(R, L, p)), \tag{44}$$

with

$$L' = RA(p)\circ L, \quad \omega(R,p)=A^{-1}(Rp)\, RA(p) \in S_j,$$
$$X(R, L, p)=f^{-1}(\omega(R,p)\circ L)\, \omega(R,p) f(L).$$

In the case of odd flavour the simplification comes from the fact that the flavour orbit consists of a single point only. Therefore we may omit in (40) the index $L$, and we can put $f(L)=1$. Then a similar calculation as above leads to the result

$$U(\varepsilon d^K, -\tfrac{1}{2}e_K+a, R)\left|\begin{matrix} j, & 0, & \sigma \\ p, & a, & n \end{matrix}\right\rangle = e^{i(Rp, a-\frac{1}{2}e_K)} \sum_{a', n'} \left|\begin{matrix} j, & 0, & \sigma \\ Rp, & a', & n' \end{matrix}\right\rangle \cdot \tilde{\Gamma}^0_{a',a}(\varepsilon' d^{K'}, \omega(R,p))\, D^\sigma_{n'n}(\omega(R,p)), \tag{45}$$

with

$$\varepsilon' = \varepsilon\rho(A(Rp), K), \quad K' = A^{-1}(Rp)\circ K,$$
$$\omega(R,p)=A^{-1}(Rp)\, RA(p).$$

Finally we want to emphasize that in the treatment of simple problems the general transformation formulas must be rarely used. It is one of the essential points of symmetry considerations that we can use an appropriate coordinate system. Thus if we choose for $p=\bar{p}_j$ and $L=\bar{L}_k$ the formulas above simplify considerable as a consequence of the Wigner-Mackey construction.

## 4 Outlook

It was successful program to base the kinematics of quantum mechanical systems on the representation theory of its space and internal symmetry group. In this spirit we presented in this paper a rather complete discussion of the irreducible, unitary representations of the symmetry group of the lattice approximation to Dirac-Kähler fermions. According to the pattern given by the treatment of other systems with other symmetry groups, a discussion of the reduction of product representations with help of generalized Clebsch Gordan coefficients could follow. However, there is a particular important group theoretical problem connected to the lattice approximation of a continuum theory. In our consideration of staggered fermion fields as a systematic geometric approximation of the Dirac-Kähler fields in the continuum, the symmetry group of lattice fermions LFG appears in a straightforward manner as a subgroup of the symmetry group $\mathscr{G}$ of the continuum theory. Therefore one may pose the problem of how an irreducible representation of $\mathscr{G}$ restricted to the LFG decomposes into irreducible representations of the LFG. In the framework of the Wigner-Mackey theory, the 'Subgroup Theorem' by Mackey [13] helps to solve this problem. In the context of some application, we did some preliminary calculations along this lines. The irreducible representations of $\mathscr{G} \simeq \mathscr{S}\mathscr{E} \times SU(4)$ are characterized by the (imaginary) mass, spin, parity, and the $SU(4)$-multiplet character. We find that the $\mathscr{G}$-irreps with spin parity $0^\pm$, $1^\pm$, $(2^\pm)$ in the case of an $SU(4)$ singlet and an $SU(4)$ 15-plet contain the following LFG irreps with momentum star $St_4$ of Table 1:

$$
\begin{aligned}
(E^2, 0^\pm, 1)|_{\mathrm{LFG}} &= \mathscr{R}(\bar{p}_4, (0,0,0,0), (1^\pm)_{W_3}) \oplus (others)\\
(E^2, 1^\pm, 1)|_{\mathrm{LFG}} &= \mathscr{R}(\bar{p}_4, (0,0,0,0), (3^\pm)_{W_3}) \oplus (others)\\
(E^2, 2^\pm, 1)|_{\mathrm{LFG}} &= \mathscr{R}(\bar{p}_4, (0,0,0,0), (3'^\pm)_{W_3}) \oplus \mathscr{R}(\bar{p}_4, (0,0,0,0), (2^\pm)_{W_3}) \oplus (others)\\
(E^2, 0^\pm, 15)|_{\mathrm{LFG}} &= \mathscr{R}(\bar{p}_4, (0,0,0,1), (1^\mp)_{W_3}) \oplus \mathscr{R}(\bar{p}_4, (1,1,1,0), (1^\pm)_{W_3}) \oplus \mathscr{R}(\bar{p}_4, (1,1,1,1), (1^\mp)_{W_3})\\
&\oplus \mathscr{R}(\bar{p}_4, (1,0,0,0), (1'^\pm)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (1,0,0,1), (1'^\mp)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (0,1,1,0), (1'^\pm)_{D_4})\\
&\oplus \mathscr{R}(\bar{p}_4, (0,1,1,1), (1'^\mp)_{D_4}) \oplus (others)\\
(E^2, 1^+, 15)|_{\mathrm{LFG}} &= \mathscr{R}(\bar{p}_4, (0,0,0,1), (3^-)_{W_3}) \oplus \mathscr{R}(\bar{p}_4, (1,1,1,0), (3^+)_{W_3}) \oplus \mathscr{R}(\bar{p}_4, (1,1,1,1), (3^-)_{W_3})\\
&\oplus \mathscr{R}(\bar{p}_4, (1,0,0,0), (1^+)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (1,0,0,1), (1'^-)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (0,1,1,0), (1^+)_{D_4})\\
&\oplus \mathscr{R}(\bar{p}_4, (0,1,1,1), (1'^-)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (1,0,0,0), (2^-)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (1,0,0,1), (2^+)_{D_4})\\
&\oplus \mathscr{R}(\bar{p}_4, (0,1,1,0), (2^-)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (0,1,1,1), (2^+)_{D_4}) \oplus (others)\\
(E^2, 1^-, 15)|_{\mathrm{LFG}} &= \mathscr{R}(\bar{p}_4, (0,0,0,1), (3^+)_{W_3}) \oplus \mathscr{R}(\bar{p}_4, (1,1,1,0), (3^-)_{W_3}) \oplus \mathscr{R}(\bar{p}_4, (1,1,1,1), (3^+)_{W_3})\\
&\oplus \mathscr{R}(\bar{p}_4, (1,0,0,0), (1'^-)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (1,0,0,1), (1^+)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (0,1,1,0), (1'^-)_{D_4})\\
&\oplus \mathscr{R}(\bar{p}_4, (0,1,1,1), (1^+)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (1,0,0,0), (2^+)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (1,0,0,1), (2^-)_{D_4})\\
&\oplus \mathscr{R}(\bar{p}_4, (0,1,1,0), (2^+)_{D_4}) \oplus \mathscr{R}(\bar{p}_4, (0,1,1,1), (2^-)_{D_4}) \oplus (others).
\end{aligned}
$$

These results agree with other calculations [14]. The symbol '$\oplus$(*others*)' indicates that the 'continuous spectrum decomposition' of $(M^2, s^\pm, (\dim)_{SU(4)})|_{\mathrm{LFG}}$ contains many more irreps of the LFG. The notations of $W_3$-and $D_4$-representations are those explained in Sect. 3.2 (5). More complete calculations are in progress.

The importance of such 'branching rules' in the framework of lattice approximation is twofold. On the one hand, a lattice state of a particle characterized by the quantum numbers of a LFG-irrep can be associated with a continuum particle state characterized by the quantum numbers of a $\mathscr{G}$-irrep, if the LFG-irrep is contained in the $\mathscr{G}$-irrep. Such a procedure is well known from the discussion of the glueball spectrum in lattice QCD [20]. On the other hand, it is a signal of the lattice approximation being close to the continuum, if the different lattice states contained in a $\mathscr{G}$-irrep are dynamically degenerate. It is the importance of these criteria which makes it worthwhile to study the representation theory of the LFG in full generality.

One of the obvious applications of the group theoretical methods described here, is the calculation of the hadron spectrum in lattice QCD with Dirac-Kähler fermions. The aim of such an investigation should be to reproduce the equivalent of a nonrelativistic quark model with Susskind flavour. This may shed some light on the physical meaning of these new degrees of freedom associated with Dirac-Kähler quarks. We have gained some first experience with such calculations performed in strong coupling approximation combined with a resummed hopping parameter expansion [21]. There the methods described here were of great help

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