CHIRAL ANOMALY FROM THE FOKKER–PLANCK FORMALISM

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We show how chiral anomalies arise in the Fokker–Planck formulation of stochastic quantization. Starting from a noise correlation function which is non-local in real space–time, a gauge invariantly regularized Fokker–Planck hamiltonian is derived and used to compute the anomaly.

1. Introduction. Recently various authors [1,2] treated the problem of anomalous symmetry breaking in the framework of stochastic quantization [2–5]. All these derivations of the chiral anomaly, for instance, have been done using the Langevin formulation. In this letter we look at the same problem from the Fokker–Planck point of view. At first glance, it might seem that this can be done trivially because the relationship between the Fokker–Planck and the Langevin formulation is well established. However, actually this is not the case. The reason is that, to regulate the quantum field theory, one uses the Breit–Gupta–Zaks [3] regularized noise which basically makes the process non-Markov. Thus, it is not evident if there is a Fokker–Planck formulation at all because the equivalence between the two formulations is defined by a single-"time" equation [4]. To circumvent this problem we propose to use the following different regularization scheme: instead of smearing out the noise correlation in the τ-direction, we replace the space–time δ-function by a smooth, Lorentz-invariant regulator function. This leads to a Fokker–Planck hamiltonian which is a non-local, second-order functional operator. Calculating the anomaly in this scheme turns out to be basically equivalent to the well-known point-splitting method [6,7].

2. The gauge-invariant, regularized Fokker–Planck hamiltonian. In this section we will derive the gauge-invariant, regularized Fokker–Planck hamiltonian using the canonical procedure.

For illustrative purposes consider the euclideanized action of massless QED$_4$

$$S=-\int d^4x\,\bar{\psi}(\mathrm{i}\not{D})\psi\,, \tag{1}$$

where $\not{D}=\not{\partial}+\mathrm{i}\not{A}$ and we follow the convention $g_{\mu\nu}=-\delta_{\mu\nu}$, $\gamma_\mu^+=-\gamma_\mu$. The Langevin equations are then

$$\partial\psi/\partial\tau=-\not{D}^2\psi+\mathrm{i}\not{D}\eta\,,\quad \partial\bar{\psi}/\partial\tau=-\bar{\psi}\overleftarrow{\not{D}}^2+\bar{\eta}\,, \tag{2a, b}$$

where $\overleftarrow{\not{D}}=\overleftarrow{\not{\partial}}-\mathrm{i}\not{A}$. The Langevin equations (2a), (2b) are gauge invariant provided the noise terms η and $\bar{\eta}$ transform like ψ and $\bar{\psi}$, respectively.

At this point, we specify how the noise is regularized. It is a common practice to regularize the noise in the τ direction. As pointed out already in the introduction, this makes the stochastic process non-Markov. As an alternative, we propose to regulate the noise in the x_μ direction. Because of the gauge transformation properties of η and $\bar{\eta}$, we propose the following correlation:

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$$\langle \eta_\alpha(x,\tau)\bar{\eta}_\beta(x',\tau')\rangle = 2\delta_{\alpha\beta}\delta(\tau-\tau')\beta_\Lambda(x-x')\phi(x,x') , \tag{3}$$

where $\beta_\Lambda(x-x')$ is a Lorentz-invariant smearing function which approaches the Dirac delta as Λ (which has the dimensionality of a mass) approaches ∞. We also assume that $\int\beta_\Lambda(x-x')\mathrm{d}^4x' = 1$. The phase factor

$$\phi(x,x') = \exp\left(\mathrm{i}\int_x^{x'} A\cdot\mathrm{d}y\right) \tag{4}$$

is necessary because of the gauge transformation properties of the noises. The correlation (3) implies that the noises have the distribution

$$\exp\left(-\frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^4x\,\mathrm{d}^4x'\;\bar{\eta}(x,\tau)\beta_\Lambda(x-x')\phi(x,x')\eta(x',\tau)\right).$$

To derive the Fokker–Planck hamiltonian, we consider the Wiener integral corresponding to the Langevin process given by (2a), (2b).

$$\langle \mathrm{f},\tau=\tau_\mathrm{f}|\mathrm{i},\tau=0\rangle = \int_{\substack{\psi_\eta(x,0)=\psi_\mathrm{i}(x)\\ \psi_\eta(x,\tau)=\psi_\mathrm{f}(x)}} [\mathrm{d}\bar{\eta}(x,\tau)]\;[\mathrm{d}\eta(x,\tau)]$$

$$\times\exp\left(-\frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^4x\,\mathrm{d}^4x'\;\bar{\eta}(x,\tau)\beta_\Lambda(x-x')\phi(x,x')\eta(x',\tau)\right). \tag{5}$$

Using the Langevin equations, we transform this into integrals over 4 and $\bar{4}$.

$$\langle \mathrm{f},\tau=\tau_\mathrm{f}|\mathrm{i},\tau=0\rangle = \int_{\text{end points}} [\mathrm{d}\bar{\psi}(x,\tau)]\;[\mathrm{d}\psi(x,\tau)]J\left(\frac{\eta,\bar{\eta}}{\psi,\bar{\psi}}\right)$$

$$\times\exp\left(-\frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^4x\,\mathrm{d}^4x'\,\bar{\psi}[\overleftarrow{\partial}/\partial\tau+\overleftarrow{\not{D}}^2]\beta_\Lambda(x-x')\phi(x,x')[(1/\mathrm{i}\not{D})(\partial/\partial\tau+\not{D}^2)\psi]_{x',\tau}\right), \tag{6}$$

and the jacobian of the transformation is

$$J = \det\begin{bmatrix} g(x-x')[\partial/\partial\tau+\not{D}^2(x')]\delta(\tau-\tau') & 0 \\ 0 & \delta(x-x')[\partial/\partial\tau+\overleftarrow{\not{D}}^2(x')]\delta(\tau-\tau') \end{bmatrix}. \tag{7}$$

In (7), $g(x-x')$ is the Green function of $\mathrm{i}\not{D}$. Factoring out

$$J = \det\begin{bmatrix} g(x-x')(\partial/\partial\tau)\delta(\tau-\tau') & 0 \\ 0 & (\partial/\partial\tau)\delta(x-x')\delta(\tau-\tau') \end{bmatrix},$$

and using the midpoint rule $\Theta(0)=\frac{1}{2}$, we find

$$J\sim\exp\left(\tfrac{1}{2}\delta^4(0)\int \mathrm{d}\tau\,\mathrm{d}^4x[\not{D}_x^2+\overleftarrow{\not{D}}_x^2]\right). \tag{8}$$

From (8) and the exponential term in (6) we read off the Fokker–Planck lagrangian [5]

$$L_{\mathrm{FP}} = \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'[\partial\bar{\psi}/\partial\tau+\bar{\psi}\overleftarrow{\not{D}}^2]_x\beta_\Lambda(x-x')\phi(x,x')[(1/\mathrm{i}\not{D})(\partial\psi/\partial\tau+\not{D}^2\psi)]_{x'} - \tfrac{1}{2}\delta^4(0)\int \mathrm{d}^4x[\not{D}_x^2+\overleftarrow{\not{D}}_x^2]. \tag{9}$$

We now use the canonical procedure. The conjugate momenta are

$$\pi_{\psi}=\frac{\delta L_{\mathrm{FP}}}{\delta(\partial_{\tau}\psi)}=\frac{1}{2}\int \mathrm{d}^4x_1\ \mathrm{d}^4x_2[\partial\bar{\psi}/\partial\tau+\bar{\psi}\overleftarrow{\not{D}}^2]_{x_1}\beta_{\Lambda}(x_1-x_2)\phi(x_1,x_2)g(x_2-x)\ , \tag{10a}$$

$$\pi_{\bar{\psi}}=\frac{\delta L_{\mathrm{FP}}}{\delta(\partial_{\tau}\bar{\psi})}=\frac{1}{2}\int \mathrm{d}^4x_1\ \mathrm{d}^4x_2\beta_{\Lambda}(x-x_1)\phi(x,x_1)g(x_1-x_2)(\partial\psi/\partial\tau+\not{D}^2\psi)_{x_2}\ . \tag{10b}$$

To solve for the velocities, we use

$$\int \mathrm{d}^4x_1\beta_{\Lambda}(x-x_1)\phi(x,x_1)\beta_{\Lambda}^{-1}(x_1-x_2)\phi(x_1,x_2)=\delta^4(x-x_2)\ .$$

The "hamiltonian" is given by the Legendre transformation

$$H_{\mathrm{FP}}=\int \mathrm{d}^4x(\pi_{\psi_\alpha}\partial\psi_\alpha/\partial\tau+\pi_{\bar{\psi}_\alpha}\partial\bar{\psi}_\alpha/\partial\tau)-L_{\mathrm{FP}}\ , \tag{11}$$

and after operator ordering gives

$$\hat{H}_{\mathrm{FP}}=\int \mathrm{d}^4x\,\mathrm{d}^4x'\left(\frac{\delta}{\delta\psi(x)}\mathrm{i}\not{D}_x[\beta_{\Lambda}^{-1}(x-x')\phi(x,x')]\frac{\delta}{\delta\bar{\psi}(x')}-\frac{\delta}{\delta\bar{\psi}(x)}[\phi^{-1}(x,x')\beta_{\Lambda}^{-1}(x-x')]\mathrm{i}\overleftarrow{\not{D}}_x\frac{\delta}{\delta\psi(x')}\right)$$
$$+\int \mathrm{d}^4x\left(\frac{\delta}{\delta\psi(x)}\not{D}_x^2\psi(x)+\frac{\delta}{\delta\bar{\psi}(x)}[\bar{\psi}(x)\overleftarrow{\not{D}}_x^2]\right)\ . \tag{12}$$

Eq. (12) is the gauge-invariant, regularized Fokker–Planck hamiltonian. Its Lorentz invariance is manifest since β_Λ is required to be an invariant function: $\beta_\Lambda(ax)=\beta_\Lambda(x)$ for any Lorentz transformation $a^{\mu}{}_{\nu}$. By doing a similarity transformation

$$\hat{H}'_{\mathrm{FP}}=\mathrm{e}^{\hat{A}}\hat{H}_{\mathrm{FP}}\mathrm{e}^{-\hat{A}}\ , \tag{13a}$$

$$\hat{A}=\int \mathrm{d}^4x\,\mathrm{d}^4x'\frac{\delta}{\delta\psi(x)}g(x-x')\beta_{\Lambda}^{-1}(x-x')\phi(x,x')\frac{\delta}{\delta\bar{\psi}(x')}\ , \tag{13b}$$

we find

$$\hat{H}'_{\mathrm{FP}}=\int \mathrm{d}^4x\left(\frac{\delta}{\delta\psi(x)}\not{D}_x^2\psi(x)+\frac{\delta}{\delta\bar{\psi}(x)}[\bar{\psi}(x)\overleftarrow{\not{D}}_x^2]\right)\ . \tag{14}$$

Note that $\hat{H}'_{\mathrm{FP}}$ does not depend on the regulator and thus its spectrum is independent of Λ. Sakita had shown that $\hat{H}'_{\mathrm{FP}}$ is a positive-definite operator and thus it is also true that the point-splitted, gauge-invariant Fokker–Planck hamiltonian (12) is also positive definite. Another prerequisite is the existence of a mass gap between the zero mode and the non-zero modes. Making the usual assumption that $\hat{H}'_{\mathrm{FP}}$ has a mass gap then $\hat{H}_{\mathrm{FP}}$ also has a mass gap.

3. Derivation of the anomaly. We determine the anomaly as the jacobian $J[\alpha]$ associated with an infinitesimal chiral transformation

$$\psi'(x)=\exp[\mathrm{i}\alpha(x)\gamma_5]\ \psi(x)\ ,\quad \bar{\psi}'(x)=\bar{\psi}(x)\ \exp[\mathrm{i}\alpha(x)\gamma_5]\ . \tag{15}$$

For this jacobian we make the ansatz

$$J[\alpha]=\exp\left(-\mathrm{i}\int \mathrm{d}^4x\ \alpha(x)\mathscr{A}(x)\right)\ . \tag{16}$$

Now one considers the normalization condition

$$\int [\mathrm{d}\psi]\,[\mathrm{d}\bar{\psi}]\,P[\psi,\bar{\psi};\tau]=1\,, \tag{17}$$

and changes the integration variables from ψ and $\bar{\psi}$ to ψ' and $\bar{\psi}'$ according to (15) for infinitesimal $\alpha(x)$. This implies

$$\mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\mathscr{A}(x)=\int [\mathrm{d}\psi]\,[\mathrm{d}\bar{\psi}]\,\delta P[\psi,\bar{\psi};\tau]\,, \tag{18}$$

where $\delta P[\psi,\bar{\psi};\tau]=P[\psi',\bar{\psi}';\tau]-P[\psi,\bar{\psi};\tau]$. To make this equation more transparent, we introduce a complete set $\{F_n[\psi,\bar{\psi}]\}$ of eigenfunctionals of the Fokker–Planck hamiltonian: $\hat{H}_{\mathrm{FP}}F_n=E_nF_n$. In terms of the F_n the distribution function may be expanded as

$$P[\psi,\bar{\psi};\tau]=\sum_n c_n\exp(-E_n\tau)F_n[\psi,\bar{\psi}]\,. \tag{19}$$

The coefficients $\{c_n\}$ are to be determined from the initial conditions. Thus we have

$$\mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\mathscr{A}(x)=\sum_n c_n\exp(-E_n\tau)\int [\mathrm{d}\psi]\,[\mathrm{d}\bar{\psi}]\delta F_n[\psi,\bar{\psi}]\,. \tag{20}$$

This equation nicely exhibits the τ-evolution of the anomaly. Since in general the eigenfunctionals F_n for $E_n>0$ are not known explicitly, we can evaluate the RHS of (10) only in the limit $\tau\to\infty$. In this case only the zero-mode $F_0\equiv Z^{-1}\exp(-S_\epsilon)$ with the partition function $Z\equiv\int[\mathrm{d}\psi][\mathrm{d}\bar{\psi}]\exp(-S_\epsilon)$ contributes. Hence one obtains

$$\mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\mathscr{A}(x)=-Z^{-1}\int [\mathrm{d}\psi]\,[\mathrm{d}\bar{\psi}]\,\delta S_\epsilon\exp(-S_\epsilon)=-\langle\delta S_\epsilon\rangle\,. \tag{21}$$

Contrary to Fujikawa's [8] approach, where $\mathscr{A}$ is evaluated directly from the definition of the jacobian $\delta(\psi',\bar{\psi}')/\delta(\psi,\bar{\psi})$, we here determine $\mathscr{A}$ by calculating $\langle\delta S_\epsilon\rangle$. From the hamiltonian (12) it is easy to see that S_ϵ is given by

$$S_\epsilon=-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4\epsilon\,\beta_\Lambda(\epsilon)[\bar{\psi}(x+\epsilon)\phi(x+\epsilon,x)\mathrm{i}\not{D}_x\psi(x)+\bar{\psi}(x)\mathrm{i}\not{D}_x\phi(x,x+\epsilon)\psi(x+\epsilon)]\,. \tag{22}$$

Recalling that for $\Lambda\to\infty$, the Lorentz invariant function β_Λ approaches a δ-function, we may replace $\lim_{\Lambda\to\infty}\int\mathrm{d}^4\epsilon\times\beta_\Lambda(\epsilon)(...)$ by $\lim_{\epsilon\to0}(...)$ where an average over the directions of ϵ^μ according to $\epsilon^\mu\epsilon^\nu/\epsilon^2\to g^{\mu\nu}/4$, etc. is understood. Expanding the bracket of (22) to first order in ϵ, which is equivalent to the first order in $1/\Lambda$, one obtains

$$S_\epsilon=-\lim_{\epsilon\to0}\int \mathrm{d}^4x\,\bar{\psi}(x+\epsilon/2)[\mathrm{i}\not{\partial}-\not{A}+\epsilon^\mu A_\mu\not{\partial}+\mathrm{i}\not{A}\epsilon^\mu A_\mu+\tfrac{1}{2}(\not{\partial}\epsilon^\mu A_\mu)]\psi(x-\epsilon/2)\,. \tag{23}$$

The change of (23) under an infinitesimal chiral rotation reads

$$\delta S_\epsilon=-\int \mathrm{d}^4x\,\bar{\psi}(x+\epsilon/2)[-(\not{\partial}\alpha)+\epsilon^\mu(\partial_\mu\alpha)(\not{\partial}+\mathrm{i}\not{A})+\tfrac{1}{2}\not{\partial}(\epsilon^\mu\partial_\mu\alpha)+\mathrm{i}\epsilon^\mu A_\mu(\not{\partial}\alpha)]\gamma_5\psi(x-\epsilon/2)\,. \tag{24}$$

Next, one has to compute the vacuum expectation value of (24) in the limit $\epsilon\to0$. Using the well-known matrix element [7]

$$\langle\bar{\psi}(x+\epsilon/2)\gamma_\mu\gamma_5\psi(x-\epsilon/2)\rangle=(1/4\pi^2)(\epsilon^\nu/\epsilon^2)E_{\nu\mu\alpha\beta}F^{\alpha\beta}-\mathrm{O}(\epsilon^0)\,, \tag{25}$$

and the Heisenberg equations for ψ it is straightforward to arrive at

$$\langle\delta S_\epsilon\rangle=-\mathrm{i}\lim_{\epsilon\to0}\int \mathrm{d}^4x\,F_{\mu\nu}\epsilon^\mu\langle\bar{\psi}(x+\epsilon/2)\gamma^\mu\gamma_5\psi(x-\epsilon/2)\rangle=-\mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\mathscr{A}(x)\,, \tag{26}$$

with

$$\mathscr{A}(x)=(1/8\pi^2)F^*_{\mu\nu}F^{\mu\nu} \tag{27}$$

Together with (16) this yields the standard result [8] for the jacobian of the chiral transformations (15):

$$J[\alpha]=\exp\left(-\frac{i}{8\pi^2}\int d^4x\, F^*_{\mu\nu}F^{\mu\nu}\right). \tag{28}$$

To obtain the divergence of the axial vector current one notes that using the equations of motion for ψ the expectation value of (24) can be written as

$$\langle \delta S_\epsilon \rangle = -\int d^4x\, \alpha(x) \lim_{\epsilon\to 0} \partial_\mu \langle \bar\psi(x+\epsilon/2)\gamma^\mu\gamma_5\phi(x+\epsilon/2, x-\epsilon/2)\psi(x-\epsilon/2)\rangle . \tag{29}$$

Inserting this together with (27) into (21), one ends up with the desired relation [7]

$$\lim_{\epsilon\to 0} \partial_\mu \langle \bar\psi(x+\epsilon/2)\gamma^\mu\gamma_5\phi(x+\epsilon/2, x-\epsilon/2)\psi(x-\epsilon/2)\rangle = (i/8\pi^2)F^*_{\mu\nu}(x)F^{\mu\nu}(x) . \tag{30}$$

This completes the proof that the Fokker–Planck dynamics described by our $\hat H_{FP}$ leads to the correct anomaly for $\tau\to\infty$.

4. Conclusions. In this letter we proposed a new way of regulating the noise in stochastic quantization. We find that this scheme gives rise to a (gauge-invariant) Fokker–Planck hamiltonian whereas in the Breit–Gupta–Zaks scheme it is not evident if there is a simple Fokker–Planck formulation. As a first application, we derived the chiral anomaly of QED_4. The generalization to more complicated theories is in principle straightforward.

Let us now briefly discuss other possible applications of the regulator considered here. Looking at the regularized action (23) it is clear that a perturbation theory based upon the propagator associated wit S_ϵ would be quite complicated and not very advantageous. On the other hand, if one does not evaluate single graphs but, as in the case of effective actions, say, infinite sums of Feynman diagrams, the situation is different. Within the present formalism the one-loop effective action of QED is obtained as follows. Following the discussion of Gozzi [5], for instance, it is obvious that stochastic averages of $\psi_\eta(x,\tau)$ and $\bar\psi_\eta(x,\tau)$ can be computed as derivatives of a generating functional which can be represented by a path integral whose basic ingredient is the Fokker–Planck lagrangian L_{FP}. In general the corresponding effective action is obtained by the usual Legendre transformation. If we restrict ourselves to an external vector field, we obtain from the lagrangian (9) for this (τ-dependent) effective action

$$\Gamma_\tau[A]=\mathrm{Tr}\ln[\not{D}^{-1}(-d^2/d\tau^2+\not{D}^4)]_{\rm reg} \tag{31a}$$

$$=-\tfrac{1}{2}\,\mathrm{Tr}\ln\not{D}^2_{\rm reg}+\mathrm{Tr}\ln(-d^2/d\tau^2+\not{D}^4_{\rm reg}) , \tag{31b}$$

where "reg" means that the respective matrix elements are convoluted with $\beta_\Lambda(x-x')\phi(x,x')$. To evaluate the trace with respect to the τ-dependence in the second term of (31b) we follow Haba [9] and impose periodic boundary conditions for τ on the interval $[-T, T]$ and finally perform the limit $T\to\infty$; a simple calculation yields [9]

$$\mathrm{Tr}\ln(-d^2/d\tau^2+\not{D}^4_{\rm reg})=\tfrac{1}{2}\,\mathrm{Tr}\ln\not{D}^4_{\rm reg}+\mathrm{Tr}\ln[1-\exp(-4T\not{D}^2_{\rm reg})] . \tag{32}$$

Therefore the field theory (Heisenberg–Euler) effective action is given by

$$\Gamma_{HE}[A]=\lim_{T\to\infty}\Gamma_T[A]=\tfrac{1}{2}\,\mathrm{Tr}\ln\not{D}^2_{\rm reg} . \tag{33}$$

Using an integral representation for the logarithm, this explicitly reads (for $\Lambda\to\infty$):

$$\Gamma_{HE}[A]=-\frac{1}{2}\int d^4x\, d^4x'\, \beta_\Lambda(x-x')\phi(x,x')\,\mathrm{tr}\int_0^\infty \frac{dt}{t}\langle x|\exp(-t\not{D}^2)|x'\rangle . \tag{34}$$

Obviously, at this level the effect of the regulator introduced into the noise correlation function is to replace the trace of the heat-kernel $\langle x|\exp(-t\not{D}^2)|x'\rangle$ by a quantity where the coincidence limit has not yet been performed. The expression (34) could be further evaluated using standard methods [10]. Finally we mention that the point-splitting regularization of effective actions is particularly useful in curved space–times [11].

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