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Path integrals on curved manifolds

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Abstract. A general framework for treating path integrals on curved manifolds is presented. We also show how to perform general coordinate and space-time transformations in path integrals. The main result is that one has to subtract a quantum correction $\Delta V \sim \hbar^2$ from the classical Lagrangian $\mathscr{L}$, i.e. the correct effective Lagrangian to be used in the path integral is $\mathscr{L}_{\text{eff}} = \mathscr{L} - \Delta V$. A general prescription for calculating the quantum correction ΔV is given. It is based on a canonical approach using Weyl-ordering and the Hamiltonian path integral defined by the midpoint prescription. The general framework is illustrated by several examples: The d-dimensional rotator, i.e. the motion on the sphere S^{d-1}, the path integral in d-dimensional polar coordinates, the exact treatment of the hydrogen atom in $\mathbf{R}^2$ and $\mathbf{R}^3$ by performing a Kustaanheimo-Stiefel transformation, the Langer transformation and the path integral for the Morse potential.

I Introduction

A lot of problems in theoretical physics make it desirable to have a precise formulation of path integrals on curved manifolds. Approaches towards a general theory exist due to DeWitt [4], McLaughlin and Schulman [28], Dowker and Mayes [8], Mizrahi [29], Gervais and Jevicki [11], Omote [30], Marinov and Terentyev [27], Marinov [26] and Lee [24]. The main result of these discussions is that one has to subtract from the original Lagrangian $\mathscr{L}$ some quantum correction $\Delta V \sim \hbar^2$:

$$\mathscr{L}_{\text{eff}} = \mathscr{L} - \Delta V \tag{1}$$

where $\mathscr{L}_{\text{eff}}$ is the correct expression to be used in path integrals on curved manifolds. Unfortunately, the expressions for ΔV derived by the above authors apparently do not agree. This confusion arises mainly because different lattice definitions for the path integral are used; e.g. DeWitt prefers a prepoint formulation, whereas Mizrahi and Lee use midpoints.

For special coordinate-transformations it is possible to calculate ΔV by expanding $(\Delta x)^2 \to f(\Delta q)$ up to forth order in Δq (e.g. x-cartesian coordinates, q-new (curved) coordinates), where Δx, Δq denote coordinate differences in a discrete version of the path integral. Examples have been discussed by Gerry and Inomata [10], Inomata [18] and Steiner [39].

Quantum corrections are also known for the rotator in three dimensions (motion on the S^2-sphere). They were observed first by Gutzwiller [15] and then by Patrascioiu [34]; the latter discusses this problem in the connection with lattice gauge theories.

In this paper we apply the general theory to the rotator in d dimensions (motion on the S^{d-1}-sphere), polar coordinates in d dimensions, the Kustaanheimo-Stiefel transformation in $\mathbf{R}^2$ [19] and $\mathbf{R}^4$ [7] and some further examples of one dimensional path integral problems which have become important in recent years, i.e. the Langer-transformation in a radial path integral for a semiclassical treatment of the hydrogen atom [10], the Morse-potential [6], the Coulomb problem in polar coordinates [17, 40] and general space-time transformations in radial path integrals [39]. Whereas the path integral in d-dimensional polar coordinates can be directly derived by a coordinate transformation from cartesian coordinates to polar coordinates, this is not so easy in the case of the rotator. Usually one constructs its path integral from the d-dimensional radial path integral and takes the constraint $\mathbf{x}^2 = R^2$ (R-radius of the S^{d-1}-sphere) into account by delta-functions (see e.g. [2, 37]). But this approach does not respect the Riemannian structure of the S^{d-1}-sphere. The S^{d-1}-sphere has constant positive curvature $R^{(d)} = (d-1)(d-2)/R^2$, whereas Euclidian space remains flat in whatever coordinates expressed (e.g. polar coordinates). It is quite astonishing that no correct path integral formulation for the rotator has been given up to now. The rotator is one

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of the simplest examples of a curved manifold, and, furthermore, its solution is exactly known.

Our general prescription for obtaining the correct path integral is the following. Let us consider the generic case, where the classical Lagrangian has the following form

$$\mathscr{L}(q,\dot{q})=\frac{m}{2}g_{ab}(q)\dot{q}^a\dot{q}^b-V(q). \tag{2}$$

Here g_{ab} is the metric tensor corresponding to the line element $ds^2=g_{ab}dq^a dq^b$. Then the quantum Hamiltonian reads:

$$H=-\frac{1}{2m}\Delta_{\mathrm{LB}}+V(q) \tag{3}$$

where the Laplacian (Laplace-Beltrami operator) is given by ($\hbar=1$, $g:=\det(g_{ab})$):

$$\Delta_{\mathrm{LB}}=\frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b. \tag{4}$$

In order to write down the Hamiltonian path integral, we first have to construct momentum operators [26]

$$p_a=-i(\partial_a+\Gamma_a/2),\quad \Gamma_a=\partial_a\ln\sqrt{g} \tag{5}$$

which are hermitian with respect to the scalar product

$$(f_1,f_2)=\int f_1^* f_2\sqrt{g}dq. \tag{6}$$

In terms of the momentum operators (5) we define a hermitian Hamiltonian by using the Weyl-ordering prescription [24, 29]:

$$H=\frac{1}{8m}(g^{ab}p_ap_b+2p_ag^{ab}p_b+p_ap_bg^{ab})+\Delta V(q)+V(q). \tag{7}$$

In (7) appears a well-defined quantum correction which is given by

$$\Delta V=\frac{1}{8m}(g^{ab}\Gamma^d_{ac}\Gamma^c_{bd}-R)=\frac{1}{8m}[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_a)_{,b}+g^{ab}_{\ ,ab}] \tag{8}$$

(R-scalar curvature; Γ^a_{bc}-Christoffel symbols, see Appendix A). Using the Trotter formula

$$e^{-itH}=e^{-it(A+B)}=s-\lim_{N\to\infty}(e^{-itA/N}e^{-itB/N})^N \tag{9}$$

(see, e.g. [38]) and the short time approximation to the matrix element $\langle q''|e^{-i\varepsilon H}|q'\rangle$ one obtains the Hamiltonian path integral ($\tau:=t''-t'$)

$$K(q'',q';\tau)=C\int\mathscr{D}q(t)\int\mathscr{D}p(t)\exp\left\{i\int_{t'}^{t''}[p\dot{q}-\mathscr{H}(p,q)]dt\right\} \tag{10}$$

where the normalisation C is given by (see e.g. [30])

$$C=[g(q')g(q'')]^{-1/4}. \tag{11}$$

Here the path integral is defined on a lattice using the midpoint prescription: $\bar{q}^{(j)}:=\frac{1}{2}(q^{(j)}+q^{(j-1)})$, $q^{(j)}=q(t_j)$, $t_j=t'+j\varepsilon$, $\varepsilon=(t''-t')/N$, $N\to\infty$. With this prescription the "classical" Hamiltonian to be used in the path integral (10) is on the lattice

$$\mathscr{H}(p^{(j)},\bar{q}^{(j)})=\frac{1}{2m}g^{ab}(\bar{q}^{(j)})p_a^{(j)}p_b^{(j)}+V(\bar{q}^{(j)})+\Delta V(\bar{q}^{(j)}). \tag{12}$$

Clearly, other lattice definitions of (10), like prepoint, postpoint or something in between, can be formulated. General considerations connecting these lattices with the appropriate quantum corrections can be found e.g. in [8, 16, 23]. A rigorous approach, but with the focal point on the stochastic nature of diffusion processes can be found in [5]. Finally, the role of operator ordering in quantum field theory is duscussed in [42]. The Weyl-ordering and midpoint prescription, however, express most clearly all the symmetries of the classical Lagrangian. This can be seen by an analysis of equation (7), which has been done e.g. by Leschke and Schmutz [25] and Omote and Sato [31].

Our paper is organised as follows:

In Sect. II we present the calculation for the rotator in d dimensions. We calculate ΔV, but use instead of midpoints a "product form" on the lattice, i.e. we set $\sin^2\theta\to\sin\theta^{(j)}\sin\theta^{(j-1)}$ etc. instead of $\sin^2\theta\to\sin^2\bar{\theta}^{(j)}$. These two lattice formulations turn out to be equivalent with the same ΔV. Our lattice definition makes the Lagrangian path integral simpler, but the choice of the lattice remains nevertheless a matter of taste. We then use an identity under the path integral to obtain an equivalent, but even simpler Lagrangian path integral for the rotator.

In Sect. III we shall discuss the path integral in d-dimensional polar coordinates, and in Sect. IV we shall treat the other transformations which we already noted. In Sect. V we shall discuss our results.

In Appendix A we list the various quantum corrections derived by previous authors. Appendices B–D contain the detailed proofs for deriving the Schrödinger equation from the short time kernels corresponding to different path integral representations.

II The path integral for the d-dimensional rotator

We are considering the time-dependent Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi_l^\mu=-\frac{1}{2mR^2}L^2_{(d)}\psi_l^\mu \tag{1}$$

($L^2_{(d)}$ is the Legendre operator in d dimensions) in d-dimensional polar coordinates (see Erdelyi et al. [9]):

$$x_1=R\cos\theta_1$$
$$x_2=R\sin\theta_1\cos\theta_2$$

$$x_3 = R\sin\theta_1 \sin\theta_2 \cos\theta_3 \tag{2}$$
$$\cdots$$
$$x_{d-1} = R\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-2}\cos\phi$$
$$x_d = R\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-2}\sin\phi$$

where $0 \leqq \theta_\nu \leqq \pi (\nu = 1, \ldots, d-2)$, $0 \leqq \phi \leqq 2\pi$, $R = (\sum_{\nu=1}^{d} x_\nu^2)^{1/2}$ fixed. (We shall often also use $\theta_{d-1} = \phi$.) Then with $H = -(1/2mR^2)L^2_{(d)}$:

$$H = -\frac{1}{2mR^2}\left\{\left[\frac{\partial^2}{\partial\theta_1^2} + (d-2)\cot\theta_1\frac{\partial}{\partial\theta_1}\right] + \frac{1}{\sin^2\theta_1}\left[\frac{\partial^2}{\partial\theta_2^2} + (d-3)\cot\theta_2\frac{\partial}{\partial\theta_2}\right] + \cdots + \frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{d-3}}\left[\frac{\partial^2}{\partial\theta_{d-2}^2} + \cot\theta_{d-2}\frac{\partial}{\partial\theta_{d-2}}\right] + \frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{d-2}}\frac{\partial^2}{\partial\phi^2}\right\}. \tag{3}$$

The time-independent Schrödinger equation reads

$$-\frac{1}{2mR^2}L^2_{(d)}\psi_l^\mu = E_l\psi_l^\mu. \tag{4}$$

For the eigenvalues E_l one obtains:

$$E_l = \frac{1}{2mR^2}l(l+d-2), \quad (l = 0, 1, 2, \ldots), \tag{5}$$

whereas the eigenfunctions are given by

$$\psi_l^\mu = S_l^\mu(\Omega) \tag{6}$$

where $S_l^\mu(\Omega)$ are the real hyperspherical harmonics of degree l with unit vector Ω and $l \in \mathbf{N}_0$, $\mu = 1, 2, \ldots, M$, $M = (2l+d-2)(l+d-3)!/l!(d-2)!$.

For later purposes let us consider for a moment the case where R is not fixed, $R = r$, in which case the Hamiltonian is proportional to the d-dimensional Laplacian $\Delta_{(d)}$

$$\Delta_{(d)} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}L^2_{(d)}. \tag{7}$$

Rewriting the Hamiltonian (3) yields:*

$$H(\{p_\theta, \theta\}) = \frac{1}{2mR^2}\left[p_{\theta_1}^2 + \frac{1}{\sin^2\theta_1}p_{\theta_2}^2 + \cdots + \frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{d-2}}p_\phi^2\right] + \Delta V(\{\theta\}) \tag{8}$$

with

$$\Delta V(\{\theta\}) = -\frac{1}{8mR^2}\left[(d-2)^2 + \frac{1}{\sin^2\theta_1} + \cdots + \frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{d-2}}\right] \tag{9}$$

* In H no ordering ambiguity arises, because of the special nature of g_{ab} for the rotator

and the hermitian momenta

$$\left.\begin{aligned} p_r &= \frac{1}{i}\left(\frac{\partial}{\partial r} + \frac{d-1}{2r}\right) \\ p_{\theta_\nu} &= \frac{1}{i}\left(\frac{\partial}{\partial\theta_\nu} + \frac{d-1-\nu}{2}\cot\theta_\nu\right) \\ p_\phi &= \frac{1}{i}\frac{\partial}{\partial\phi}. \end{aligned}\right\} \tag{10}$$

With the correct Hamiltonian (8) we can consider the Hamiltonian path integral for the Feynman kernel K:

$$K(\{\theta''\}, \{\theta'\}; t''-t') = C\int\mathscr{D}\{\theta\}\mathscr{D}\{p_\theta\}\exp\left\{i\int_{t'}^{t''}\left[\sum_{\nu=1}^{d-1}p_{\theta_\nu}\cdot\dot{\theta}_\nu - H(\{p_\theta, \theta\})\right]dt\right\} \tag{11}$$

with

$$\mathscr{D}\{\theta\} \to \sum_{j=1}^{N-1}(d\theta_1^{(j)}\cdots d\theta_{d-1}^{(j)}),$$

$$\mathscr{D}\{p_\theta\} \to \prod_{j=1}^{N}(2\pi)^{1-d}(dp_{\theta_1^{(j)}}\cdots dp_{\theta_{d-1}^{(j)}})$$

in the lattice formulation ($N \to \infty$). Notice that the p_θ-integrations are N-fold, whereas the θ-integrations are only $(N-1)$-fold, and, furthermore that the path integration measure $\mathscr{D}\{\theta\}\mathscr{D}\{p_\theta\}$ is in general not invariant under canonical transformations. In the following we shall use Feynman's lattice definition, i.e. $\dot{\theta} \to (\theta_\nu^{(j)} - \theta_\nu^{(j-1)})/\varepsilon$ etc, supplemented by the prescription $\sin^2\theta_\nu \to \sin\theta_\nu^{(j)}\sin\theta_\nu^{(j-1)}$. Below we shall justify this prescription by showing that it leads to the correct Schrödinger equation. This lattice definition is equivalent to the midpoint prescription. This can be seen by the identity (following DeWitt [4], we use the symbol $\doteq$ to denote "equivalence as far as use in the path integral is concerned"):

$$\prod_{j=1}^{N-1}\sqrt{g^{(j)}}\cdot\exp\left\{i\varepsilon\sum_{j=1}^{N}\mathscr{L}_{\mathrm{Cl}}^N(\{\theta^{(j)}, \theta^{(j-1)}\})\right\} \doteq (g'g'')^{-1/4}\prod_{j=1}^{N}\sqrt{\bar{g}^{(j)}}\cdot\exp\{i\varepsilon\mathscr{L}_{\mathrm{Cl}}^N(\{\bar{\theta}^{(j)}\})\} \tag{12}$$

where

$$\mathscr{L}_{\mathrm{Cl}}^N(\{\theta^{(j)}\}, \{\theta^{(j-1)}\}) = \frac{mR^2}{2\varepsilon^2}[(\theta_1^{(j)} - \theta_1^{(j-1)})^2 + \sin\theta_1^{(j)}\sin\theta_1^{(j-1)}(\theta_2^{(j)} - \theta_2^{(j-1)})^2 + \cdots + (\sin\theta_1^{(j)}\cdots\sin\theta_{d-2}^{(j-1)})(\phi^{(j)} - \phi^{(j-1)})^2] \tag{13}$$

denotes a "classical Lagrangian" on the lattice. $\mathscr{L}_{\mathrm{Cl}}^N\{\{\bar{\theta}^{(j)}\}\}$ is again defined by (13), except that one has to take all trigonometrics at midpoints. C is given by (I.11):

$$C = C(\{\theta'\}, \{\theta''\}) = \left[\prod_{\nu=1}^{d-1}\sin^\mu\theta'_\nu\sin^\mu\theta''_\nu\right]^{-1/2}. \tag{14}$$

The integrals over p_{θ_ν} are of Gaussian form and we get the following Lagrangian path integral:

$$K(\{\theta''\},\{\theta'\};t''-t') = \int \mathscr{D}\Omega(t)\exp\left\{ i\int_{t'}^{t''} [\mathscr{L}_{\mathrm{Cl}}(\{\theta,\dot\theta\}) - \Delta V(\{\theta\})]\,dt \right\} \tag{15}$$

where the classical Lagrangian and the integration measure are given by:

$$\mathscr{L}_{\mathrm{Cl}}(\{\theta,\dot\theta\}) = \frac{m}{2}R^2[\dot\theta_1^2 + \sin^2\theta_1\,\dot\theta_2^2 + \cdots + (\sin^2\theta_1\cdots\sin^2\theta_{d-2})\dot\phi^2], \tag{16}$$

$$\mathscr{D}\Omega(t) \to \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{N\cdot(d-1)/2} \prod_{j=1}^{N-1} d\Omega^{(j)},$$

$$\varepsilon = \frac{t''-t'}{N}. \tag{17}$$

Here $d\Omega^{(j)}$ denotes the $(d-1)$-dimensional surface element on the unit sphere S^{d-1}

$$d\Omega^{(j)} = \prod_{k=1}^{d-1} (\sin\theta_k^{(j)})^{d-1-k}\,d\theta_k^{(j)}. \tag{18}$$

It is worthwhile to notice that the normalisation C has been exactly cancelled, and that the path integral (15) has the standard measure (18), which can be directly derived by a transformation from Cartesian to polar coordinates.

As a final check we have to show that (15) leads to the correct Schrödinger equation. With our lattice definition we obtain from (15–17) and (9) the following short-time kernel:

$$\begin{aligned} K(\{\theta^{(j)}\},\{\theta^{(j-1)}\};\varepsilon) &= \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2} \exp\Bigg\{ i\varepsilon\mathscr{L}^N_{\mathrm{Cl}}(\{\theta^{(j)},\theta^{(j-1)}\}) \\ &+ \frac{i\varepsilon}{8mR^2}\Bigg[(d-2)^2 + \frac{1}{\sin\theta_1^{(j)}\sin\theta_1^{(j-1)}} + \cdots \\ &+ \frac{1}{\sin\theta_1^{(j)}\sin\theta_1^{(j-1)}\cdots\sin\theta_{d-2}^{(j)}\sin\theta_{d-2}^{(j-1)}}\Bigg]\Bigg\}. \end{aligned} \tag{19}$$

Using the time-evolution equation

$$\psi(\{\theta^{(j+1)}\},t+\varepsilon) = \int d\Omega^{(j)} K(\{\theta^{(j+1)}\},\{\theta^{(j)}\};\varepsilon)\psi(\{\theta^{(j)}\},t) \tag{20}$$

it is straightforward but tedious to derive the correct Schrödinger equation (1). The details are given in Appendix B. Note that the constant

$$\frac{(d-2)^2}{8mR^2} \tag{21}$$

in $\mathscr{L}_{\mathrm{eff}} = \mathscr{L}_{\mathrm{Cl}} - \Delta V$ is crucial in order to obtain the correct Schrödinger equation.

The path integral (15) with the Lagrangian given by (16) is too complicated for explicit calculations. We therefore try to replace (16) under the path integral (15) by the following expression:

$$\mathscr{L}_{\mathrm{Cl}}(\{\theta,\dot\theta\}) \to \tilde{\mathscr{L}}_{\mathrm{Cl}}(\{\theta,\dot\theta\}) := \frac{m}{2}R^2\dot\Omega^2 - V_c(\{\theta\}) \tag{22}$$

where V_c has to be determined and Ω denotes the d-dimensional unit vector on the S^{d-1}-sphere. The hope, of course, is that $V_c + \Delta V$ is simple enough so that the path integral (15) can be explicitly computed. We have

$$(\Omega^{(1)} - \Omega^{(2)})^2 = 2(1-\cos\psi^{(1,2)}) \tag{23}$$

with the well-known addition theorem:

$$\cos\psi^{(1,2)} = \cos\theta_1^{(1)}\cos\theta_1^{(2)} + \sum_{m=1}^{d-2}\cos\theta_{m+1}^{(1)}\cos\theta_{m+1}^{(2)}\prod_{n=1}^{m}\sin\theta_n^{(1)}\sin\theta_n^{(2)} + \prod_{n=1}^{d-1}\sin\theta_n^{(1)}\sin\theta_n^{(2)}. \tag{24}$$

We shall use (24) to justify the replacement (22) and thereby derive an expression for V_c. We start with the kinetic term $(x^{(j)} - x^{(j-1)})^2$ expressed in the polar coordinates (2), $R = r$ (not fixed), and expand it in terms of Δr and $\Delta\theta_\nu$. In this procedure we follow the reasoning of Pak and Sökmen [32]*. If one has an expression like $\Delta f^{(j)} = f(u_1^{(j)}\cdots u_d^{(j)}) - f(u_1^{(j-1)}\cdots u_d^{(j-1)})$, one gets for the expansion about the midpoint $\bar u^{(j)} := (1/2)\,(u^{(j)} + u^{(j-1)})$:

$$\Delta f_l^{(j)} = \sum_{m=1}^{d}\Delta u_m^{(j)}\left(\frac{\partial f_l^{(j)}}{\partial u_m^{(j)}}\right)_{\bar u^{(j)}} + \frac{1}{24}\sum_{m,n,k=1}^{d}\Delta u_m^{(j)}\Delta u_n^{(j)}\Delta u_k^{(j)}\left(\frac{\partial^3 f_l^{(j)}}{\partial u_m^{(j)}\partial u_n^{(j)}\partial u_k^{(j)}}\right)_{\bar u^{(j)}} + \cdots \tag{25}$$

Here $f_l^{(j)} = x_l^{(j)}$, $(l = 1,\ldots,d)$, $u_m = \{u_d = r,\ u_m = \theta_m (m = 1,\ldots,d-1)\}$. In a similar manner like in [32] we can state after tedious calculations the following identity

$$\exp[i\varepsilon\mathscr{L}^N_{\mathrm{Cl}}(\{\theta^{(j)}\},\{\theta^{(j-1)}\})] \doteq \exp\left\{\frac{im}{\varepsilon}R^2(1-\cos\psi^{(j,j-1)}) - i\varepsilon V_c(\{\theta^{(j)}\})\right\} \tag{26}$$

with

$$V_c(\{\theta^{(j)}\}) = \frac{1}{8mR^2}\left[1 + \frac{1}{\sin\theta_1^{(j)}\sin\theta_1^{(j-1)}} + \cdots + \frac{1}{\sin\theta_1^{(j)}\sin\theta_1^{(j-1)}\cdots\sin\theta_{d-2}^{(j)}\sin\theta_{d-2}^{(j-1)}}\right]. \tag{27}$$

(V_c is the same whether or not $\Delta r^{(j)} = 0$; so we have set $r^{(j)} = R$ in V_c). In the final equations the midpoints

* This method goes back to DeWitt [4], McLaughlin and Schulman [28] and Gervais and Jevicki [11]; we prefer the formulation [32] because it seems more explicit to us in the rotator case

don't appear; they are used as a tool in the expansions in order to derive (26) and (27). From (9) and (27) we obtain:

$$V_c + \Delta V = -\frac{1}{8mR^2}(d-1)(d-3) \tag{28}$$

and thus obtain our final form of the *path integral for the d-dimensional rotator*:

$$K(\{\theta''\}, \{\theta'\}; t''-t') = \int \mathscr{D}\Omega(t) \exp\left\{ i \int_{t'}^{t''} \left[\frac{m}{2} R^2 \dot{\Omega}^2 + \frac{(d-1)(d-3)}{8mR^2} \right] dt \right\}. \tag{29}$$

Equation (29) is our main result in this section. Let us make some comments:

1) In Appendix C we show that one obtains the correct Schrödinger equation from the short-time kernel of (29).

2) The usual way to construct the path integral for the rotator has been to start with the free particle Euclidean path integral in d dimensions, to introduce the polar coordinates (2) (but r not fixed) and to implement in the path integral in polar coordinates the rotator-constraint by $\delta(r^{(j)} - R)$ (all j)—see [37]. But that works only for $d = 3$, otherwise it is wrong and the quantum correction $V_R = (d-1)(d-3)/8mR^2$ is missing. The right result in [37] for $d = 3$ is just an accident. The naïve implementation of the rotator constraint by δ-distributions does not take into account the huge difference between d-dimensional Cartesian space which remains a flat manifold in whatever coordinate system expressed and the d-dimensional rotator which corresponds to a *curved* manifold. The term V_R is due to that curvilinear nature of the rotator.

3) The missing of V_R in the naïve calculation was first observed by Marinov and Terentyev. Instead of V_R they have $\bar{V}_R = [(d-2)^2 + 2/3]/8mR^2$ for the path integral (29). They got $\bar{V}_R$ by starting with the known solution of the rotator (see below equation (40)) and deriving the short-time kernel from it. This was done by an asymptotic expansion of the modified Bessel-function I_ν for large ν. But $\bar{V}_R$ is not the correct quantum correction.

4) Junker and Inomata [21] have deduced V_R by expanding $\cos\psi$ (see equations (23) and (24)), $\cos\psi \simeq 1 - \psi^2/2$, and by stating that this expansion is effectively correct up to $O(\varepsilon^2)$. With $\Delta^2 x^{(j)} \simeq 2(1 - \cos\psi^{(j,j-1)} + [\psi^{(j,j-1)}]^4/4!)$ they derived V_R. Well, the validity of the expansion is a consequence once V_R is known, but this is not a proof, respectively a rigorous path integral treatment.

Starting from (29) we can calculate the path integral for the rotator. For that purpose we need the following formula [12, p. 980]:

$$e^{z\cos\psi} = \left(\frac{z}{2}\right)^{-\nu} \Gamma(\nu) \sum_{l=0}^{\infty} (l+\nu) I_{l+\nu}(z) C_l^\nu(\cos\psi) \tag{30}$$

for $\nu = (d-2)/2$. The C_l^ν's are Gegenbauer polynomials and I_μ modified Bessel functions. Equation (30) is a generalisation of the well-known expansion in three dimensions where $\nu = 1/2$, $C_l^{1/2} = P_l$:

$$e^{z\cos\psi} = \sqrt{\frac{\pi}{2z}} \sum_{l=0}^{\infty} (2l+1) I_{l+1/2}(z) P_l(\cos\psi) \tag{31}$$

[12, p. 980].

Note. It is possible to include the case $d = 2$, i.e. $\nu = 0$ if one uses $l \lim_{\nu\to 0} \Gamma(\nu)$. $C_l^\nu(\cos\psi) = \varepsilon_l \cos l\psi$ ($\varepsilon_l = 1$ for $l = 0$, $\varepsilon_l = 2$ for $l = 1, 2, \dots$ [12, p. 1030]), yielding [12, p. 973]:

$$e^{z\cos\psi} = \sum_{k=-\infty}^{\infty} I_k(z) e^{ik\psi}. \tag{32}$$

The addition theorem for the surface (or hyperspherical) harmonics S_l^μ on the S^{d-1}-sphere (see [9]) reads

$$\sum_{\mu=1}^{M} S_l^\mu(\Omega^{(1)}) S_l^\mu(\Omega^{(2)}) = \frac{1}{\Omega(d)} \frac{2l+d-2}{d-2} C_l^{(d-2)/2}(\cos\psi^{(1,2)}), \tag{33}$$

($\Omega(d) = 2\pi^{d/2}/\Gamma(d/2)$). The orthonormality relation is

$$\int d\Omega\, S_l^\mu(\Omega) S_{l'}^{\mu'}(\Omega) = \delta_{ll'} \delta_{\mu\mu'}. \tag{34}$$

Combining (30) and (33) we get the expansion formula

$$e^{z(\Omega^{(1)}\cdot\Omega^{(2)})} = e^{z\cos\psi^{(1,2)}} = 2\pi \left(\frac{2\pi}{z}\right)^{(d-2)/2} \sum_{l=0}^{\infty} \sum_{\mu=1}^{M} S_l^\mu(\Omega^{(1)}) S_l^\mu(\Omega^{(2)}) I_{l+(d-2)/2}(z). \tag{35}$$

Using (35) in each j-integration, the angular integrations can be easily carried out ($\tau := t'' - t'$):

$$\begin{aligned} K(\psi^{(\prime,\prime\prime)}; \tau) &= \int \mathscr{D}\Omega(t) \exp\left\{ i \int_{t'}^{t''} \left[\frac{m}{2} R^2 \dot{\Omega}^2 + \frac{(d-1)(d-3)}{8mR^2} \right] dt \right\} \\ &= e^{i\tau(d-1)(d-3)/8mR^2} \lim_{N\to\infty} \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{N\cdot(d-1)/2} \\ &\quad \cdot \int d\Omega^{(1)} \cdots \int d\Omega^{(N-1)} \\ &\quad \cdot \exp\left[\frac{imR^2}{\varepsilon} \sum_{j=1}^{N} (1 - \cos\psi^{(j,j-1)}) \right] \\ &= \frac{1}{\Omega(d)} e^{i\tau(d-1)(d-3)/8mR^2} \sum_{l=0}^{\infty} \frac{2l+d-2}{d-2} C_l^{(d-2)/2}(\cos\psi^{(\prime\prime,\prime)}) \\ &\quad \mu_l^{(d)}[R] \end{aligned} \tag{36}$$

with

$$\mu_l^{(d)}[R] = \lim_{N\to\infty} \left[\sqrt{\frac{2\pi m}{i\varepsilon} R^2}\, e^{imR^2/\varepsilon} I_{l+(d-2)/2}\left(\frac{m}{i\varepsilon} R^2\right) \right]^N. \tag{37}$$

Notice that the expression (37) is just the functional weight needed in the radial path integral (evaluated for paths satisfying the constraint $r(t)=R=fixed$ for all t) which we have already introduced in our earlier work [41]*. Using the asymptotic expansion $I_\nu(z)\simeq(2\pi z)^{-1/2}e^{z-(\nu^2-1/4)/2z}$ for the modified Bessel functions**, the above limit is easily performed resulting in

$$\mu_l^{(d)}[R]=\exp\left[-i\tau\frac{l(l+d-2)+(1/4)(d-1)(d-3)}{2mR^2}\right]. \tag{38}$$

We thus obtain our final result

$$K(\psi^{('',')};\tau)=\sum_{l=0}^{\infty}\sum_{\mu=1}^{M}S_l^\mu(\Omega')S_l^\mu(\Omega'')\cdot\exp\left\{-\frac{i\tau}{2mR^2}l(l+d-2)\right\} \tag{39}$$

$$=\frac{1}{\Omega(d)}\sum_{l=0}^{\infty}\frac{2l+d-2}{d-2}C_l^{(d-2)/2}(\cos\psi^{('',')})\cdot\exp\left\{-\frac{i\tau}{2mR^2}l(l+d-2)\right\}. \tag{40}$$

This is the correct result, from which one easily reads off the wave functions and energyvalues, see (5) and (6).

Concluding the discussion of the rotator we summarise our procedure:

1) We started with the Hamiltonian $H=-(1/2mR^2)L^2_{(d)}$,

2) obtained from the Hamiltonian

$$H(\{p_\theta,\theta\})=\frac{1}{2mR^2}\sum_{\nu=1}^{d-1}g^{\nu\nu}(\{\theta\})p^2_{\theta_\nu}+\Delta V(\{\theta\}) \tag{41}$$

the quantum correction

$$\Delta V(\{\theta\})=-\frac{1}{8mR^2}\sum_{\nu=1}^{d-2}g^{\nu\nu}\left(\mu^2+\frac{2\mu-\mu^2}{\sin^2\theta_\nu}\right)=-\frac{1}{8mR^2}\left[(d-2)^2+\frac{1}{\sin^2\theta_1}+\cdots+\frac{1}{(\sin^2\theta_1\cdots\sin^2\theta_{d-2})}\right]. \tag{42}$$

3) This quantum correction was used to define the correct Lagrangian path integral with an effective Lagrangian $\mathscr{L}_{\text{eff}}=\mathscr{L}_{\text{Cl}}-\Delta V$

$$K(\{\theta''\},\{\theta'\};t''-t')=\int\mathscr{D}\Omega(t)\exp\left\{i\int_{t'}^{t''}[\mathscr{L}_{\text{Cl}}(\{\theta,\dot\theta\})-\Delta V(\{\theta\})]\,dt\right\}. \tag{43}$$

4) Finally we used the identity (26) to cast the path integral (15) into the simple form

$$K(\{\theta''\},\{\theta'\};t''-t')=\int\mathscr{D}\Omega(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}R^2\dot\Omega^2+\frac{(d-1)(d-3)}{8mR^2}\right]dt\right\}. \tag{44}$$

III The path integral in d-dimensional polar coordinates

We consider the Schrödinger equation in d dimensions with a potential $V(|x|)=V(r)$:

$$i\frac{\partial}{\partial t}\psi(r,\{\theta\};t)=\left[-\frac{1}{2m}\Delta_{(d)}+V(r)\right]\psi(r,\{\theta\};t) \tag{1}$$

with $\Delta_{(d)}$ as defined in (II.7). The Hamiltonian is just $H=-(1/2m)\Delta_{(d)}+V(r)$. H rewritten with (I.10) yields:

$$H=\frac{1}{2m}p_r^2+\frac{1}{2mr^2}\left[p^2_{\theta_1}+\frac{1}{\sin^2\theta_1}p^2_{\theta_2}+\cdots+\frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{d-2}}p^2_\phi\right]+V(r)+\Delta V(r,\{\theta\}) \tag{2}$$

with:

$$\Delta V(r,\{\theta\})=-\frac{1}{8mr^2}\left[1+\frac{1}{\sin^2\theta_1}+\cdots+\frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{d-2}}\right]. \tag{3}$$

We now repeat the reasoning of Sect. II. We start with a Hamiltonian path integral similar to (II.11) and get after the integration over all momenta:

$$K^{(d)}(r'',r',\{\theta''\},\{\theta'\};t''-t')=\int\mathscr{D}r(t)\mathscr{D}\Omega(t)\exp\left\{i\int_{t'}^{t''}[\mathscr{L}_{\text{Cl}}(r,\dot r,\{\theta,\dot\theta\})-\Delta V(r,\{\theta\})]\,dt\right\}, \tag{4}$$

with classical Lagrangian and measure, respectively:

$$\mathscr{L}_{\text{Cl}}(r,\dot r,\{\theta,\dot\theta\})=\frac{m}{2}[\dot r^2+r^2\dot\theta_1^2+r^2\sin^2\theta_1\dot\theta_2^2+\cdots+r^2(\sin^2\theta_1\cdots\sin^2\theta_{d-2})\dot\phi^2]-V(r) \tag{5}$$

$$\mathscr{D}r(t)\mathscr{D}\Omega(t)\to\left(\frac{m}{2\pi i\varepsilon}\right)^{N\cdot(d/2)}\sum_{j=1}^{N-1}r_{(j)}^{d-1}dr_{(j)}d\Omega_{(j)},\qquad \varepsilon=\frac{t''-t'}{N} \tag{6}$$

($N\to\infty$). Next we try to replace $\mathscr{L}_{\text{Cl}}$ by a simpler expression and hope that the resulting path integral is simple enough so that the angular integrations can be exactly carried out. We therefore repeat the steps from (II.23) to (II.25) with the only difference that $r_{(j)}$

---

* See also Sect. III

** This asymptotic expansion is valid for $|z|\to\infty$, $|\arg z|<\pi/2$. In order to apply it in (37) we perform a Wick rotation, i.e., analytically continue to purely (negative) imaginary time, $\varepsilon\to-i\delta(\delta>0)$. For more details see [41] is not restricted to $r_{(j)}=R$. The result is that the potential V_c (see (II.25)) generated by these steps cancells exactly $\Delta V(r,\{\theta\})$! Therefore we get:

$$K^{(d)}(r'',r',\{\theta''\},\{\theta'\};t''-t') = \int \mathscr{D}r(t)\mathscr{D}\Omega(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(r)\right]dt\right\}, \quad (7)$$

where $\dot{x}^2$ has to be expressed in polar coordinates. In the lattice formulation $\dot{x}^2$ reads

$$\dot{x}^2 \to [r_{(j)}^2+r_{(j-1)}^2-2r_{(j)}r_{(j-1)}\cos\psi_{(j,j-1)}]/\varepsilon^2. \quad (8)$$

To carry out the angular integrations we use (II.35) in each j-integration:

$$\begin{aligned} K^{(d)}&(r'',r',\{\theta''\},\{\theta'\};t''-t') \\ &= \int \mathscr{D}r(t)\mathscr{D}\Omega(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(r)\right]dt\right\} \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{Nd/2}\int_0^\infty r_{(1)}^{d-1}\,dr_{(1)}\int d\Omega_{(1)}\cdots \\ &\quad\cdot\int_0^\infty r_{(N-1)}^{d-1}\,dr_{(N-1)}\int d\Omega_{(N-1)} \\ &\quad\cdot\prod_{j=1}^{N}\exp\left\{\frac{im}{2\varepsilon}[r_{(j)}^2+r_{(j-1)}^2-2r_{(j)}r_{(j-1)}\right. \\ &\quad\left.\cdot\cos\psi_{(j,j-1)}]-i\varepsilon V(r_{(j)})\right\} \\ &= (r'r'')^{-(d-2)/2}\sum_{l=0}^{\infty}\sum_{\mu=1}^{M}S_l^\mu(\Omega')S_l^\mu(\Omega'') \\ &\quad\cdot\lim_{N\to\infty}\left(\frac{m}{i\varepsilon}\right)^N\int_0^\infty r_{(1)}\,dr_{(1)}\cdots \\ &\quad\cdot\int_0^\infty r_{(N-1)}\,dr_{(N-1)}\prod_{j=1}^{N}\exp\left\{\frac{im}{2\varepsilon}(r_{(j)}^2+r_{(j-1)}^2)\right. \\ &\quad\left.-i\varepsilon V(r_{(j)})\right\}I_{l+(d-2)/2}\left(\frac{m}{i\varepsilon}r_{(j)}r_{(j-1)}\right). \end{aligned} \quad (9)$$

Thus we can separate the radial part of the path integral (partial wave expansion):

$$\begin{aligned} K^{(d)}&(r'',r',\{\theta''\},\{\theta'\};\tau) \\ &= \Omega^{-1}(d)\sum_{l=0}^{\infty}\frac{2l+d-2}{d-2}C_l^{(d-2)/2}(\cos\psi^{(',")})K_l^{(d)}(r'',r';\tau) \end{aligned} \quad (10)$$

with the *radial kernel* given by ($z_{(j)} := (m/i\varepsilon)r_{(j)}r_{(j-1)}$)

$$\begin{aligned} K_l^{(d)}&(r'',r';\tau) \\ &= (r'r'')^{-(d-1)/2}\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{N/2}\int_0^\infty dr_{(1)}\cdots\int_0^\infty dr_{(N-1)} \\ &\quad\cdot\left[\prod_{j=1}^{N}\sqrt{2\pi z_{(j)}}\,e^{-z_{(j)}}I_{l+(d-2)/2}(z_{(j)})\right] \\ &\quad\cdot\exp\left\{i\sum_{j=1}^{N}\left[\frac{m}{2\varepsilon}(r_{(j)}-r_{(j-1)})^2-\varepsilon V(r_{(j)})\right]\right\}. \end{aligned} \quad (11)$$

This is the correct path integral in d-dimensional polar coordinates as presented in a previous paper [41]. Thus we obtain our final form for the *radial path integral in d dimensions*

$$K_l^{(d)}(r'',r';\tau) = (r'r'')^{-(d-1)/2}\int \mathscr{D}r(t)\mu_l^{(d)}[r]\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2-V(r)\right]dt\right\} \quad (12)$$

with the *functional measure*

$$\mu_l^{(d)}[r] \to \prod_{j=1}^{N}[\sqrt{2\pi z_{(j)}}\,e^{-z_{(j)}}I_{l+(d-2)/2}(z_{(j)})] \quad (13)$$

and the one-dimensional measure defined by

$$\mathscr{D}r(t) \to \left(\frac{m}{2\pi i\varepsilon}\right)^{N/2}\prod_{j=1}^{N-1}dr_{(j)}. \quad (14)$$

Notice that the radial path integral (12) contains only the S-wave ($l=0$) part of the classical Lagrangian

$$\mathscr{L}_{\mathrm{Cl}}(r,\dot{r}) = \frac{m}{2}\dot{r}^2 - V(r) - \frac{l(l+1)}{2mr^2}, \quad (15)$$

i.e. it does not explicitly contain the centrifugal potential. Instead the l-dependence of (12) is determined by the functional measure (13) (for more details, see [41]).

IV Other examples of path integrals on curved manifolds

The program stated in the introduction and illustrated in Sect. II and III in the cases of the d-dimensional rotator and d-dimensional polar coordinates, challenges to be applied to other examples which have become important in recent path integral calculations. These calculations include not only a coordinate- but also a time-transformation $dt=f\,ds$ with a new "time" ds.

The method goes as follows (see also [39]): One starts with the path integral

$$K(x'',x';\tau) = \int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}x(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\} \quad (1)$$

where it is assumed that the potential $V(x)$ is so complicated, that a direct evaluation of (1) is not possible. One then defines a new "time" s together with a coordinate transformation $x(t)\to q(s)$

$$s(t) = \int_{t'}^{t}\frac{d\sigma}{f(x(\sigma))} \quad \text{and} \quad x = F(q) \quad (2)$$

with some well-defined positive functions f and F (we shall restrict to the case* $f(F(q)) = [F'(q)]^2$, $F'>0$).

* For a d-dimensional path integral, F' has to be interpreted as the Jacobian

Let us assume that the constraint ($s(t'')=s''$)

$$\int_0^{s''} ds f[F(q(s))] = \tau \tag{3}$$

has for all admissible paths a unique solution $s'' \geqq 0$. Of course, since τ is fixed, the "time" s'' will be path-dependent. In order to incorporate the constraint (3) we use the identity

$$\begin{aligned} 1 &= f(x'') \int_0^\infty ds'' \delta\left(\int_0^{s''} ds f[F(q(s))] - \tau \right) \\ &= f(x'') \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-i\tau E} \int_0^\infty ds'' \\ &\quad \cdot \exp\left\{ i \int_0^{s''} ds f[F(q(s))] E \right\} \end{aligned} \tag{4}$$

under the path integral (1). Defining the energy-dependent Feynman kernel $G(x'', x'; E)$ via the Fourier transformation

$$K(x'', x'; \tau) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-i\tau E} G(x'', x'; E) dE \tag{5}$$

one obtains finally the *transformation formula*

$$G(x'', x'; E) = i[f(x'')f(x')]^{1/4} \int_0^\infty \tilde{K}(q'', q'; s'') ds'' \tag{6}$$

which gives the energy-dependent kernel G as a time integral over the transformed Feynman path integral $\tilde{K}$:

$$\begin{aligned} &\tilde{K}(q'', q'; s'') \\ &= \int_{q(0)=q'}^{q(s'')=q''} \mathcal{D}q(s) \exp\left\{ i \int_0^{s''} \left[\frac{m}{2}\dot{q}^2 - f(F(q))V(F(q)) \right.\right. \\ &\quad \left.\left. + Ef(F(q)) - \Delta V(q) \right] ds \right\} \end{aligned} \tag{7}$$

($\dot{q} = dq(s)/ds$, $q' = F^{-1}(x')$, $q'' = F^{-1}(x'')$). Here the measure $\mathcal{D}q(s)$ is defined in the same way as $\mathcal{D}x(t)$ in the path integral (1). The crucial point is now the calculation of the correct quantum correction $\Delta V(q)$*.

Our program to calculate ΔV will be as follows;

i) Consider the "Legendre transformed" Hamiltonian

$$H_E(\partial_x, x) = -\frac{1}{2m}\Delta_{LB} + V(x) - E$$

ii) transform it to the Hamiltonian $\hat{H}_E(\partial_q, q)$ via the transformation $x = F(q)$,

iii) make a time-transformation $dt = f(F(q))ds$ which yields the new Hamiltonian

$$\tilde{H}(\partial_q, q) = f(F(q))\hat{H}_E(\partial_q, q),$$

iv) construct hermitian momentum operators

$$p_q = \frac{1}{i}\left(\frac{d}{dq} + \frac{\Gamma(q)}{2} \right), \quad \Gamma(q) = \frac{d \ln J}{dq} \tag{8}$$

with a measure $J\,dq$ with respect to the scalar product

$$(f, g) = \int f^* g J\, dq. \tag{9}$$

In the generic case, the new Hamiltonian $\hat{H}$ will be proportional to the d-dimensional Laplace–Beltrami operator (see (I.4)).

v) Finally we get the quantum correction from the formula:

$$\left.\begin{aligned} \tilde{H}(\partial_q, q) &= -\frac{1}{2m}\Delta_{LB} + f(F(q))[V(F(q)) - E] \\ &= \frac{1}{8m}(g^{ab}p_a p_b + 2p_a g^{ab} p_b + p_a p_b g^{ab}) \\ &\quad + f(F(q))[V(q) - E] + \Delta V(q) \\ &= H_{\text{eff}}(p_q, q) \end{aligned}\right\} \tag{10}$$

($g^{ab} = \delta^{ab}$ in the one-dimensional case) and ΔV as in (I.8).

We shall illustrate our program with

A) the Kustaanheimo–Stiefel transformation in $\mathbf{R}^2$ and $\mathbf{R}^3$, and

B) coordinate transformations in a general one dimensional Hamiltonian.

A1) The Kustaanheimo–Stiefel Transformation in $\mathbf{R}^2$. In [19] Inomata calculated the path integral for the two dimensional "Coulomb"-problem. The calculation was carried out without any quantum correction. Here we start with:

$$H_E = -\frac{1}{2m}\Delta_{(2)} - \frac{e^2}{r} - E. \tag{11}$$

The transformation is: $x = \xi^2 - \eta^2$, $y = 2\xi\eta$. The transformed Laplacian is:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{4v^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \tag{12}$$

with $v^2 = \xi^2 + \eta^2$. The appropriate new Hamiltonian reads $\tilde{H} = 4v^2 \hat{H}_E$ (i.e. $f(x, y) = 4r$ with $g_{ab} = \delta_{ab}$, $J = 1$, $\Gamma_a = 0$, $p_\xi = -i\partial_\xi$, $p_\eta = -i\partial_\eta$)

$$H_{\text{eff}}(p_\xi, p_\eta, \xi, \eta) = \frac{1}{2m}(p_\xi^2 + p_\eta^2) - 4e^2 - 4E(\xi^2 + \eta^2). \tag{13}$$

Thus the quantum correction vanishes, $\Delta V = 0$, as it should be.

* The transformation formulae (6) and (7) were originally derived in [39] for radial path integrals using the lattice definition of the various path integrals. Although the resulting transformation formula could easily be shown to be exact by inserting it directly into the corresponding Schrödinger equations for the kernels, the lattice derivation is far from being trivial and not without problems from a rigorous mathematical point of view. (An attempt to justify the lattice derivation can be found in [20] and [22]). We therefore rather prefer to *define* the transformation formulae by (6) and (7), which reduces the problem to the determination of the correct quantum correction ΔV

A2) The Kustaanheimo–Stiefel transformation in $\mathbf{R}^3$. We consider the Hamiltonian for the Coulomb problem in $\mathbf{R}^3$:

$$H_E = -\frac{1}{2m}\Delta_{(3)} - \frac{e^2}{r} - E. \tag{14}$$

The path integral for the Coulomb problem has been first calculated by Duru and Kleinert [7], followed by several discussions concerning the details about simultaneous coordinate and time-transformations [17, 39–41]. Let us write H_E in the coordinates [3]:

$$\begin{aligned} q_1 &= q\cos\alpha\cos\beta \\ q_2 &= q\cos\alpha\sin\beta \quad \left(q=|q|=\sqrt{r}, \alpha=\frac{\theta}{2}, \beta\pm\gamma=\phi\right) \\ q_3 &= q\sin\alpha\cos\gamma \quad (0\leqq\theta\leqq\pi, 0\leqq\phi\leqq 2\pi). \\ q_4 &= q\sin\alpha\sin\gamma \end{aligned} \tag{15}$$

Then we get for H_E in the coordinates $q\in\mathbf{R}^4$:

$$\tilde{H} = 4q^2\hat{H}_E = -\frac{1}{2m}\Delta_4 - 4e^2 - 4Eq^2$$

and with $J=\sqrt{g}=1$, $\Gamma_a=0$ and $pq_k=-i\partial/\partial q_k$:

$$H_{\text{eff}}(p_q, q) = \frac{1}{2m}\sum_{k=1}^{4} p^2 q_k - 4e^2 - 4Eq^2. \tag{16}$$

No quantum correction appears! This is the reason why the calculation in [7] was correct. Note that in both cases, A1) and A2), the Coulomb problem has been transformed into a simple harmonic oscillator problem.

B) The general one-dimensional Hamiltonian. Let us consider the space-time transformations:

$$x = F(y), \quad dt = f(x)ds \tag{17}$$

and the Hamiltonian:

$$H_E = -\frac{1}{2m}\left(\frac{d^2}{dx^2} + h(x)\frac{d}{dx}\right) + V(x) - E, \tag{18}$$

which is hermitian with respect to the inner product

$$(f_1, f_2) = \int f_1^*(x) f_2(x) J(x)dx, \quad J(x) = e^{\int h(x)dx}. \tag{19}$$

Let $G(y) = h(F(y))$, then:

$$\hat{H}_E = -\frac{1}{2m}\frac{1}{F'^2}\left[\frac{d^2}{dy^2} + \left(G(y)F'(y) - \frac{F''(y)}{F'(y)}\right)\frac{d}{dy}\right] + V(F(y)) - E. \tag{20}$$

With $f(F(y)) = F'^2(y)$ we get for $\tilde{H} = f\hat{H}_E$:

$$\begin{aligned} \tilde{H} &= -\frac{1}{2m}\left[\frac{d^2}{dy^2} + \left(G(y)F'(y) - \frac{F''(y)}{F'(y)}\right)\frac{d}{dy}\right] + f(F(y))[V(F(y)) - E] \\ &= -\frac{1}{2m}\left[\frac{d^2}{dy^2} + \Gamma(y)\frac{d}{dy}\right] + f(F(y))[V(F(y)) - E] \end{aligned} \tag{21}$$

with

$$\Gamma(y) = G(y)F'(y) - F''(y)/F'(y). \tag{22}$$

It is easy to see that the Hamiltonian (21) is of the generic type (see (I.7) and (10) with

$$\sqrt{g} = \tilde{J}(y) = e^{\int_{y_0}^{y}\Gamma(y')dy'} \tag{23}$$

$$p_y = \frac{1}{i}\left(\frac{d}{dy} + \frac{\Gamma(y)}{2}\right), \tag{24}$$

and we thus obtain the space-time transformed Hamiltonian

$$H_{\text{eff}}(p_y, y) = \frac{1}{2m}p_y^2 + f(F(y))[V(F(y)) - E] + \Delta V(y) \tag{25}$$

with the quantum correction

$$\Delta V(y) = \frac{1}{8m}\left[3\left(\frac{F''(y)}{F'(y)}\right)^2 - 2\frac{F'''(y)}{F'(y)} + (G(y)F'(y))^2 + 2G'(y)F'(y)\right] \tag{26}$$

(Pak and Sökmen [33] have got the same result for $h(x) = 0$.)

We shall now apply this general space-time transformation to four examples.

B1) The coulomb problem in polar coordinates. Let us rewrite $\tilde{H}$ of example A2) in coordinates q, α, β, γ. One gets:

$$\tilde{H} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial q^2} + \frac{3}{q}\frac{\partial}{\partial q} - \frac{4K^2}{q^2}\right) - 4e^2 - 4Eq^2 \tag{27}$$

where K^2 is defined in [3] and has eigenvalue $l(l+1)$. The Hamiltonian (27) has the form (21) if we make the following identifications:

$$y = q, \quad \Gamma = \frac{3}{q}, \quad f(q) = 4q, \quad F(q) = q^2,$$

$$V(q) = \frac{l(l+1)}{2mq^2} - \frac{e^2}{q}. \tag{28}$$

We thus obtain:

$$\sqrt{g} = \tilde{J}(q) = q^3, \quad p_q = \frac{1}{i}\left(\frac{\partial}{\partial q} + \frac{3}{2q}\right), \quad \Delta V = \frac{3}{8mq^2} \tag{29}$$

which yields the effective Lagrangian

$$\mathscr{L}_{\text{eff}}(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - \frac{4l(l+1) + \frac{3}{4}}{2mq^2} + 4e^2 + 4Eq^2 \tag{30}$$

to be used in the transformed path integral (7) (see [18], [41]). Equation (30) describes the radial motion of a three dimensional harmonic oscillator with frequency $\omega=\sqrt{-8E/m}$ and effective angular momentum $L_{\text{eff}}=2l+1/2$.

B2) The Langer transformation. Gerry and Inomata [10] used the Langer transformation $r=e^x$ to perform a semiclassical calculation for the hydrogen atom. We start with:

$$H_E=-\frac{1}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)}{2mr^2}-\frac{e^2}{r}-E. \tag{31}$$

With $r=e^x=F(x)$ we obtain

$$\hat{H}_E=-\frac{1}{2m}e^{-2x}\left(\frac{d^2}{dx^2}+\frac{d}{dx}\right)+e^{-2x}\frac{l(l+1)}{2m}$$
$$-e^{-x}e^2-E. \tag{32}$$

Now the natural choice for $f(x)$ is $f(x)=x^2$ yielding

$$\tilde{H}=-\frac{1}{2m}\left(\frac{d^2}{dx^2}+\frac{d}{dx}\right)+\frac{l(l+1)}{2m}-e^2e^x-Ee^{2x}. \tag{33}$$

A comparison with (21) gives

$$y=x,\quad \Gamma=1,\quad \sqrt{g}=\tilde{J}(x)=e^x,$$
$$p_x=-i(d/dx+1/2),$$
$$\Delta V=\frac{1}{8m}. \tag{34}$$

Thus we end up with the Hamiltonian

$$H_{\text{eff}}(p_x,x)=\frac{1}{2m}p_x^2+\frac{(l+1/2)^2}{2m}-e^xe^2-e^{2x}E, \tag{35}$$

which is hermitian with respect to the inner product $(f,g)=\int_{-\infty}^{\infty}f^*ge^xdx$. In this case the quantum correction is well-known as the Langer modification and the result coincides with the one given by Gerry and Inomata.

B3) The Morse potential. Duru [6] has calculated the path integral for the Morse potential. He has used the transformation $x=-(2/a)\ln y$ and got by a heuristic argument concerning initial- and final points in the path integral the quantum correction $\Delta V=-1/8my^2$.

Consider the Hamiltonian

$$H_E=-\frac{1}{2m}\frac{d^2}{dx^2}+V(x)-E \tag{36}$$

with $V(x)=V_0(e^{-2ax}-2e^{-ax})$. With the transformation $x=-(2/a)\ln y=F(y)$, i.e. $y=e^{-(a/2)x}$, we get:

$$\hat{H}_E=-\frac{a^2}{8m}y^2\left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}\right)+V(F(y))-E \tag{37}$$

which can be brought to the generic form (21) if we choose $f(x)=(4/a^2)e^{ax}$. We then obtain

$$\Gamma=\frac{1}{y},\quad \sqrt{g}=\tilde{J}(y)=y,\quad p_y=-i(d/dy+1/2y),$$
$$\Delta V=-\frac{1}{8my^2} \tag{38}$$

which finally leads to the effective Hamiltonian

$$H_{\text{eff}}(p_y,y)=\frac{1}{2m}p_y^2-\frac{1}{2my^2}\left(\frac{8mE}{a^2}+\frac{1}{4}\right)$$
$$+\frac{4V_0}{a^2}(y^2-2), \tag{39}$$

which is hermitian with respect to the scalar product $(f,g)=\int_0^{\infty}yf^*gdy$.

B4) General space-time transformations in radial path integrals. Steiner [39, 41] performed in a general radial path integral the simultaneous space-time transformations defined by

$$r=R^{\mu}=F(R),\quad dt=f(r)ds \tag{40}$$

with $f(r)=\mu^2r^{\nu}$, $\mu=2/(2-\nu)$, where ν is an arbitrary real parameter with $\nu<2$. He got a quantum correction

$$\Delta V(R)=\frac{\nu(4-\nu)}{8mR^2(2-\nu)^2}, \tag{41}$$

which leads to a modification of the centrifugal barrier. The lattice definition reads: $R^2\to R_{(j)}R_{(j-1)}$. We start with the Hamiltonian:

$$H_E=-\frac{1}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)}{2mr^2}+V(r)-E. \tag{42}$$

Performing the transformation (40) we arrive at a Hamiltonian $\tilde{H}$ which has the generic form (21) with

$$\Gamma=\frac{1+\mu}{R},\quad \sqrt{g}=\tilde{J}(R)=R^{1+\mu},$$
$$p_R=\frac{1}{i}\left(\frac{d}{dR}+\frac{1+\mu}{2R}\right), \tag{43}$$

$\Delta V(R)$ as in (41). For the space-time transformed effective Hamiltonian we obtain

$$H_{\text{eff}}(p_R,R)=\frac{1}{2m}p_R^2+\frac{L_\nu(L_\nu+1)}{2mR^2}$$
$$+\frac{4}{(2-\nu)^2}R^{2\nu/(2-\nu)}[V(R^{2/(2-\nu)})-E] \tag{44}$$

with an effective angular momentum

$$L_\nu=\frac{4l+\nu}{2(2-\nu)}. \tag{45}$$

(Notice that L_ν will not be in general an integer or half integer). From (44) we obtain the effective

Lagrangian

$$\mathscr{L}_{\text{eff}}(R,\dot{R})=\frac{m}{2}\dot{R}^2-\frac{4}{(2-\nu)^2}R^{2\nu/(2-\nu)}[V(R^{2/(2-\nu)})-E]-\frac{L_\nu(L_\nu+1)}{2mR^2} \tag{46}$$

which has the generic form (III.15). From Section III we know, that the corresponding radial path integral will only involve the S-wave part of (46), i.e. the part with $L_\nu=0$, while the dependence on L_ν will be determined by the functional measure (III.13) evaluated at $l=L_\nu$. We thus obtain from (6) and (7) the *transformation formula for radial path integrals* ($d=3$, $\nu<2$)

$$G_l(r'',r';E)=\frac{2i}{2-\nu}(r'r'')^{\nu/4}\int_0^\infty \tilde{K}_{L_\nu}(R'',R';s'')\,ds'' \tag{47}$$

with the *transformed radial path integral*

$$\tilde{K}_{L_\nu}(R'',R';s'')=\int_{R(0)=R'}^{R(s'')=R''}\mathscr{D}R(s)\mu_{L_\nu}^{(3)}[R]\cdot\exp\left\{i\int_0^{s''}\left[\frac{m}{2}\dot{R}^2-\frac{4}{(2-\nu)^2}R^{2\nu/(2-\nu)}\{V(R^{2/(2-\nu)})-E\}\right]ds\right\} \tag{48}$$

($R'=r'^{(2-\nu)/2}$, $R''=r''^{(2-\nu)/2}$). For applications see [40, 41].

V Discussion

In this paper we have presented several examples of path integrals on curved spaces. The examples have been the d-dimensional rotator, d-dimensional polar coordinates, the exact treatment of the H-atom in $\mathbf{R}^2$ and $\mathbf{R}^3$ by performing a Kustaanheimo-Stiefel transformation and some one-dimensional path integral problems, i.e. the Langer-transformation, the Morse-potential, the Coulomb problem in polar coordinates and general space-time transformations in radial path integrals. Except the rotator, all the examples have been treated recently by other means, but never under the aspect of an application of a path integral on a curved manifold. The d-dimensional rotator has been discussed by Marinov and Terentyev, Inomata and Junker and Böhm and Junker. But these authors never calculated the path integral within the framework of a general theory which is necessary to handle a path integral correctly on a curved manifold.*

It is interesting that another approach, i.e. defining the quantum Hamiltonian by

$$H=\frac{1}{2m}p_a g^{ab}p_b+\frac{1}{8m}[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_a)_{,b}] \tag{1}$$

which is mentioned by Marinov [26], gives in all our examples the correct result. A detailed analysis shows, however, that this corresponds to the symmetrisation rule

$$\mathscr{H}(p,q'',q')=\tfrac{1}{2}[\mathscr{H}_{\text{eff}}(p,q'')+\mathscr{H}_{\text{eff}}(p,q')] \tag{2}$$

and a quantum correction

$$\Delta\tilde{V}=\frac{1}{8m}[(g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_a)_{,b}+2g^{ab}{}_{,ab}]. \tag{3}$$

This has been discussed e.g. by Dowker and Mayes [8].

In a forthcoming paper we shall show that the Weyl prescription yields the correct path integral formulation for the pseudosphere Λ^{d-1} [14] and for three further Riemannian manifolds which are analytically equivalent to Λ^2, i.e. the Poincaré upper half plane U [13], the Poincaré disc D and the hyperbolic strip S [14]. The pseudosphere Λ^{d-1} has also been discussed recently by Böhm and Junker, but in their treatment the question of quantum corrections due to the Riemannian structure is not discussed with the consequence that the energy spectrum comes out to be wrong.

In summary: we have presented a complete and consistent treatment of path integrals on curved manifolds based on Weyl correspondence and the midpoint lattice definition. Within our framework there exists a closed expression for the quantum correction ΔV which has to be subtracted from the original Lagrangian. We hope that our paper will contribute to a clarification of the apparent confusion in the existing path integral literature.

Acknowledgement. We want to thank N.K. Falck for discussions on the operator ordering problem.

Appendix A

The quantum corrections to the classical Lagrangian $\mathscr{L}$ proposed by the authors cited in the introduction read:*

$$\Delta V_{[4]}=-\frac{R^{(d)}}{12m}=\frac{1}{12m}g^{ab}(\Gamma_{a,b}-\Gamma^c_{ab,c}+\Gamma^d_{ac}\Gamma^c_{bd}-\Gamma^c_{ab}\Gamma_c), \tag{A1}$$

$$\Delta V_{[28]}=-\frac{1}{48m}(g_{ab,cd}-2g^{mn}\Gamma_{abm}\Gamma_{cdn})\cdot(g^{ab}g^{cd}+g^{ac}g^{bd}+g^{ad}g^{bc})+\frac{R^{(d)}}{12}, \tag{A2}$$

* The only exception known to us is Arthurs [1], who discussed two dimensional polar coordinates by rewriting the Hamiltonian in a similar manner like (III.2)

* We do not claim to present a complete list. Be careful with the signs in the definition of $R^{(d)}$, the scalar curvature. We use $R^{(d)}=g^{ij}(\Gamma^k_{ij,k}-\Gamma^k_{kj,i}+\Gamma^l_{ij}\Gamma^k_{kl}-\Gamma^k_{lj}\Gamma^l_{ik})$

$$\Delta V_{[26]} = \Delta V_{[30]} = \frac{1}{8m}(g^{ab}\Gamma^d_{ac}\Gamma^c_{bd} - R^{(d)}) \quad \text{(A3)}$$

$$= \frac{1}{8m} g^{ab}(\Gamma_{a,b} - \Gamma^c_{ab,c} + 2\Gamma^d_{ac}\Gamma^c_{bd} - \Gamma^c_{ab}\Gamma_c)$$

$$= \frac{1}{8m}(g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + g^{ab}_{,ab}),$$

$$\Delta V_{[8]} = \frac{1}{8m}[(g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + 2g^{ab}_{,ab}], \quad \text{(A4)}$$

$$\Delta V_{[24]} = \frac{1}{8m}\left(\frac{\partial}{\partial q^j}\frac{\partial q^i}{\partial x_\nu}\right)\left(\frac{\partial}{\partial q^i}\frac{\partial q^j}{\partial x^\nu}\right). \quad \text{(A5)}$$

Let us make some comments:

1) DeWitt [4] uses a prepoint lattice definition.
2) McLaughlin and Schulman [28] derive their quantum correction by evaluating g_{ab} in the Lagrangian at midpoints, but the measure term $\sqrt{g}$ at prepoints, which is at first sight rather puzzling. We do not express their result in Christoffels Γ^a_{ab}, because $\Delta V_{[28]}$ is rather complicated.
3) The quantum corrections corresponding to the Weyl-ordering rule lead to a lattice-prescription where *all* metric expressions – except the normalisation C – have to be evaluated at midpoints. The appropriate Hamiltonian to start with reads then

$$H = \frac{1}{2m} g^{-1/4} p_a g^{1/2} g^{ab} p_b g^{-1/4} = -\frac{1}{2m}\Delta_{LB} \quad \text{(A6)}$$

and $p_a = -i(\partial_a + \Gamma_a/2)$ (see e.g. [24, 29, 30] for details). We have stated different formulations for $\Delta V_{[26,30]}$ to simplify a comparison with the other quantum corrections.
4) Dowker and Mayes [8] derive their result by the symmetrisation rule:

$$\mathscr{H}(p, q'', q') = \tfrac{1}{2}[\mathscr{H}_{\text{eff}}(p, q'') + \mathscr{H}_{\text{eff}}(p, q')] \quad \text{(A7)}$$

($\mathscr{H}$ denotes the Hamilton function to be used in the lattice version of the Hamilton path integral and $\mathscr{H}_{\text{eff}}$ denotes the effective Hamilton function).
5) Lee's [24] very compact form is also based on Weyl-ordering (x, q denote cartesian and curvilinear coordinates, respectively). It is for dimensions $d > 1$ equivalent to our ΔV. But for one dimensional cases difficulties arise, which can be seen e.g. in the case of the Morse-potential, where the sign in $\Delta V_{[24]}$ is wrong.

Appendix B

We want to prove that with the short-time kernel

$K(\{\theta^{(j+1)}\}, \{\theta^{(j)}\}; \varepsilon)$

$$= \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2} \exp\left\{i\varepsilon \mathscr{L}^N_{\text{Cl}}(\{\theta^{(j+1)}\}, \{\theta^{(j)}\}) + \frac{i\varepsilon}{8mR^2}\left[(d-2)^2 + \frac{1}{\sin\theta_1^{(j+1)}\sin\theta_1^{(j)}} + \cdots + \frac{1}{\sin\theta_1^{(j+1)}\sin\theta_1^{(j)}\cdots\sin\theta_{d-2}^{(j+1)}\sin\theta_{d-2}^{(j)}}\right]\right\} \quad \text{(B1)}$$

and the time-evolution equation

$$\psi(\{\theta^{(j+1)}\}, t+\varepsilon) = \int d\Omega^{(j)} K(\{\theta^{(j+1)}\}, \{\theta^{(j)}\}; \varepsilon)\psi(\{\theta^{(j)}\}, t) \quad \text{(B2)}$$

the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi^\mu_l = -\frac{1}{2mR^2} L^2_{(d)}\psi^\mu_l \quad \text{(B3)}$$

can be derived.* For this purpose, a Taylor expansion has to be performed in (B2) yielding (identify $\theta_{d-1} = \phi$, $\theta' := \theta^{(j)}$ and $\theta'' := \theta^{(j+1)}$):

$$\psi(\{\theta''\}; t) + \varepsilon\frac{\partial\psi(\{\theta''\}, t)}{\partial t} = \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2} e^{-i\varepsilon\Delta V(\{\theta''\})}\left\{\psi(\{\theta''\}; t)B_0 + \sum_{\nu=1}^{d-1}\frac{\partial\psi(\{\theta''\}; t)}{\partial\theta''}(B_{\theta_\nu} - \theta''_\nu B_0) + \frac{1}{2}\sum_{\substack{\mu,\nu=1\\ \mu\geqq\nu}}^{d-1}\frac{\partial^2\psi(\{\theta''\}; t)}{\partial\theta''_\mu\partial\theta''_\nu}(B_{\theta_\mu\theta_\nu} - \theta''_\mu B_{\theta_\nu} - \theta''_\nu B_{\theta_\mu} + \theta''_\mu\theta''_\nu B_0)\right\} \quad \text{(B4)}$$

where we have used the abbreviations

$$\left.\begin{aligned}
B_0 &= \int d\Omega' e^{i\varepsilon\mathscr{L}^N_{Cl}(\{\theta''\}, \{\theta'\})} \simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-1)/2} e^{i\varepsilon\Delta V(\{\theta''\})}\\
B_\phi &= \int d\Omega' \phi' e^{i\varepsilon\mathscr{L}^N_{Cl}} \simeq \phi'' B_0\\
B_{\phi^2} &= \int d\Omega' \phi'^2 e^{i\varepsilon\mathscr{L}^N_{Cl}} \simeq \phi''^2 B_0 + \frac{1}{\sin^2\theta''_1\cdots\sin^2\theta''_{d-2}}\frac{i\varepsilon}{mR^2}B_0\\
B_{\theta_\nu} &= \int d\Omega' \theta'_\nu e^{i\varepsilon\mathscr{L}^N_{Cl}}\\
&\simeq \theta''_\nu B_0 + \frac{1}{2}\frac{d-\nu-1}{\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}}\cot\theta''_\nu\frac{i\varepsilon}{mR^2}B_0\\
B_{\phi\theta_\nu} &= \int d\Omega' \phi'\theta'_\nu e^{i\varepsilon\mathscr{L}^N_{Cl}} \simeq \phi''\theta''_\nu B_0\\
&\quad + \frac{1}{2}\frac{d-\nu-1}{\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}}\phi''\cot\theta''_\nu\frac{i\varepsilon}{mR^2}B_0
\end{aligned}\right\} \quad \text{(B5)}$$

---

* A similar calculation for $d = 3$ was done by Patrascioiu and Richard [35]

$$B_{\theta_\nu\theta_\mu}=\int d\Omega'\theta'_\nu\theta'_\mu e^{i\varepsilon\mathscr{L}^N_{Cl}}\simeq\theta''_\nu\theta''_\mu B_0+\frac{1}{2}\left[\frac{d-\nu-1}{\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}}\cot\theta''_\nu+\frac{d-\mu-1}{\sin^2\theta''_1\cdots\sin^2\theta''_{\mu-1}}\cot\theta''_\mu\right]\frac{i\varepsilon}{mR^2}B_0$$

$$B_{\theta^2_\nu}=\int d\Omega'\theta'^2_\nu e^{i\varepsilon\mathscr{L}^N_{Cl}}\simeq\theta''_\nu B_0+\frac{1+(d-\nu-1)\cot\theta''_\nu}{\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}}\frac{i\varepsilon}{mR^2}B_0.$$

Here the equations are valid up to terms of $O(\varepsilon^{(d+1)/2})$, and

$$\mathscr{L}^N_{Cl}(\{\theta''\},\{\theta'\})=\frac{mR^2}{2\varepsilon^2}[(\theta'_1-\theta''_1)^2+\cdots+(\sin\theta'_1\sin\theta''_1\cdots\sin\theta'_{d-2}\sin\theta''_{d-2})(\phi'-\phi'')^2] \tag{B6}$$

denotes a "classical Lagrangian" on the lattice. In order to make the calculation manageable, we have taken the ΔV-term at the argument $\{\theta''\}$ and have factored out this term in (B4). This is legitimate, because changing $\sin\theta'_\nu(\theta'_\nu$ is an integration variable) to $\sin\theta''_\nu$ in ΔV gives a correction of $O(\varepsilon)$, hence of order ε^2 in the short time-kernel and, therefore, can be omitted.

We shall only illustrate how to calculate the integral B_0 in (B5). All other integrals containing powers of θ'_ν are of similar type because they are of Gaussian form. For simplification we use the abbreviations ($\nu=1,\ldots,d-1$):

$$E(\theta_\nu)=\exp\left\{\frac{imR^2}{2\varepsilon}[(\theta'_1-\theta''_1)^2+\cdots+(\sin\theta'_1\sin\theta''_1\cdots\sin\theta'_{\nu-1}\sin\theta''_{\nu-1})(\theta'_\nu-\theta''_\nu)^2)\right\} \tag{B7}$$

and $\alpha=mR^2/2i\varepsilon$.

We consider now the integral

$$B_0=\int_0^\pi d\theta'_1\sin^{d-2}\theta'_1\cdots\int_0^\pi d\theta'_{d-2}\sin\theta'_{d-2}\int_0^{2\pi}d\phi' E(\theta_{d-1})\simeq\int_0^\pi d\theta'_1\sin^{d-2}\theta'_1\cdots\int_0^\pi d\theta'_{d-2}\sin\theta'_{d-2}E(\theta_{d-2})\cdot\int_{-\infty}^{\infty}dx\,e^{-\alpha(\sin\theta'_1\cdots\sin\theta''_{d-2})x^2} \tag{B8}$$

where we have set $x:=\phi'-\phi''$ which varies from $-\infty$ to $+\infty$, and "$\simeq$" denotes that this is correct in the limit $\varepsilon\to 0$. The x-integration is of Gaussian form, and we get

$$B_0\simeq\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{1/2}\int_0^\pi d\theta'_1\frac{\sin^{d-2}\theta'_1}{\sqrt{\sin\theta'_1\sin\theta''_1}}\cdots\int_0^\pi d\theta'_{d-2}\sqrt{\frac{\sin\theta'_{d-2}}{\sin\theta''_{d-2}}}E(\theta_{d-2})$$

$$\simeq\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{1/2}\int_0^\pi d\theta'_1\frac{\sin^{d-2}\theta'_1}{\sqrt{\sin\theta'_1\sin\theta''_1}}\cdots\int_0^\pi d\theta'_{d-3}\frac{\sin^2\theta'_{d-3}}{\sqrt{\sin\theta'_{d-3}\sin\theta''_{d-3}}}E(\theta_{d-3})\cdot\int_{-\infty}^{\infty}dx\left[1+\tfrac{1}{2}\cot\theta''_{d-2}\cdot x-\frac{x^2}{8}\left(1+\frac{1}{\sin^2\theta''_{d-2}}\right)\right]\cdot e^{-\alpha(\sin\theta'_1\cdots\sin\theta''_{d-3})x^2} \tag{B9}$$

where we have performed a Taylor expansion around θ''_{d-2} in the last step. The integral is again Gaussian, the term linear in x vanishes* and we get

$$\left.\begin{aligned}
B_0&\simeq\left(\frac{2\pi i\varepsilon}{mR^2}\right)\int_0^\pi d\theta'_1\frac{\sin^{d-2}\theta'_1}{\sin\theta'_1\sin\theta''_1}\cdots\int_0^\pi d\theta'_{d-3}\frac{\sin^2\theta'_{d-3}}{\sin\theta'_{d-3}\sin\theta''_{d-3}}E(\theta_{d-3})\cdot\left[1-\frac{i\varepsilon}{8mR^2}\cdot\frac{1}{\sin\theta'_1\cdots\sin\theta''_{d-3}}\left(1+\frac{1}{\sin^2\theta''_{d-2}}\right)\right]\\
&\simeq\left(\frac{2\pi i\varepsilon}{mR^2}\right)\int_0^\pi d\theta'_1\frac{\sin^{d-2}\theta'_1}{\sin\theta'_1\sin\theta''_1}\cdots\int_0^\pi d\theta'_{d-4}\frac{\sin^3\theta'_{d-4}}{\sin\theta'_{d-4}\sin\theta''_{d-4}}E(\theta_{d-4})\cdot\int_{-\infty}^{\infty}dx\left(1+\cot\theta''_{d-3}\cdot x-\frac{x^2}{2}\right)\cdot e^{-\alpha(\sin\theta'_1\cdots\sin\theta''_{d-4})x^2}\\
&\quad-\frac{i\varepsilon}{8mR^2}\left(\frac{2\pi i\varepsilon}{mR^2}\right)\int_0^\pi d\theta'_1\frac{\sin^{d-2}\theta'_1}{(\sin\theta'_1\sin\theta''_1)^2}\cdots\int_0^\pi d\theta'_{d-4}\frac{\sin^3\theta'_{d-4}}{(\sin\theta'_{d-4}\sin\theta''_{d-4})^2}E(\theta_{d-4})\cdot\frac{1}{\sin^2\theta''_{d-3}}\left(1+\frac{1}{\sin^2\theta''_{d-2}}\right)\cdot\int_{-\infty}^{\infty}dx\,e^{-\alpha(\sin\theta'_1\cdots\sin\theta''_{d-4})x^2}\\
&\simeq\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{3/2}\int_0^\pi d\theta'_1\frac{\sin^{d-2}\theta'_1}{(\sin\theta'_1\sin\theta''_1)^{3/2}}\cdots\int_0^\pi d\theta'_{d-4}\frac{\sin^3\theta'_{d-4}}{(\sin\theta'_{d-4}\sin\theta''_{d-4})^{3/2}}E(\theta_{d-4})\\
&\quad\cdot\left[1-\frac{i\varepsilon}{8mR^2}\cdot\frac{1}{\sin\theta'_1\cdots\sin\theta''_{d-4}}\cdot\left(4+\frac{1}{\sin^2\theta''_{d-3}}+\frac{1}{\sin^2\theta''_{d-3}\sin^2\theta''_{d-2}}\right)\right],
\end{aligned}\right\} \tag{B.10}$$

* It will become important in the calculation of the other integrals, e.g. in B_{θ_ν}, where it generates the term proportional to $\cot\theta_\nu$

and so on up to the kth step:

$$B_0 \simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{k/2} \int_0^{\pi} d\theta'_1 \frac{\sin^{d-2}\theta'_1}{(\sin\theta'_1 \sin\theta''_1)^{k/2}} \cdots \int_0^{\pi} d\theta'_{d-1-k}$$

$$\cdot \frac{\sin^k \theta'_{d-1-k}}{(\sin\theta'_{d-1-k}\sin\theta''_{d-1-k})^{k/2}} E(\theta_{d-1-k})$$

$$\cdot\left\{1 - \frac{i\varepsilon}{8mR^2} \cdot \frac{1}{\sin\theta'_1 \cdots \sin\theta''_{d-1-k}} \left[(k-1)^2 + \frac{1}{\sin^2\theta''_{d-k}} + \frac{1}{\sin^2\theta''_{d-k}\cdots\sin^2\theta''_{d-2}}\right]\right\}, \tag{B11}$$

and finally after d–1 steps:

$$B_0 \simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-1)/2} + \left\{1 - \frac{i\varepsilon}{8mR^2}\left[(d-2)^2 + \frac{1}{\sin^2\theta''_1} + \cdots + \frac{1}{\sin^2\theta''_1\cdots\sin^2\theta''_{d-2}}\right]\right\} \tag{B12}$$

which gives in the required approximation the result quoted in (B5). Substituting the expressions (B5) in the Taylor expansion (B4), one obtains in the limit $\varepsilon \to 0$ the correct Schrödinger equation (B3).

Appendix C

In deriving the Schrödinger equation (II.1) from (II.8) with the help of the short-time kernel of (II.27) one has to perform a Taylor expansion in (II.18) yielding ($z = mR^2/\varepsilon$-identify $\theta_{d-1} = \phi$):

$$\psi(\{\theta''\};t) + \varepsilon \frac{\partial\psi(\{\theta''\},t)}{\partial t}$$

$$= \left(\frac{z}{2\pi i}\right)^{(d-1)/2} e^{iz + (i/8z)(d-1)(d-3)}$$

$$\cdot\left\{\psi(\{\theta''\};t)B_0 + \sum_{\nu=1}^{d-1} \frac{\partial\psi(\{\theta''\};t)}{\partial\theta''}(B_{\theta_\nu} - \theta''_\nu B_0) + \frac{1}{2}\sum_{\substack{\mu,\nu=1\\ \mu\geq\nu}}^{d-1} \frac{\partial^2\psi(\{\theta''\};t)}{\partial\theta''_\mu\partial\theta''_\nu}\right.$$

$$\left.\cdot(B_{\theta_\mu\theta_\nu} - \theta''_\mu B_{\theta_\nu} - \theta''_\nu B_{\theta_\mu} + \theta''_\mu\theta''_\nu B_0)\right\}. \tag{C1}$$

We have used the abbreviations ($\cos\psi$ as in (II.22)):*

$$B_0 = \int d\Omega' e^{-iz\cos\psi} = 2\pi\left(\frac{2\pi i}{z}\right)^{(d-2)/2} I_{(d-2)/2}(z/i)$$

$$B_\phi = \int d\Omega' \phi' e^{-iz\cos\psi} = \phi'' B_0$$

$$B_{\phi^2} = \int d\Omega' \phi'^2 e^{-iz\cos\psi} \simeq \phi''^2 B_0 + \frac{1}{\sin^2\theta''_1\cdots\sin^2\theta''_{d-2}}\left(\frac{2\pi i}{z}\right)^{d/2} I_{(d-1)/2}(z/i)$$

$$B_{\theta_\nu} = \int d\Omega' \theta'_\nu e^{-iz\cos\psi} \simeq \theta''_\nu B_0 + \frac{1}{2\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}} \frac{d-\nu-1}{}\cot\theta''_\nu \left(\frac{2\pi i}{z}\right)^{d/2} I_{(d-2)/2}(z/i)$$

$$B_{\phi\theta_\nu} = \int d\Omega' \phi'\theta'_\nu e^{-iz\cos\psi} \simeq \phi''\theta''_\nu B_0 + \frac{1}{2}\frac{d-\nu-1}{\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}}\phi'' \cdot\cot\theta''_\nu\left(\frac{2\pi i}{z}\right)^{d/2} I_{(d-2)/2}(z/i)$$

$$B_{\theta_\nu\theta_\mu} = \int d\Omega' \theta'_\nu\theta'_\mu e^{-iz\cos\psi} \simeq \theta''_\nu\theta''_\mu B_0 + \frac{1}{2}\left[\frac{d-\nu-1}{\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}}\cot\theta''_\nu + \frac{d-\mu-1}{(\sin^2\theta''_1\cdots\sin^2\theta''_{\mu-1})}\cot\theta''_\mu\right]\left(\frac{2\pi i}{z}\right)^{d/2} I_{(d-2)/2}(z/i)$$

$$B_{\theta_\nu^2} = \int d\Omega' \theta'^2_\nu e^{-iz\cos\psi} \simeq \theta''^2_\nu B_0 + \frac{1 + (d-\nu-1)\cot\theta''_\nu}{\sin^2\theta''_1\cdots\sin^2\theta''_{\nu-1}} \cdot\left(\frac{2\pi i}{z}\right)^{d/2} I_{(d-2)/2}(z/i) \tag{C2}$$

where the equations are valid up to terms of $O(\varepsilon^{(d+1)/2})$. The integrals are now more complicated than in Appendix B. Nevertheless, (C2a) is relatively easy to prove. With the use of [12, p. 488]

$$\int_0^{2\pi} e^{p\cos x + q\sin x}\begin{pmatrix}\cos mx\\ \sin mx\end{pmatrix} = \begin{pmatrix}1\\ i\end{pmatrix}\pi(p^2+q^2)^{-m/2} I_m(\sqrt{p^2+q^2}) \cdot\left[(p+iq)^m + \begin{pmatrix}+1\\ -1\end{pmatrix}(p-iq)^m\right], \tag{C3}$$

one gets for the ϕ-integration in $B_0(f_\nu = \sin\theta'_1 \cdot\sin\theta''_1\cdots\sin\theta'_{\nu-1}\sin\theta''_{\nu-1})$:

$$\int_0^{2\pi} e^{izf_{d-1}\cos(\phi''-\phi')} d\phi' = 2\pi J_0(zf_{d-1}), \tag{C4}$$

so one has to consider in the next steps the iterated integrals:

$$B_0 = 2\pi\int_0^{\pi} d\theta'_1 \sin^{d-2}\theta'_1 e^{-iz\cos\theta'_1\cos\theta''_1}\cdots$$

$$\cdots\int_0^{\pi} d\theta'_{d-2}\sin\theta'_{d-2} e^{-izf_{d-2}\cos\theta'_{d-2}\cos\theta''_{d-2}}$$

$$\cdot J_0(zf_{d-2}\sin\theta'_{d-2}\sin\theta''_{d-2}). \tag{C5}$$

These integrals can be calculated using the formulas

* The abbreviations (C2) should not be mixed up with the abbreviations used in (B5)

(see [12, pp. 1031, 830, 938], respectively):

$$J_{1/2-\lambda}(z\sin\alpha\sin\beta)(z\sin\alpha\sin\beta)^{1/2-\lambda}e^{-iz}\cos\alpha\cos\beta$$
$$=\frac{\sqrt{2}\Gamma(\lambda)}{\Gamma(\lambda+1/2)}\sum_{k=0}^{\infty}(k+\lambda)i^{-k}\cdot\frac{J_{k+\lambda}(z)C_k^\lambda(\cos\alpha)C_k^\lambda(\cos\beta)}{z^\lambda C_k^\lambda(1)} \tag{C6}$$

$$\int_0^\pi C_k^\lambda(\cos\alpha)\sin^{2\lambda}\alpha\,d\alpha=\frac{\pi}{2^{2\lambda}}\frac{\Gamma(2\lambda+1)}{\Gamma^2(\lambda+1)}\delta_{k,0}, \tag{C7}$$

$$\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+1/2) \tag{C8}$$

in each step, yielding finally:

$$B_0=(2\pi)^{d/2}\left(\frac{i}{z}\right)^{(d-2)/2}I_{(d-2)/2}(z/i). \tag{C9}$$

The integrals (C2b, c) are calculated by expanding (C3) on both sides about $m=0$ and using the asymtotic form of the modified Bessel functions

$$I_k(z)\simeq\frac{1}{\sqrt{2\pi z}}\exp\left\{z-\frac{k^2-1/4}{2z}\right\}\quad(|z|\gg 1) \tag{C10}$$

together with the completeness relation

$$\sum_{k=0}^{\infty}U_k^\lambda(x)U_k^\lambda(y)=\delta(x-y)(1-x^2)^{1/2-\lambda} \tag{C11}$$

for the orthonormal Gegenbauer polynomials:

$$U_k^\lambda(x)=\sqrt{\frac{\Gamma(\lambda)(k+\lambda)}{\sqrt{\pi}\Gamma(\lambda+1/2)}}\frac{C_k^\lambda(x)}{C_k^\lambda(1)}. \tag{C12}$$

The integrals in (C2) containing powers of θ_ν are calculated in the same manner up to the νth integration. There one has to use the approximation

$$I_{k+\lambda}(z)\simeq I_\lambda(z)[1+k(k+2\lambda)/z]\quad(|z|\gg 1) \tag{C13}$$

and the defining differential equation for the Gegenbauer polynomials

$$F''(x)+\frac{x}{x^2-1}(2\lambda+1)F'(x)-\frac{1}{x^2-1}k(k+2\lambda)F(x)=0 \tag{C14}$$

in order to perform the summation:

$$\sum_{k=0}^{\infty}k(k+2\lambda)U_k^\lambda(x)U_k^\lambda(y)=(x^2+1)\frac{d^2}{dx^2}[\delta(x-y)(1-x^2)^{1/2-\lambda}]+(2\lambda+1)\frac{d}{dx}[\delta(x-y)(1-x^2)^{1/2-\lambda}]. \tag{C15}$$

With these means the equations (C2) are easily derived and the Schrödinger equation is proved.

Appendix D

In this appendix we shall derive the Schrödinger equation from the short-time kernel of the radial path integral (III.11). For the time evolution from t to $t+\varepsilon$ we have:

$$\psi(r'',t+\varepsilon)=\left(\frac{m}{i\varepsilon}\right)r''^{(2-d)/2}e^{(im/2\varepsilon)r''^2-i\varepsilon V(r'')}\cdot\int_0^\infty e^{(im/2\varepsilon)r^2}I_{l+(d-2)/2}((m/i\varepsilon)rr'')r^{d/2}\psi(r,t)\,dr. \tag{D1}$$

Taylor expansion yields:

$$\psi(r'',t)+\varepsilon\frac{\partial\psi(r'',t)}{\partial t}+\cdots=\left(\frac{m}{i\varepsilon}\right)r''^{(2-d)/2}e^{(im/2\varepsilon)r''^2-i\varepsilon V(r'')}\cdot\left\{\psi(r'',t)B_0+\frac{\partial\psi(r'',t)}{\partial r''}[B_1-r''B_0]+\frac{1}{2}\frac{\partial^2\psi(r'',t)}{\partial r''^2}[B_2-2r''B_1+r''^2B_0]\right\} \tag{D2}$$

with ($n=0$, 1, 2):

$$B_n=\int_0^\infty r^{n+d/2}I_{l+(d-2)/2}\left(\frac{m}{i\varepsilon}rr''\right)e^{-(m/2i\varepsilon)r^2}dr=i^{-(l+d/2-1)}\left(\frac{\varepsilon}{mr}\right)\left(\frac{2\varepsilon i}{m}\right)^{n/2+d/4}\frac{\Gamma((n+d+l)/2)}{\Gamma(l+d/2)}\cdot e^{-imr''^2/4\varepsilon}M_{n/2+d/4,(1/2)(l+d/2-1)}\left(\frac{imr''^2}{2\varepsilon}\right).$$

$$B_n=\left(\frac{\varepsilon}{mr''}\right)\left(\frac{2i\varepsilon}{m}\right)^{n/2+d/4}e^{-imr''^2/4\varepsilon}i^{1-d/2-n}\cdot\left\{W_{n/2+d/4,(1/2)(l+d/2-1)}\left(\frac{imr''^2}{2\varepsilon}\right)+i^{-(l+d/2)}\cdot\frac{\Gamma((l+d+n)/2)}{\Gamma((l-n)/2)}W_{-(n/2)-(d/4),(1/2)(l+(d/2)-1)}\cdot\left(-\frac{imr''^2}{2\varepsilon}\right)\right\}$$
$$\simeq\left(\frac{i\varepsilon}{m}\right)r''^{(d-2)/2+n}e^{-imr''^2/2\varepsilon}\cdot\left\{1-\frac{i\varepsilon}{2mr''^2}[l(l+d-2)-n(n+d-2)]\right\}+\left(\frac{2\varepsilon}{mi}\right)^{n+d/2}\left(\frac{\varepsilon}{m}\right)i^{-l+n+1}r^{-n-d/2-1}e^{imr''^2/2\varepsilon}\cdot\left\{1+\frac{i\varepsilon}{2mr''^2}[l(l+d-2)-n(n+d+2)-2d]\right\} \tag{D3}$$

where we have used the integral formula

$$\int_0^\infty x^\mu e^{-\alpha x^2} J_\nu(\beta x)\,dx = \frac{\Gamma((\mu+\nu+1)/2)}{\beta\alpha^{\mu/2}\Gamma(\nu+1)} e^{-\beta^2/8\alpha} M_{\mu/2,\nu/2}\left(\frac{\beta^2}{4\alpha}\right) \tag{D4}$$

and some properties of the Whittaker functions:

$$M_{\lambda,\mu}(z) = \Gamma(2\mu+1)e^{-i\pi\lambda} \cdot \left[\frac{e^{i\pi(\mu+1/2)}W_{\lambda,\mu}(z)}{\Gamma(\mu+\lambda+1/2)} + \frac{W_{-\lambda,\mu}(-z)}{\Gamma(\mu-\lambda+1/2)}\right] \tag{D5}$$

$$W_{\lambda,\mu}(z) \simeq z^\lambda e^{-z/2} \cdot \left\{1 + \sum_{k=1}^\infty \frac{(\mu^2-(\lambda-1/2)^2)\cdots(\mu^2-(\lambda-k+1/2)^2)}{k!\,z^k}\right\}. \tag{D6}$$

Here the last equation is valid for $|z| \gg 1$, $|\arg(z)| < \pi$ (see [12, pp. 716, 1062, 1061], respectively). In (D3) we have to take into account only the terms $O(\varepsilon)$. This implies that only the first term in (D3) is relevant:

$$B_n \simeq \left(\frac{i\varepsilon}{m}\right) r''^{n+(d-2)/2} e^{-imr''^2/2\varepsilon} \cdot \left\{1 - \frac{i\varepsilon}{2mr''^2}[l(l+d-2) - n(n+d-2)]\right\}. \tag{D7}$$

In particular:

$$B_0 \simeq \left(\frac{i\varepsilon}{m}\right) r''^{(d-2)/2} e^{(im/2\varepsilon)r''^2}\left[1 - \frac{i\varepsilon}{2mr''^2} l(l+d-2)\right]$$

$$B_1 \simeq B_0 r''\left[1 - \frac{i\varepsilon}{2mr''^2}(d-1)\right] \tag{D8}$$

$$B_2 \simeq B_0 r''^2\left[1 - \frac{i\varepsilon}{2mr''^2} 2d\right].$$

These equations inserted in the Taylor expansion (D2) yield in the limit $\varepsilon \to 0$:

$$i\frac{\partial\psi(r,t)}{\partial t} = \left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right) + \frac{l(l+d-2)}{2mr^2} + V(r)\right]\psi(r,t) \tag{D9}$$

which is the correct Schrödinger equation.

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