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SCALING LAWS AND TRIVIALITY BOUNDS IN THE LATTICE ϕ^4 THEORY
(I). One-component model in the symmetric phase

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The lattice ϕ^4 theory in four space-time dimensions is most likely "trivial", i.e. its continuum limit is a free field theory. However, for small but positive lattice spacing a and at energies well below the cutoff mass $\Lambda = 1/a$, the theory effectively behaves like a continuum theory with particle interactions, which may be appreciable. By a combination of known analytical methods, we here determine the maximal value of the renormalized coupling at zero momentum as a function of Λ/m, where m denotes the mass of the scalar particle in the theory. Moreover, a complete solution of the model is obtained in the sense that all low energy amplitudes can be computed with reasonable estimated accuracy for arbitrarily chosen bare coupling and mass in the symmetric phase region.

1. Introduction

The triviality of the lattice ϕ^4 theory in four dimensions has not yet been established rigorously, but the perturbative renormalization group analysis [1] together with the accumulated evidence from high temperature series expansions [2–4], numerical simulations [5–9], exact inequalities [10–14] and rigorous block spin renormalization group studies [15–17] leaves little doubt that the conjecture, first formulated by Wilson many years ago [2], is in fact true. Other field theories including (pure) QED and the standard SU(2) Higgs model [18, 19] are also believed to be trivial, although the arguments given in these cases are less convincing.

Trivial field theories are not necessarily useless for the description of elementary particles and their interactions, nor does triviality imply that the renormalized perturbation expansion is meaningless. The crucial point to note is that trivial field theories exist and may be far from being free provided we allow for a large but

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finite ultraviolet cutoff Λ, which may be introduced explicitly in the form of a lattice, for example, or which can arise dynamically in the context of an embracing more complete theory. For low energy processes, the precise specification of the cutoff is unimportant, because the corresponding amplitudes are universal up to tiny effects of order E^2/Λ^2, where E is a typical energy in the process. Thus, as long as the cutoff Λ is well beyond the experimentally accessible region, a trivial theory may provide an accurate and mathematically well-defined model of elementary particle interactions.

Nevertheless, triviality signals a serious defect of the theory, because the cutoff Λ cannot be pushed to arbitrary high values once the renormalized coupling g_R at zero momentum has been fixed experimentally (or otherwise). Quantitatively, this is expressed by an upper bound on Λ, which, at small g_R, assumes the form

$$\ln(\Lambda/m_R) \leqslant A/g_R + B \ln g_R + O(1), \tag{1.1}$$

where A and B are known constants and m_R denotes a renormalized particle mass. We emphasize that this bound is only meaningful, when the details of the cutoff procedure have been specified, in particular, the O(1) term in eq. (1.1) is not universal and cannot be calculated in perturbation theory.

In the context of the theory of electroweak interactions, the above considerations have led to upper bounds on the mass of the elusive Higgs particle [20–26]. A weak point in the derivation of these bounds usually is that only the first two terms in eq. (1.1) are kept, which is an approximation of unknown quality, especially so, if one is also interested in situations where the renormalized Higgs self-coupling is large and the cutoff Λ is as low as a few TeV. It has thus been proposed [25], to determine the exact upper bound for all values of g_R in the lattice theory and this is what will be achieved in this paper for the simpler case of the one-component ϕ^4 theory in the symmetric phase, hoping that the methods developed will also be useful in the physically relevant cases.

Other results on the ϕ^4 theory obtained in the present work include a quantitative plot of the renormalization group trajectories (curves of constant coupling g_R) in the plane of bare parameters and a determination of the region in the phase diagram, where renormalized perturbation theory may be expected to apply. To our surprise, this region turns out to be rather large and apparently includes the scaling region* defined by the inequality

$$\Lambda \geqslant 2m_R\,. \tag{1.2}$$

Thus, as long as one is only interested in situations where the cutoff is reasonably large, the low energy amplitudes seem to be essentially given by renormalized perturbation theory.

* We use the term "scaling region" to denote an area in the phase diagram, where the low energy amplitudes depend only weakly on the cutoff (cf. subsect. 4.2).

The initial motivation for our study was that we wanted to compare the stock of known analytic results on the ϕ^4 theory with high precision Monte Carlo data, which were generated in the course of an investigation of finite volume effects on the energy spectrum in this model [27]. As a consequence, the numerical aspects of our work are emphasized, in particular, we have taken care to estimate and quote errors throughout.

To a large extent, our paper is a review of known methods and results. However, we feel that the perturbative techniques of ref. [1], for example, are perhaps not as well-known to the lattice gauge theory community as they deserve to be. Also, in the past 10 years or so, many new results on different aspects of the ϕ^4 theory have been obtained which we here attempt to integrate in a unified picture.

The organization of our paper is as follows. After introducing our notations on the lattice ϕ^4 theory in sect. 2, the idea of how to "solve" it is sketched in sect. 3. One of the basic tools used is the lattice Callan-Symanzik equation, which we derive in sect. 4. This section also contains an extensive discussion of the scaling behaviour of the theory near the critical line. In sect. 5, we show that there is an interesting connection between the divergence of the renormalized perturbation series and the existence of scaling violations (the terms of order E^2/Λ^2 mentioned above). The other basic tool used is the "high temperature" expansion, which allows us to determine the renormalized coupling g_R and other relevant quantities in the non-scaling region $\Lambda \leqslant 2m_R$ (sect. 6). By considering a number of examples, we then argue (sect. 7) that renormalized perturbation theory should be applicable when g_R is smaller than the tree level unitarity bound. In sects. 8 and 9, our main results, the triviality bound and a quantitative plot of the renormalization group trajectories, are obtained by combining the renormalization group equations with the data provided by the high temperature series analysis. The paper ends with a few concluding remarks (sect. 10) and three appendices, where we list the perturbation expansion coefficients for various quantities.

2. Basic definitions

We consider real valued fields $\phi(x)$ on the hypercubic lattice with points $x \in \mathbb{Z}^4$. For notational convenience, we choose lattice units throughout this paper, which means that all length scales are measured in numbers of lattice spacings.

A popular way to write the action of the lattice ϕ^4 theory is

$$S = \sum_x \left\{ -\kappa \sum_{\mu=0}^{3} \left(\phi(x)\phi(x+\hat{\mu}) + \phi(x)\phi(x-\hat{\mu}) \right) + \phi(x)^2 + \lambda\left(\phi(x)^2 - 1\right)^2 \right\}, \tag{2.1}$$

where $\hat{\mu}$ denotes the unit vector in the positive μ-direction and the bare parameters

κ, λ are restricted to the range $\kappa \geqslant 0$, $\lambda \geqslant 0$. This formulation, which is convenient for Monte Carlo simulations and the high temperature expansion, for example, is completely equivalent to the more traditional expression

$$S = \sum_x \left\{ \tfrac{1}{2} \sum_{\mu=0}^{3} \left(\partial_\mu \phi_0(x) \right)^2 + \tfrac{1}{2} m_0^2 \phi_0(x)^2 + \frac{g_0}{4!} \phi_0(x)^4 \right\}, \tag{2.2}$$

where

$$\partial_\mu \phi_0(x) = \phi_0(x+\hat{\mu}) - \phi_0(x) \tag{2.3}$$

denotes the lattice derivative of ϕ_0. Indeed, eq. (2.2) is obtained from eq. (2.1) by setting

$$\phi_0(x) = \sqrt{2\kappa}\,\phi(x), \tag{2.4a}$$

$$m_0^2 = (1-2\lambda)/\kappa - 8, \tag{2.4b}$$

$$g_0 = 6\lambda/\kappa^2. \tag{2.4c}$$

Note that for $\lambda \to \infty$ and fixed κ, the theory reduces to the Ising model, whereas for $\lambda = 0$ one has a free field theory with mass m_0 (in this limit, the theory is actually only defined if $\kappa \leqslant \frac{1}{8}$).

The lattice ϕ^4 theory is known to exist in two phases, one where the reflection symmetry $\phi \to -\phi$ is spontaneously broken and the other where it is not. The corresponding regions in the κ, λ-plane are separated by a critical curve $\kappa_c(\lambda)$, which qualitatively looks as in fig. 1 (accurate numbers will be given later). In what follows, we shall only discuss the symmetric phase, which is characterized by $\kappa < \kappa_c$.

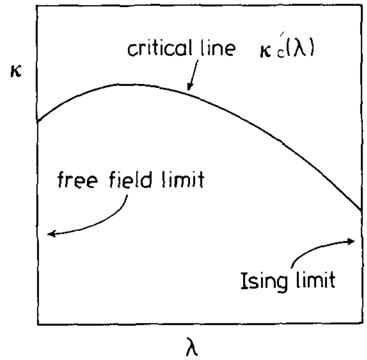

Fig. 1. Qualitative plot of the phase diagram of the lattice ϕ^4 theory. The broken symmetry phase ($\kappa > \kappa_c$) and the symmetric phase ($\kappa < \kappa_c$) are separated by a critical line, where the mass gap of the theory vanishes.

Apart from $\phi(x)$, the composite field

$$\mathcal{O}(x) = \sum_{\mu=0}^{3} \{\phi(x)\phi(x+\hat{\mu}) + \phi(x)\phi(x-\hat{\mu})\}, \tag{2.5}$$

which is proportional to $\phi(x)^2$ in the formal continuum limit, will later play an important rôle in the renormalization group analysis. We thus consider the generating functional $W(H, K)$ defined by

$$e^W = \frac{1}{Z}\int \prod_x \mathrm{d}\phi(x) \exp\left\{-S + \sum_x (H(x)\phi(x) + K(x)\mathcal{O}(x))\right\}, \tag{2.6}$$

where H, K denote the source fields and Z is determined by the requirement $W(0,0) = 0$. The coefficients in the expansion of $W(H, K)$ in powers of H and K are the connected correlation functions of ϕ and $\mathcal{O}$. To obtain the corresponding vertex functions, we introduce the local magnetization

$$M(x) = \frac{\partial W}{\partial H(x)} \tag{2.7}$$

and define the Legendre transform $\Gamma(M, K)$ of $W(H, K)$ through

$$\Gamma = W - \sum_x H(x)M(x), \tag{2.8}$$

where H is to be expressed as a function of M and K by solving eq. (2.7). Then, as is well-known, the coefficients $\Gamma^{(n,l)}$ in the expansion

$$\Gamma = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{n!l!}\int_{-\pi}^{\pi} \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4 p_n}{(2\pi)^4}\frac{\mathrm{d}^4 q_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4 q_l}{(2\pi)^4} (2\pi)^4 \delta_{\mathrm{P}}\left(\sum_{i=1}^{n} p_i + \sum_{j=1}^{l} q_j\right)$$
$$\times \Gamma^{(n,l)}(p_1,\ldots,p_n; q_1,\ldots,q_l)\tilde{M}(p_1)\cdots\tilde{M}(p_n)\tilde{K}(q_1)\cdots\tilde{K}(q_l) \tag{2.9}$$

are the (unrenormalized) vertex functions of n fields ϕ and l fields $\mathcal{O}$. In eq. (2.9), the Fourier transform of the source fields is defined by

$$\tilde{F}(k) = \sum_x e^{-ikx} F(x), \tag{2.10}$$

and δ_{P} denotes the periodic δ-function expressing total momentum conservation modulo 2π.

$\Gamma^{(n,l)}$ vanishes for odd n and for $n = l = 0$. Except for $\Gamma^{(2,0)}(p, -p)$, which is equal to the negative inverse propagator of ϕ, all other vertex functions are the one-particle irreducible parts of the connected $n+l$-point functions with full propagator amputated external ϕ-legs. For later use, we note

$$\frac{\partial}{\partial\kappa}\Gamma^{(n,l)}(p_1,\ldots,p_n;q_1,\ldots,q_l)=\Gamma^{(n,l+1)}(p_1,\ldots,p_n;q_1,\ldots,q_l,0)$$

$$-\delta_{n0}\delta_{l0}\Gamma^{(0,1)}(0), \qquad (2.11)$$

which is an easy consequence of the definitions of S, $\mathcal{O}$ and Γ.

We finally introduce a wave function renormalization constant Z_R, a renormalized mass parameter m_R and a renormalized coupling g_R through

$$\Gamma^{(2,0)}(p,-p)=-Z_R^{-1}\{m_R^2+p^2+O(p^4)\} \qquad (p\to 0), \qquad (2.12)$$

$$\Gamma^{(4,0)}(0,0,0,0)=-Z_R^{-2}g_R. \qquad (2.13)$$

These quantities are well-defined functions of κ, λ in the whole region $0\leqslant\lambda\leqslant\infty$, $0<\kappa<\kappa_c(\lambda)$, and one may show, using correlation inequalities, that $g_R\geqslant 0$ [10–14]. Note that m_R and g_R are distinct from the physical particle mass m and coupling g, which are defined through the pole of the propagator in the complex energy plane and the on-shell four-point function, respectively. However, as we shall see later, these two sets of renormalized parameters are numerically close when κ is close to κ_c, in particular, $m=m_R$ is an excellent approximation in the "scaling region" characterized by the inequality (1.2).

3. Tools and strategy

When κ approaches κ_c, the theory becomes critical and the mass m_R goes to zero. In other words, the vicinity of the critical line in the phase diagram is a region where the ultraviolet cutoff Λ (which is equal to 1 in lattice units) is large compared to the mass of the ϕ-particle and the theory may be expected to effectively behave like a continuum theory at low momenta. It is also in this region, where the triviality bound (1.1), which we would like to make more precise, should apply. Our aim is thus, to determine the minimal possible value of m_R along the curves in the phase diagram with fixed g_R (i.e. along the "renormalization group trajectories"). To achieve this goal, we shall calculate the functions $m_R(\kappa,\lambda)$ and $g_R(\kappa,\lambda)$ with reasonable estimated accuracy using the analytic techniques described below.

The classical tool of statistical mechanics is the high "temperature" expansion, which here amounts to an expansion of correlation functions in powers of κ at fixed (arbitrary) λ. For the correlation functions entering the definition of m_R and g_R, the expansion coefficients are known up to 10th order [3]. For $\kappa<\kappa_c$ the high temperature series is convergent, but the rate of convergence is of course slow when κ is very close to κ_c so that the truncation of the series after the 10th order would then not be a sensible approximation. As we shall show in sect. 5, the truncation

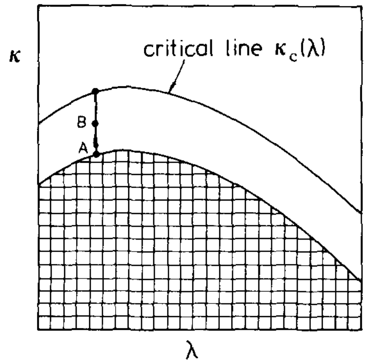

Fig. 2. Same as fig. 1, but showing the region where the theory will be solved by the high temperature expansion (cross-hatched area). To calculate m_R and g_R at point B, for example, the renormalization group equations will be integrated at constant λ using the high temperature series results at point A as initial data.

error can however be reliably estimated for $\kappa \leqslant 0.95\kappa_c$ by choosing a good expansion variable and by taking into account the structure of the singularity at $\kappa = \kappa_c$ as predicted by the renormalization group. At first sight, $\kappa = 0.95\kappa_c$ seems to be rather close to κ_c, but it turns out that $m_R \simeq 0.5$ along this line so that in fact we only use the high temperature expansion in a region where the correlation length is less than two lattice spacings, i.e. in a region, which conforms with the typical size of the high temperature graphs included.

With the help of the high temperature series, we can thus "solve" the theory in the cross-hatched part of fig. 2. To calculate m_R and g_R in the remaining white part of the symmetric phase region, we shall make use of the renormalization group equations, which are first order differential equations describing the evolution of m_R, g_R and other quantities, when κ increases towards κ_c and λ is held fixed. The right-hand sides of these equations involve the Callan-Symanzik coefficients (the β-function in particular) and are thus not known exactly, but for sufficiently small renormalized coupling g_R, we may expect the perturbation expansion in powers of g_R to apply.

At this point, the following observations are crucial. The first one is that along the line $\kappa = 0.95\kappa_c$, the coupling g_R (as calculated by the high temperature expansion) turns out to be rather small with a maximal value of about $\frac{2}{3}$ of the tree level unitarity bound. For these values of g_R, the perturbation expansion of the Callan-Symanzik coefficients and also of many other quantities appears to be reasonably convergent. A second remark is that if the integration of the renormalization group equations is started at say point A of fig. 2 and if the coupling g_R is small there, it will be even smaller after the integration at e.g. point B, because the β-function is positive in the perturbative region. Putting these facts together, the use of perturbation theory for the calculation of the Callan-Symanzik coefficients in the relevant range of g_R appears to be justified and we can thus evaluate $m_R(\kappa, \lambda)$ and $g_R(\kappa, \lambda)$ by integrating the renormalization group equations starting from the line $\kappa = 0.95\kappa_c$.

In the following sections, the theoretical background for our calculations will be recalled and the details, including a number of checks, will then be worked out.

4. Scaling behaviour in the critical region and the Callan-Symanzik equation

As already mentioned in the introduction, this section is basically a review of known results. It is included here to make our paper self-contained and because we would like to state clearly what (qualitative) assumptions have to be made for the analysis to work.

4.1. RENORMALIZED VERTEX FUNCTIONS

The renormalized vertex functions $\Gamma_R^{(n,l)}$ are defined as follows:

$$\Gamma_R^{(n,l)} = 0 \qquad \text{for odd } n \text{ and for } n = 0, l \leqslant 1, \tag{4.1}$$

$$\Gamma_R^{(0,2)}(q, -q) = \left(Z_R^{\mathcal{O}}\right)^2 \left\{ \Gamma^{(0,2)}(q, -q) - \Gamma^{(0,2)}(0,0) \right\}, \tag{4.2}$$

$$\Gamma_R^{(n,l)} = \left(Z_R\right)^{n/2} \left(Z_R^{\mathcal{O}}\right)^l \Gamma^{(n,l)} \qquad \text{for all other } n, l. \tag{4.3}$$

Here, the wave function renormalization constant $Z_R^{\mathcal{O}}$ of the operator $\mathcal{O}$ is determined by

$$\Gamma^{(2,1)}(0,0;0) = \left(Z_R Z_R^{\mathcal{O}}\right)^{-1} \tag{4.4}$$

so that by construction, the renormalized vertex functions satisfy the normalization conditions

$$\Gamma_R^{(2,0)}(p, -p) = -\left\{ m_R^2 + p^2 + \mathrm{O}(p^4) \right\} \qquad (p \to 0), \tag{4.5}$$

$$\Gamma_R^{(4,0)}(0,0,0,0) = -g_R, \tag{4.6}$$

$$\Gamma_R^{(0,2)}(0,0) = 0, \tag{4.7}$$

$$\Gamma_R^{(2,1)}(0,0;0) = 1 \tag{4.8}$$

(and eq. (4.1)). By eliminating κ and λ in favour of m_R and g_R, we shall always consider $\Gamma_R^{(n,l)}$ to be a function of m_R, g_R and the momenta. A subtle point here is that the mapping $(\kappa, \lambda) \to (m_R, g_R)$ may not be globally invertible in the symmetric phase region, in which case $\Gamma_R^{(n,l)}$ would be a multi-valued function of m_R, g_R. This question has been discussed extensively in the literature (see ref. [28] and references quoted therein), but no definite conclusions were reached. Although the mapping is probably invertible for the simple lattice action we have chosen, there is in fact no deep reason for this. But since the following considerations do not crucially depend on the single-valuedness of the renormalized vertex functions, we shall for the moment ignore this problem. Also, we shall argue later (sects. 8, 9) that our numerical results would in fact not be affected significantly in case the mapping should not be invertible after all.

4.2. SCALING PROPERTIES OF THE VERTEX FUNCTIONS

For small g_R, the renormalized vertex functions can be expanded in powers of g_R with coefficients, which are finite sums of lattice Feynman diagrams. From renormalization theory, one knows that to all orders the continuum limit of these coefficients exists and is universal, i.e. it does not depend on the details of the lattice action chosen. Actually, a complete and rigorous proof of this important property of the lattice theory has only recently been given [29] (the old power counting theorems do not apply to Feynman diagrams with a lattice cutoff).

To explain what precisely the existence of the continuum limit in perturbation theory means, we suppose the lattice spacing a is measured in some (externally defined) physical units. The renormalized mass in these units is then given by

$$\bar{m}_R = m_R/a \tag{4.9}$$

and, similarly, the momenta p_i, q_j and the vertex functions are related to dimensionful quantities through

$$\bar{p}_i = p_i/a\,, \qquad \bar{q}_j = q_j/a\,, \tag{4.10}$$

$$\bar{\Gamma}_R^{(n,l)} = a^{n+2l-4}\Gamma_R^{(n,l)}\,. \tag{4.11}$$

The continuum limit is now obtained by letting a, m_R, p_i and q_j go to zero in such a way that $\bar{m}_R$, $\bar{p}_i$ and $\bar{q}_j$ remain fixed. To all orders of g_R, we then have

$$\lim_{a\to 0}\bar{\Gamma}_R^{(n,l)} = \Gamma_{as}^{(n,l)}(\bar{p}_1,\dots,\bar{p}_n;\bar{q}_1,\dots,\bar{q}_l;\bar{m}_R, g_R)\,, \tag{4.12}$$

where $\Gamma_{as}^{(n,l)}$ is the universal (and non-trivial) continuum amplitude, which could be calculated directly with continuum Feynman rules using the BPHZ finite part prescription, for example. We emphasize that $\Gamma_{as}^{(n,l)}$ only exists as a formal power series in g_R and the limit (4.12) is to be taken only after expanding $\bar{\Gamma}_R^{(n,l)}$ in powers of g_R.

In perturbation theory, the rate of convergence to the continuum limit is described by the asymptotic formula [30]

$$a\frac{\partial}{\partial a}\bar{\Gamma}_R^{(n,l)} = \mathrm{O}\big(a^2(\ln a)^r\big)\,, \tag{4.13}$$

where r is the maximal number of loops in the diagrams considered. These "O(a^2)

scaling violations" are not universal, i.e. they are lattice artefacts, which depend on the choice of lattice action. Still, as we shall discuss in more detail in sect. 5, their structure is not totally arbitrary but may always be described by a local effective lagrangian with a finite number of parameters.

For our choice of lattice action, the scaling violations are apparently rather small. Typically, what one finds at the tree and 1-loop level is that they are less than 10% of the continuum amplitude and decrease according to eq. (4.13) provided

$$a^{-1} \geqslant 2\Big(\bar{m}_R + \sum_i |\bar{p}_i| + \sum_j |\bar{q}_j|\Big). \tag{4.14}$$

Thus, continuum behaviour sets in rather early and if the cutoff a^{-1} is a few orders of magnitude larger than the particle mass and the momenta, the scaling violations are totally insignificant.

Because of the triviality bound (1.1) (which will be derived later), we do not expect to be able to take the continuum limit of the full vertex functions for a fixed coupling $g_R > 0$. As a working hypothesis, we shall however assume in this paper that the vertex functions at low momenta (say $|\bar{p}_i|, |\bar{q}_j| \leqslant \bar{m}_R$) scale in the sense that

$$\left| a\frac{\partial}{\partial a}\bar{\Gamma}_R^{(n,l)} \right| \ll |\bar{\Gamma}_R^{(n,l)}| \tag{4.15}$$

holds, whenever the cutoff a^{-1} is large compared to the mass $\bar{m}_R$. More precisely, we expect the scaling violations to satisfy a bound of the form

$$\left| a\frac{\partial}{\partial a}\bar{\Gamma}_R^{(n,l)} \right| \leqslant \bar{m}_R^{4-n-2l} C_{n,l}(g_R)(a\bar{m}_R)^{\varepsilon}, \tag{4.16}$$

where $C_{n,l}(g_R)$ is continuous for $g_R \geqslant 0$ and ε is positive (the perturbative result (4.13) suggests $\varepsilon \simeq 2$, but this additional information is not really needed). We shall also take it for granted that the renormalized perturbation series is indeed an asymptotic expansion of the renormalized vertex functions for $g_R \to 0$ no matter how large the bare coupling λ and the cutoff a^{-1} are.

The above assumptions will later be referred to as the "scaling hypothesis" and a region in the phase diagram where (4.15) holds will be called a "scaling region". We emphasize that our scaling hypothesis does in no way contradict the expected triviality of the theory, because triviality only implies a logarithmically vanishing coupling g_R for $a \to 0$ whereas the scaling violations go down with a power of a. Thus, eq. (4.15) will always be valid sufficiently close to the critical line even though $\bar{\Gamma}_R^{(n,l)}$ may actually vanish for $\kappa = \kappa_c$.

4.3. THE CALLAN-SYMANZIK EQUATION

Following ref. [1], we now derive the Callan-Symanzik equation by studying the variation of the renormalized vertex functions with respect to κ at fixed λ. Since $\Gamma_R^{(n,l)}$ depends on κ only through m_R and g_R, the chain rule gives

$$\frac{\partial}{\partial\kappa}\Gamma_R^{(n,l)} = \left\{\frac{\partial m_R}{\partial\kappa}\frac{\partial}{\partial m_R} + \frac{\partial g_R}{\partial\kappa}\frac{\partial}{\partial g_R}\right\}\Gamma_R^{(n,l)}. \tag{4.17}$$

On the other hand, for $n+1 \geqslant 2$ we have

$$\Gamma_R^{(n,l)} = (Z_R)^{n/2}(Z_R^{\mathcal{O}})^l\left\{\Gamma^{(n,l)} - \delta_{n0}\delta_{l2}\Gamma^{(0,2)}(0,0)\right\} \tag{4.18}$$

and, using (2.11), one obtains

$$\frac{\partial}{\partial\kappa}\Gamma_R^{(n,l)} = \left\{\frac{n}{2}\frac{\partial \ln Z_R}{\partial\kappa} + l\frac{\partial \ln Z_R^{\mathcal{O}}}{\partial\kappa}\right\}\Gamma_R^{(n,l)}$$

$$+ Z_R^{\mathcal{O}-1}\left\{\Gamma_R^{(n,l+1)}\big|_{q_{l+1}=0} - \delta_{n0}\delta_{l2}\Gamma_R^{(0,3)}(0,0,0)\right\}. \tag{4.19}$$

The combination of (4.17) and (4.19) now yields the Callan-Symanzik equation

$$\left\{m_R\frac{\partial}{\partial m_R} + \beta\frac{\partial}{\partial g_R} - n\gamma - l\delta\right\}\Gamma_R^{(n,l)}$$

$$= \varepsilon m_R^2\left\{\Gamma_R^{(n,l+1)}\big|_{q_{l+1}=0} - \delta_{n0}\delta_{l2}\Gamma_R^{(0,3)}(0,0,0)\right\}, \tag{4.20}$$

where the coefficients are given by

$$\beta(m_R, g_R) = m_R\frac{\partial g_R}{\partial\kappa}\Big/\frac{\partial m_R}{\partial\kappa}, \tag{4.21}$$

$$\gamma(m_R, g_R) = \tfrac{1}{2}m_R\frac{\partial \ln Z_R}{\partial\kappa}\Big/\frac{\partial m_R}{\partial\kappa}, \tag{4.22}$$

$$\delta(m_R, g_R) = m_R\frac{\partial \ln Z_R^{\mathcal{O}}}{\partial\kappa}\Big/\frac{\partial m_R}{\partial\kappa}, \tag{4.23}$$

$$\varepsilon(m_R, g_R) = \left(m_R Z_R^{\mathcal{O}}\frac{\partial m_R}{\partial\kappa}\right)^{-1} \tag{4.24}$$

(derivatives with respect to κ are at fixed λ). Actually, ε is not an independent coefficient, but is related to γ through

$$\varepsilon = 2\gamma - 2 \tag{4.25}$$

as one may show by evaluating (4.20) for $n = 2$, $l = 0$, $p_1 = p_2 = 0$ and using the normalization conditions (4.5), (4.8).

So far no non-trivial property of the theory has been taken into account, i.e. the Callan-Symanzik equation (4.20) is just an exact identity, which follows from our definition of the renormalized vertex functions. However, if we now add the scaling hypothesis, the formalism becomes a powerful tool for the analysis of the theory near the critical line. The basic observation is that the coefficients β, γ, δ (and ε) can be expressed through the renormalized vertex functions by writing eq. (4.20) for three independent choices of n, l and the momenta and solving the resulting linear system for β, γ, δ. From this we conclude that β, γ, δ scale in the same way as the vertex functions so that in a scaling region (i.e. sufficiently close to the critical line), they are independent of m_R up to scaling violation terms, which vanish like a power of m_R. Taking this property into account, eqs. (4.21)–(4.24) become non-trivial differential equations for m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ as functions of κ at fixed λ, which allow us to determine the behaviour of these quantities for $\kappa \to \kappa_c$ as we shall now explain.

4.4. SCALING-LAWS FOR m_R, g_R, Z_R AND $Z_R^{\mathcal{O}}$

For the solution of eqs. (4.21)–(4.24), it is more convenient to write them in the form

$$m_R \frac{\partial g_R}{\partial m_R} = \beta, \tag{4.26}$$

$$m_R \frac{\partial \ln Z_R}{\partial m_R} = 2\gamma, \tag{4.27}$$

$$m_R \frac{\partial \ln Z_R^{\mathcal{O}}}{\partial m_R} = \delta, \tag{4.28}$$

$$m_R \frac{\partial \kappa}{\partial m_R} = 2m_R^2(\gamma - 1) Z_R^{\mathcal{O}}, \tag{4.29}$$

where m_R is regarded as the independent variable and the differentiations are at fixed λ as before. Since we are interested in the asymptotic behaviour of the solution for $m_R \to 0$, we may neglect scaling violations and shall thus assume that β, γ and δ are independent of m_R.

For small g_R, the β-function can be calculated in perturbation theory and is found to be positive (the expansion coefficients for β, γ, δ are tabulated up to 3 loops in appendix A). This implies that once the coupling g_R is in the perturbative range, the differential equation (4.26) monotonically drives it to smaller values as m_R decreases and, for $m_R \to 0$, g_R is eventually forced to zero according to the asymptotic formula

$$m_R = C_1(\beta_1 g_R)^{-\beta_2/\beta_1^2} e^{-1/\beta_1 g_R}\{1 + O(g_R)\}, \tag{4.30}$$

where C_1 is a constant (depending on λ) and β_1, β_2 are the one- and two-loop coefficients of the β-function. As we have pointed out, the validity of (4.30) depends on the assumption that the initial value of g_R is sufficiently close to the origin. We shall later argue that this condition is indeed fulfilled along the line $\kappa = 0.95\kappa_c$, where the high temperature series results are available. We thus conclude that the scaling law (4.30) is correct for all λ, in particular, the triviality of the theory follows since g_R vanishes for $\kappa = \kappa_c$.

Other scaling laws may now easily be derived by solving eqs. (4.27)–(4.29) for $m_R \to 0$. Because g_R goes to zero in this limit, we may use perturbation theory to evaluate γ, δ and the asymptotic form of the solutions is then found to be

$$Z_R = C_2\{1 + O(g_R)\}, \tag{4.31}$$

$$Z_R^{\mathcal{O}} = C_3 g_R^{-1/3}\{1 + O(g_R)\}, \tag{4.32}$$

$$\kappa_c - \kappa = C_3 m_R^2 g_R^{-1/3}\{1 + O(g_R)\}, \tag{4.33}$$

where C_2 and C_3 are integration constants. Finally, combining eqs. (4.30)–(4.33) and setting $\tau = 1 - \kappa/\kappa_c$, we obtain the well-known scaling laws [1]

$$m_R \underset{\tau\to 0}{\sim} C_4 \tau^{1/2} |\ln \tau|^{-1/6}, \tag{4.34}$$

$$g_R \underset{\tau\to 0}{\sim} \tfrac{1}{3} \cdot 32\pi^2 |\ln \tau|^{-1}, \tag{4.35}$$

$$Z_R \underset{\tau\to 0}{\sim} C_2, \tag{4.36}$$

$$Z_R^{\mathcal{O}} \underset{\tau\to 0}{\sim} C_5 |\ln \tau|^{1/3}. \tag{4.37}$$

Of course, given the perturbation expansion of β, γ, δ up to 3 loops, subleading terms in these asymptotic relations could be worked out easily. We also note that scaling laws for the bare vertex functions $\Gamma^{(n,l)}$ may now be obtained from eqs. (4.34)–(4.37) and the defining equations (4.1)–(4.3) of the renormalized vertex functions. For example, the 2-point susceptibility

$$\chi_2 = \sum_x \left. \frac{\partial^2 W}{\partial H(x) \partial H(0)} \right|_{H=K=0} \tag{4.38}$$

(cf. eq. (2.6)) may be shown to diverge according to

$$\chi_2 \underset{\tau \to 0}{\sim} C_6 \tau^{-1} |\ln \tau|^{1/3} . \tag{4.39}$$

5. Scaling violations and the divergence of perturbation theory*

In this section, we would like to show that the existence of $O(a^2)$ scaling violations, the triviality of the theory as expressed by the scaling law (4.30) and the divergence of the renormalized perturbation series are apparently related aspects of the ϕ^4 theory. To explain what the connection is, we first summarize what is known about the structure of scaling violations and how precisely the perturbation series is expected to diverge.

Scaling violations in the ϕ^4 theory were studied in great detail by Symanzik [30, 35]. The most important result of his work, which later gave rise to the "improvement programme" (for a review see ref. [36]), is that the leading scaling violation terms may be described by a local effective lagrangian involving a linear combination of composite fields with canonical dimensions up to 6. Explicitly, for the vertex functions $\bar{\Gamma}_{\mathrm{R}}^{(n,l)}$ with $l = 0$, we have to all orders of perturbation theory

$$a \frac{\partial}{\partial a} \bar{\Gamma}_{\mathrm{R}}^{(n,0)} = a^2 \sum_{A=1}^{7} c_A \bar{m}_{\mathrm{R}}^{6-d_A} \Delta_A \bar{\Gamma}_{\mathrm{R}}^{(n,0)} + O\left(a^4 (\ln a)^r\right), \tag{5.1}$$

where Δ_A denotes a (renormalized) insertion of a composite operator $\mathcal{O}_A$ of dimension $d_A \leqslant 6$ at zero momentum. There are exactly 7 such operators respecting the discrete lattice symmetries, which are linearly independent up to scaling violation and derivative terms (for a discussion of the renormalization of operator insertions see ref. [1], for example). The coefficients c_A in eq. (5.1) depend on the lattice action chosen and are of the form

$$c_A = \sum_{\nu=0}^{\infty} c_A^{(\nu)} g_{\mathrm{R}}^{\nu}, \tag{5.2}$$

$c_A^{(\nu)}$ being a polynomial of $\ln(a\bar{m}_{\mathrm{R}})$ of degree r at r-loop order.

It is of course possible to generalize (5.1) to all vertex functions $\bar{\Gamma}_{\mathrm{R}}^{(n,l)}$, $l \geqslant 0$, although further terms are then needed to account for the intrinsic cutoff dependence of the composite field $\mathcal{O}$. We also note that a complete proof of eq. (5.1) to all orders of perturbation theory is still lacking and that the summation of the logarithms in the loop expansion (5.2), using a Callan-Symanzik-type equation, has not yet been achieved.

From a low energy, "phenomenological" point of view, eq. (5.1) says that a large cutoff signals its presence through tiny effects, which may be described by a local

* This section will not be referred to later and may be skipped in a first reading.

effective lagrangian. Put differently, if we think in terms of the whole universality class of lattice ϕ^4 theories, eq. (5.1) is a measure for the ambiguities at low energies, which arise from the incomplete specification of the theory at very high energies. One could also say that the universal part of the renormalized vertex functions is only defined up to such ambiguities.

We now proceed to discuss the divergence of the renormalized perturbation series in the continuum limit (cf. subsect. 4.2). According to the famous Lipatov analysis (for a review see ref. [37]), the growth of the coefficients in the perturbation expansion

$$\Gamma_{\rm as}^{(n,l)} = \sum_{\nu=0}^{\infty} \left(\Gamma_{\rm as}^{(n,l)}\right)_\nu g_{\rm R}^\nu \tag{5.3}$$

is expected to be given by the asymptotic formula

$$\left(\Gamma_{\rm as}^{(n,l)}\right)_\nu \underset{\nu\to\infty}{\sim} A^{(n,l)} \nu^s b^\nu \nu! (1 + {\rm O}(1/\nu)), \tag{5.4}$$

where $A^{(n,l)}$ is a momentum dependent amplitude, s some power and

$$b = -\frac{1}{16\pi^2}. \tag{5.5}$$

However, it was later realized [38–41] that renormalization upsets the naive Lipatov argument and that the divergence of perturbation theory is probably dominated by the "renomalon" singularity in the Borel plane. This implies an asymptotic behaviour of the form (5.4) but with

$$b = \tfrac{1}{2}\beta_1 = \frac{3}{32\pi^2}, \tag{5.6}$$

where β_1 is the one-loop coefficient of the β-function. Recent rigorous results [42] also support this value of b and we shall thus assume here that eqs. (5.4) and (5.6) describe the true large order behaviour of perturbation theory*.

The divergence of the series (5.3) implies that unless further information is supplied, it is not possible to sum it unambiguously, i.e. different reasonable summation procedures (modified Borel sums, for example) yield different results. However, since all these sums must have the same perturbation expansion, we may expect that the difference $\delta\Gamma_{\rm as}^{(n,l)}$ between any two of them satisfies

$$\left|\delta\Gamma_{\rm as}^{(n,l)}\right| \lesssim \left|\left(\Gamma_{\rm as}^{(n,l)}\right)_\nu\right| g_{\rm R}^\nu \tag{5.7}$$

* We note, however, that up to five loops, the perturbation coefficients of the renormalization group functions in the minimal subtraction scheme are alternating and do not appear to grow more rapidly than a power. Thus, it seems that the asymptotic behaviour (5.4) only sets in at very high orders.

for small g_R and large ν. Using (5.4) and minimizing over ν then leads to the bound

$$\left|\delta\Gamma_{\rm as}^{(n,l)}\right| \lesssim \sqrt{2\pi}\,|A^{(n,l)}|(bg_R)^{-s-1/2}\,e^{-1/bg_R}\left\{1+O(g_R)\right\}. \tag{5.8}$$

Thus, we may say that the renormalized perturbation series in the continuum limit determines a universal set of full vertex functions up to exponentially small ambiguities as described by eq. (5.8).

This observation obviously parallels our discussion above of the significance of the scaling violations in the lattice theory. In both cases we seem to be able to extract full continuum vertex functions only up to certain (small) ambiguities. The point we wish to make now is that these two kinds of ambiguities are apparently of the same size and have a similar structure. Indeed, from the scaling law (4.30) we see that as one approaches the critical line, the $O(a^2)$ scaling violation terms (i.e. the contributions proportional to m_R^2) decrease with exactly the same exponential as the ambiguities $\delta\Gamma_{\rm as}^{(n,l)}$ do (cf. eqs. (5.6), (5.8)). Moreover, as was pointed out by Parisi [40,41], the amplitudes $A^{(n,l)}$ together with the higher order corrections in eq. (5.8) are proportional to a linear combination of vertex functions with operator insertions similar to the right-hand side of eq. (5.1), although here only Lorentz invariant operators $\mathcal{O}_A$ contribute of course. The conclusion then is that without further insight, the precision with which universal, continuum vertex functions can be extracted either from the lattice regularized theory or by applying a summation procedure to the renormalized perturbation series, is about the same in both cases. We may also take this as an indication that there are no universal physical effects in this theory beyond those covered by perturbation theory.

6. Results from the high temperature expansion

As we have already mentioned in sect. 3, the high temperature expansion may be used to "solve" the theory for $\kappa \leqslant 0.95\kappa_c$. Our goal in this section is mainly to calculate m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ for $\kappa = 0.95\kappa_c$ and $0 \leqslant \lambda \leqslant \infty$. These results will then be used in sect. 8 as initial data for the integration of the renormalization group equations (4.26)–(4.29).

6.1. SUMMARY OF NOTATIONS

In their paper, Baker and Kincaid [3] have tabulated the high temperature expansion coefficients up to 10th order for the susceptibilities

$$\chi_2 = \sum_x \langle\phi(x)\phi(0)\rangle^c, \tag{6.1}$$

$$\mu_2 = \sum_x x^2\langle\phi(x)\phi(0)\rangle^c, \tag{6.2}$$

$$\chi_4 = \sum_{x,y,z} \langle\phi(x)\phi(y)\phi(z)\phi(0)\rangle^c, \tag{6.3}$$

where

$$\langle\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle^c = \left.\frac{\partial^n W(H,K)}{\partial H(x_1)\,\partial H(x_2)\dots\partial H(x_n)}\right|_{H=K=0} \tag{6.4}$$

denotes the connected n-point correlation function of ϕ (cf. sect. 2). At each order, the expansion coefficients are polynomials of the "one-point" expectation values

$$\langle\phi^n\rangle_1 = I_n/I_0, \tag{6.5}$$

$$I_n(\lambda) = \int_0^\infty \mathrm{d}\phi\,\phi^n\,\mathrm{e}^{-\phi^2-\lambda(\phi^2-1)^2}. \tag{6.6}$$

It is not possible to evaluate these integrals for all n and λ analytically, but up to 10th order in κ, only the integrals with $n \leqslant 14$ are required and these can be calculated numerically with high precision.

The susceptibilities (6.1)–(6.3) are related to m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ through

$$m_R = \left(\frac{8\chi_2}{\mu_2}\right)^{1/2}, \tag{6.7}$$

$$g_R = -64\frac{\chi_4}{\mu_2^2}, \tag{6.8}$$

$$Z_R = 8\frac{\chi_2^2}{\mu_2}, \tag{6.9}$$

$$Z_R^{\mathcal{O}} = \frac{\mu_2}{8}\left(\frac{\partial\chi_2}{\partial\kappa}\right)^{-1}. \tag{6.10}$$

Thus, for all these quantities, the high temperature expansion may be deduced easily from the Baker-Kincaid tables.

Instead of the bare coupling λ, we have decided to present our results in terms of the parameter

$$\bar\lambda = -\tfrac{1}{2}\left\{\langle\phi^4\rangle_1 - 3\left(\langle\phi^2\rangle_1\right)^2\right\}\Big/\left(\langle\phi^2\rangle_1\right)^2, \tag{6.11}$$

which is more natural from the point of view of the high temperature expansion. $\bar\lambda$ is a strictly monotonic function of λ rising from 0 to 1 in the range $0 \leqslant \lambda \leqslant \infty$. Furthermore, at small λ we have

$$\bar\lambda = 3\lambda + \mathrm{O}(\lambda^2), \tag{6.12}$$

and a list of values of λ versus $\bar\lambda$ is given in table 1, p. 45.

It is a well-known fact that in the case of the Ising model ($\lambda = \infty$), the character variable

$$v = \operatorname{tgh} 2\kappa \tag{6.13}$$

is a better high temperature expansion parameter than κ in the sense that when the series are reexpanded in powers v, they become more regular and the convergence properties are improved. For $\lambda < \infty$, a similar improvement is observed for the choice

$$v = \frac{1}{2}\frac{\partial}{\partial\kappa}\ln\left\{\int d\phi_1\, d\phi_2 \exp(2\kappa\phi_1\phi_2 - V(\phi_1) - V(\phi_2))\right\}, \tag{6.14}$$

$$V(\phi) = \phi^2 + \lambda(\phi^2 - 1)^2, \tag{6.15}$$

which reduces to (6.13) in the limit $\lambda \to \infty$. Note that v is just the internal energy of the ϕ^4 theory on a lattice with only 2 points and the substitution $\kappa \to v$ hence corresponds to an efficient partial summation of the high temperature series.

6.2. CALCULATION OF $\kappa_c(\lambda)$

Following ref. [4], we now determine the critical line from the large order behaviour of the series

$$\chi_2 = \sum_{i=0}^{\infty} \chi_2^{(i)} v^i . \tag{6.16}$$

A first observation is that apparently $\chi_2^{(i)} \geqslant 0$ for all i and the singularity of χ_2 closest to the origin is therefore on the positive real axis. It follows that the critical value of v is given by

$$v_c = \lim_{i\to\infty} r_i, \qquad r_i = \chi_2^{(i-1)}/\chi_2^{(i)}, \tag{6.17}$$

if this limit exists. Actually, for $5 \leqslant i \leqslant 10$ and all λ, the ratios r_i are already nearly constant with a small monotonic trend and a superposed oscillation in the third significant digit.

The behaviour of r_i for large i is correlated with the form of the singularity of the susceptibility χ_2 at the critical line. In the following, the aim is then to stabilize the extrapolation of the ratios r_i for $i \to \infty$ by taking the scaling laws derived in sect. 4 into account. Because the logarithmic corrections to the free field singularity of χ_2 are suppressed for small λ, the ranges $0 \leqslant \bar{\lambda} \leqslant 0.25$ and $0.25 \leqslant \bar{\lambda} \leqslant 1.00$ are treated differently.

(a) *$0 \leqslant \bar{\lambda} \leqslant 0.25$.* For all $\lambda > 0$, the behaviour of χ_2 in the limit $\kappa \to \kappa_c$ is described by the scaling law (4.39). However, when λ is small, the logarithmic modification of the free field scaling law $\chi_2 \propto \tau^{-1}$ is only seen when κ is very close to κ_c. Indeed, integrating the renormalization group equations taking into account that the initial value of g_R is proportional to λ, yields the improved formula

$$\chi_2 \propto \left(1 - \frac{v}{v_c}\right)^{-1}\left(1 - f(\lambda)\ln\left(1 - \frac{v}{v_c}\right)\right)^{1/3}(1 + O(g_R \ln g_R)), \tag{6.18}$$

$$f(\lambda) = \frac{36}{\pi^2}\lambda + O(\lambda^2), \tag{6.19}$$

which is equivalent to eq. (4.39) when κ is close enough to κ_c to overwhelm the suppressing factor $f(\lambda)$.

If we now define an auxiliary function $h(z)$ through

$$\begin{aligned} h(z) &= (1-z)^{-1}\left(1 - \frac{36}{\pi^2}\lambda \ln(1-z)\right)^{1/3} \\ &= \sum_{i=0}^{\infty} h^{(i)} z^i, \end{aligned} \tag{6.20}$$

we may thus expect the ratios r_i and $r_i' = h^{(i-1)}/h^{(i)}$ to behave similarly for $i \to \infty$. In particular, compared to r_i, the sequence of improved ratios

$$R_i = r_i/r_i' \tag{6.21}$$

should be more easy to extrapolate from the range $5 \leqslant i \leqslant 10$ to the limit $i \to \infty$.

The R_i's are actually rapidly convergent and, by inspection, the estimate

$$v_c = v_c^*(1 + \varepsilon), \qquad |\varepsilon| \leqslant 10^{-3}, \tag{6.22}$$

is obtained, where

$$v_c^* = v_{10} \equiv \tfrac{1}{2}(R_9 + R_{10}). \tag{6.23}$$

We have also tried more sophisticated extrapolations of the sequence R_i, but did not find results outside the error band quoted in (6.22). A further check on our method is obtained by comparing (6.23) with the expansion of κ_c in powers of λ, which we have worked out up to two loops (appendix C). The agreement found is perfect for $\bar{\lambda} \leqslant 0.05$ and at $\bar{\lambda} = 0.1$, where the higher loop terms are no longer negligible, the deviation is about 1%.

(b) *$0.25 \leqslant \bar{\lambda} \leqslant 1$.* The procedure here is the same as above with two small changes. One is that the auxiliary function (6.20) is replaced by

$$h(z) = (1-z)^{-1}\left(-\frac{1}{z}\ln(1-z)\right)^{1/3}. \tag{6.24}$$

The other is that we choose

$$x = 2v(1 + v/v_{10})^{-1} \tag{6.25}$$

as a new expansion variable. The effect of this substitution is that the (weak) anti-ferromagnetic singularity of χ_2 at $\kappa = -\kappa_c$ is shifted far away from the origin and the oscillations in the ratios R_i disappear. The remaining monotonic trend may then be fitted very well by

$$R_i = x_c^* + \frac{\delta}{i^2}, \qquad i = 5, \dots, 10. \tag{6.26}$$

Thus, the result is again eq. (6.22) with v_c^* calculated from the fit (6.26) and the transformation (6.25). In the Ising limit, the final value for κ_c obtained in this way agrees with the more accurate estimate

$$\kappa_c = 0.074834(15), \tag{6.27}$$

which Gaunt et al. [4] extracted from the 17th order susceptibility series.

A list of values of κ_c versus $\bar{\lambda}$ as determined by the above methods is given in table 1. As a last consistency check, we mention that at and around $\bar{\lambda} = 0.25$, the procedures (a) and (b) give identical results within the quoted errors.

6.3. CALCULATION OF m_R, g_R, Z_R AND Z_R^φ AT $\kappa = 0.95\kappa_c$

The method of computation and error estimation is similar for all these quantities and we shall therefore present the details only for the mass m_R. Furthermore, only the large λ range $0.25 \leqslant \bar{\lambda} \leqslant 1$ will be considered, the necessary modifications for the small λ's being rather obvious from our discussion in the preceding subsection.

The scaling law (4.34) suggests writing the high temperature expansion of m_R in the form

$$m_R = \left(1 - \frac{v}{v_c}\right)^{1/2} v^{-1/2} \hat{m}_R(x), \tag{6.28}$$

$$\hat{m}_R(x) = \sum_{i=0}^{\infty} \hat{m}_R^{(i)} x^i, \tag{6.29}$$

where $x = 2v/(1 + v/v_c)$. In the neighborhood of the critical line, $\hat{m}_R$ is expected to diverge according to

$$\hat{m}_R(x) \propto \left|\ln\left(1 - \frac{x}{x_c}\right)\right|^{-1/6} (1 + \mathrm{O}(g_R \ln g_R)). \tag{6.30}$$

This is a rather weak singularity and it is therefore not surprising that at the value of

TABLE 1
Values of λ and κ_c versus $\bar{\lambda}$ (eq. (6.11))

$\bar{\lambda}$	λ	κ_c
0.00	0.0	0.1250(1)
0.01	3.4574×10^{-3}	0.1257(1)
0.02	7.1709×10^{-3}	0.1264(1)
0.03	1.1153×10^{-2}	0.1272(1)
0.04	1.5416×10^{-2}	0.1279(1)
0.05	1.9974×10^{-2}	0.1286(1)
0.06	2.4841×10^{-2}	0.1294(1)
0.07	3.0032×10^{-2}	0.1301(1)
0.08	3.5562×10^{-2}	0.1308(1)
0.09	4.1445×10^{-2}	0.1315(1)
0.10	4.7699×10^{-2}	0.1322(1)
0.20	1.3418×10^{-1}	0.1385(1)
0.30	2.7538×10^{-1}	0.1421(1)
0.40	4.8548×10^{-1}	0.1418(1)
0.50	7.7841×10^{-1}	0.1376(1)
0.60	1.1769	0.1299(1)
0.70	1.7320	0.1194(1)
0.80	2.5836	0.1067(1)
0.90	4.3303	0.09220(9)
1.00	∞	0.07475(7)

x corresponding to $\kappa = 0.95\kappa_c$, the series (6.29) is well convergent*. When truncated at 10th order, the mass m_R comes out to be around 0.5, which is rather large and thus gives us additional confidence that we may safely use the high temperature expansion at this value of κ.

The systematic error which results from the truncation of the series (6.29) may be estimated as follows. Let

$$h(z) = \left(-\frac{1}{z}\ln(1-z)\right)^{-1/6} = \sum_{i=0}^{\infty} h^{(i)} z^i \tag{6.31}$$

be an auxiliary function simulating the singularity (6.30). By inspection, one observes that for $i \geqslant 5$

$$\hat{m}_R^{(i)} \simeq C h^{(i)} / x_c^i, \tag{6.32}$$

where the constant C may be determined with an accuracy of 1–2 significant digits.

* Here and in what follows, we take $\kappa_c = \kappa_c^*$, where κ_c^* is the critical value determined in subsect. 6.2.

Thus, if we define δ through

$$\delta = C\left\{h(z) - \sum_{i=0}^{9} h^{(i)}z^i\right\}_{z=x/x_c}, \tag{6.33}$$

the truncation error is approximately given by

$$\sum_{i=10}^{\infty} \hat{m}_{\mathrm{R}}^{(i)}x^i \simeq \delta. \tag{6.34}$$

In this way we are led to take

$$\hat{m}_{\mathrm{R}}^{*}(x) = \sum_{i=0}^{9} \hat{m}_{\mathrm{R}}^{(i)}x^i + \delta \tag{6.35}$$

as our best estimate for $\hat{m}_{\mathrm{R}}(x)$ and to quote $\frac{1}{2}|\delta|$ as a realistic upper bound on the systematic error $|\hat{m}_{\mathrm{R}}^{*}(x) - \hat{m}_{\mathrm{R}}(x)|$.

The result of our calculations is displayed in table 2. In all cases, the errors are largest in the Ising limit, which is perhaps understandable given that the coupling g_{R} is maximal there. Anyway, they are reasonably small ($\leqslant 15\%$) and, if required, it would not be impossible to decrease them by working out a few higher orders in the high temperature expansion. At small $\bar{\lambda}$, the numbers in table 2 agree very well with the two-loop perturbation expansions summarized in appendix C. They also compare favourably with a large scale Monte Carlo simulation of the Ising model [27] and we are thus confident that our methods of calculation and error estimation are reliable.

6.4. FURTHER REMARKS AND RESULTS

For all fixed $\bar{\lambda}$ and for $0 < \kappa \leqslant 0.95\kappa_c$, we have found that m_{R} and g_{R} are monotonically decreasing when κ is growing. Thus, in this region we have (cf. subsect. 4.3)

$$\beta(m_{\mathrm{R}}, g_{\mathrm{R}}) > 0. \tag{6.36}$$

Furthermore, one also observes that $Z_{\mathrm{R}}^{\emptyset}$ is positive and monotonically increasing, which implies

$$\gamma(m_{\mathrm{R}}, g_{\mathrm{R}}) < 1, \tag{6.37}$$

$$\delta(m_{\mathrm{R}}, g_{\mathrm{R}}) < 0. \tag{6.38}$$

Finally, since Z_{R} is always positive and slowly decreasing, we conclude that

$$\gamma(m_{\mathrm{R}}, g_{\mathrm{R}}) > 0 \tag{6.39}$$

in the high temperature region.

TABLE 2
Values of m_R, g_R, Z_R and Z_R^φ at $\kappa = 0.95\kappa_c$ as calculated by the high temperature expansion

$\bar{\lambda}$	κ	m_R	g_R	Z_R	Z_R^φ
0.00	0.118752	0.64886(3)	0.0	4.210(1)	0.0148(1)
0.01	0.119436	0.6453(5)	1.380(5)	4.186(1)	0.0150(1)
0.02	0.120125	0.6417(9)	2.68(3)	4.162(1)	0.0151(1)
0.03	0.120818	0.638(1)	3.91(6)	4.138(1)	0.0153(1)
0.04	0.121513	0.635(2)	5.1(1)	4.115(1)	0.0154(1)
0.05	0.122208	0.632(2)	6.2(1)	4.091(1)	0.0156(1)
0.06	0.122902	0.628(3)	7.3(2)	4.068(1)	0.0160(1)
0.07	0.123593	0.625(3)	8.3(3)	4.045(1)	0.0162(1)
0.08	0.124279	0.622(4)	9.3(3)	4.022(1)	0.0163(1)
0.09	0.124960	0.619(4)	10.2(4)	4.000(1)	0.0165(2)
0.10	0.125633	0.617(5)	11.1(5)	3.978(1)	0.0167(2)
0.20	0.131591	0.592(7)	19 (1)	3.794(2)	0.0185(3)
0.30	0.134986	0.575(8)	24 (2)	3.692(3)	0.0200(5)
0.40	0.134737	0.561(9)	29 (3)	3.691(5)	0.0210(7)
0.50	0.130740	0.55(1)	32 (4)	3.794(7)	0.0213(8)
0.60	0.123399	0.53(1)	35 (4)	4.01(1)	0.0211(9)
0.70	0.113398	0.52(1)	37 (5)	4.35(1)	0.0202(9)
0.80	0.101333	0.51(1)	38 (6)	4.84(2)	0.0189(9)
0.90	0.087593	0.50(1)	40 (6)	5.58(2)	0.0171(9)
1.00	0.071017	0.49(1)	41 (6)	6.85(3)	0.0144(8)

Column 2 contains the actual value of κ used, which is equal to $0.95\kappa_c^*$, where κ_c^* is the estimate for κ_c defined in subsect. 6.2.

Another interesting remark is that the curves in the $\kappa, \bar{\lambda}$-plane corresponding to a fixed value of m_R are simple lines, which are to a good approximation given by $\kappa/\kappa_c =$ constant. Along these curves, g_R is monotonically increasing with $\bar{\lambda}$ and it thus follows that the mapping $(\kappa, \bar{\lambda}) \to (m_R, g_R)$ is globally invertible in the region $\kappa \leqslant 0.95\kappa_c$ (cf. the discussion in subsect. 4.1). It also follows that the maximal value of g_R at fixed m_R is attained in the Ising limit. In other words, at least in the high temperature region, the triviality bound (1.1) is saturated by the theory with the largest possible value of the bare coupling (as expected naively).

With a canonical normalization of the bare lattice field as in eq. (2.2), the wave function renormalization constant would be given by

$$Z_R' = 2\kappa Z_R \tag{6.40}$$

(cf. eq. (2.4a)). Curiously, it turns out that Z_R' is very nearly constant in the whole high temperature region and just a little (a few percent) smaller than 1. This is very much in line with what was observed in Monte Carlo simulations [6, 8] and the bounds $0 \leqslant Z_R' \leqslant 1$ are also suggested by formal continuum arguments (ref. [43], subsect. 16.4). Nevertheless, it is surprising that the difference $1 - Z_R'$ is so small even in situations where the coupling g_R is large.

7. The tree level unitarity bound and the applicability of renormalized perturbation theory

Neglecting higher order corrections and scaling violation terms, the S-wave phase shift δ_0 for elastic particle scattering in the ϕ^4 theory is given by

$$\frac{1}{2i}\left(e^{2i\delta_0}-1\right)=-\frac{g_R}{32\pi}\left(\frac{s-4m_R^2}{s}\right)^{1/2}, \tag{7.1}$$

where s is the centre-of-mass energy squared ($4m_R^2 \leqslant s \leqslant 16m_R^2$ in the elastic region). Since the real part of the left-hand side is bounded by $\frac{1}{2}$, this relation can only be a valid approximation at low energies provided

$$g_R \lesssim 32\pi/\sqrt{3} = 58. \tag{7.2}$$

In other words, if g_R is close or above this value, higher orders in the renormalized perturbation expansion must be expected to be non-negligible and it is then even possible that perturbation theory breaks down.

Actually, as is born out by explicit calculations up to two loops, the perturbation expansion of low energy quantities is in general rather well convergent when the unitarity bound (7.2) is satisfied. For example, for the true particle mass m defined through

$$\Gamma_R^{(2,0)}(p,-p)=0, \qquad p=(im,0,0,0), \tag{7.3}$$

we have

$$m=m_R\left\{1-0.001287\alpha_R^2+O\left(g_R^3\right)\right\}, \tag{7.4}$$

$$\alpha_R=g_R/16\pi^2. \tag{7.5}$$

There is no one-loop term in this case and the two-loop correction is very small for $g_R \lesssim 58$ so that m is practically equal to m_R. Another quantity we have considered is the scattering length

$$\begin{aligned} a_0 &= \lim_{p\to 0}\frac{1}{2ip}\left(e^{2i\delta_0}-1\right) \\ &= -\frac{g_R}{32\pi m_R}\left\{1-\alpha_R+0.9270\alpha_R^2+O\left(g_R^3\right)\right\} \end{aligned} \tag{7.6}$$

(p is the magnitude of the particle momentum in the centre-of-mass system). Here, the higher order terms are not negligible, but the series still appears to be convergent. The situation is slightly worse for the 6-point coupling

$$h_R = \Gamma_R^{(6,0)}(0,0,0,0,0,0) + \frac{10}{m_R^2}\left(\Gamma_R^{(4,0)}(0,0,0,0)\right)^2$$

$$= 10\frac{g_R^2}{m_R^2}\left\{1 - \tfrac{3}{4}\alpha_R + \tfrac{9}{4}\alpha_R^2 + O\left(g_R^3\right)\right\}, \tag{7.7}$$

but this only shows that the tree level unitarity bound (7.2) is a realistic estimate of the "radius of convergence" of the renormalized perturbation expansion.

As we have shown in sect. 6, the maximal value of g_R along the line $\kappa = 0.95\kappa_c$ is about 41, which is roughly $\frac{2}{3}$ of the tree level unitarity bound. By integrating the renormalization group equations, g_R will turn out to be even smaller for $\kappa > 0.95\kappa_c$ (sect. 8). Thus, we conclude that renormalized perturbation theory is applicable for $\kappa \geqslant 0.95\kappa_c$ and the low energy properties of the theory are hence calculable in this region once m_R and g_R are known. In particular, our discussion in subsect. 4.2 of scaling violations in perturbation theory should be meaningful for the full amplitudes, i.e. we expect they are $\lesssim 10\%$ at low energies. In this sense, the region $\kappa \geqslant 0.95\kappa_c$ (which is practically equivalent to $m_R \leqslant 0.5$) is also a scaling region.

8. Integration of the renormalization group equations

We now proceed to integrate eqs. (4.26)–(4.29) using our results from the high temperature expansion as initial data along the line $\kappa = 0.95\kappa_c$. Although we have just remarked that renormalized perturbation theory should be applicable in the range $0 < m_R \leqslant 0.5$, $0 \leqslant g_R \leqslant 41$, the series for the β-function is actually not so well convergent (see fig. 3). Still, for $0 \leqslant g_R \leqslant 20$ the higher order corrections are small and for the larger values of g_R it seems reasonable to expect that the true β-function is sandwiched between the 2- and 3-loop curves since the perturbation coefficients are alternating in sign. As we shall see, this uncertainty has fortunately no big effect on the final result. What is really important, however, is that the β-function is positive in perturbation theory and since the same is also true in the high temperature region (cf. subsect. 6.4), there is little doubt that the full lattice β-function is positive for all possible values of $m_R > 0$ and $g_R \geqslant 0$. Actually, the β-functions calculated in perturbation theory and by the high temperature expansion even agree quantitatively (within large errors) along the line $\kappa = 0.95\kappa_c$ (cf. fig. 4 below).

The perturbation series for the other Callan-Symanzik coefficients γ and δ are rather well convergent so that their evaluation in the region where the renormalization group equations will be integrated presents no problem. In accordance with what was found in the high temperature region, δ is negative and γ is small ($\leqslant 0.05$)

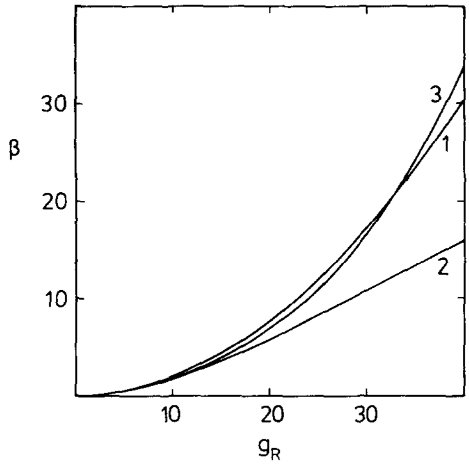

Fig. 3. Plot of the l-loop approximations to the β-function neglecting scaling violation terms (cf. appendix A). The curves are labelled by $l = 1, 2, 3$.

and positive. Thus, when m_R decreases, g_R and Z_R also decrease while $Z_R^{\varnothing}$ and κ monotonically increase.

With β, γ, δ and the initial data from the high temperature expansion (cf. table 2) at our disposal, it is easy to integrate eqs. (4.26)–(4.29) numerically with negligible error using a Runge-Kutta algorithm, for example. In table 3, the result of the integration is shown for three values of $\bar{\lambda}$. The errors quoted are obtained by propagating the errors of the initial data. For β, γ, δ we have taken the 3-loop formulae with the exact lattice expressions for the tree level and the 1-loop coefficients (eqs. (A.4)–(A.12)). The inclusion of the tree level coefficients is actually rather important at small $\bar{\lambda}$ although they are just scaling violation terms. The reason is that when the initial value of g_R is small, the perturbation expansion is dominated by the first term and it is only when m_R has decreased substantially that the universal 1-loop term takes over. For $m_R \lesssim 0.5$, the scaling violations in the 1-loop coefficients are less than 15% of the universal part and their omission would therefore have only a small effect on the result of the integration. For this reason, we are also confident that the scaling violation terms in the higher loop coefficients may be safely neglected.

If one used the 2-loop instead of the 3-loop approximation for β, γ and δ, the numbers in table 3 would not be affected except for the values of g_R at $\bar{\lambda} = 1$, which would change by up to twice the error quoted. We nevertheless take the 3-loop solution with errors as in table 3 for our final result, because it fits better with the extension of the high temperature curves down to $m_R = 0.2$ (see fig. 4) and because it agrees very well with the Monte Carlo simulation data of ref. [27] at $m_R = 0.2$ and $m_R = 0.5$.

TABLE 3
Results from the solution of the renormalization group equations at $\bar{\lambda} = 0.01, 0.10$ and 1.00

$\bar{\lambda}$	m_R	g_R	Z_R	Z_R^c	κ
0.01	1.00	1.590(3)	4.478(1)	0.01401(5)	0.11165(1)
	0.90	1.521(4)	4.383(1)	0.01432(6)	0.11407(1)
	0.80	1.459(4)	4.298(1)	0.01460(7)	0.11632(1)
	0.70	1.406(5)	4.223(1)	0.01486(9)	0.11839(1)
	0.60	1.360(6)	4.158(1)	0.0151(1)	0.12025(1)
	0.50	1.320(8)	4.103(1)	0.0153(1)	0.12186(1)
	0.40	1.283(5)	4.058(1)	0.0155(1)	0.12321(3)
	0.30	1.253(5)	4.022(1)	0.0157(1)	0.12428(4)
	0.20	1.226(5)	3.997(1)	0.0158(1)	0.12507(5)
	0.10	1.198(5)	3.982(1)	0.0160(1)	0.12554(5)
	0.09	1.194(5)	3.980(1)	0.0160(1)	0.12557(5)
	0.08	1.191(5)	3.980(1)	0.0160(1)	0.12560(5)
	0.07	1.187(5)	3.979(1)	0.0160(1)	0.12562(5)
	0.06	1.182(5)	3.979(1)	0.0160(1)	0.12564(5)
	0.05	1.177(5)	3.978(1)	0.0161(1)	0.12566(5)
	0.04	1.171(4)	3.977(1)	0.0161(1)	0.12568(5)
	0.03	1.164(4)	3.977(1)	0.0161(1)	0.12569(5)
	0.02	1.153(4)	3.977(1)	0.0162(1)	0.12570(5)
	0.01	1.136(4)	3.977(1)	0.0163(1)	0.12570(5)
0.10	1.00	13.8(3)	4.291(4)	0.01523(8)	0.11650(8)
	0.90	13.0(3)	4.196(4)	0.0156(1)	0.11914(9)
	0.80	12.3(4)	4.110(4)	0.0160(1)	0.12161(9)
	0.70	11.6(4)	4.034(4)	0.0164(2)	0.12389(9)
	0.60	11.0(5)	3.968(4)	0.0168(2)	0.12596(9)
	0.50	10.4(6)	3.911(3)	0.0171(3)	0.12778(8)
	0.40	9.8(4)	3.865(4)	0.0176(2)	0.1293(1)
	0.30	9.2(4)	3.828(4)	0.0180(2)	0.1305(1)
	0.20	8.5(3)	3.802(4)	0.0185(3)	0.1314(2)
	0.10	7.7(3)	3.784(4)	0.0192(3)	0.1320(2)
	0.09	7.6(2)	3.783(5)	0.0193(3)	0.1320(2)
	0.08	7.4(2)	3.782(5)	0.0194(3)	0.1320(2)
	0.07	7.3(2)	3.781(5)	0.0195(3)	0.1321(2)
	0.06	7.2(2)	3.780(5)	0.0197(3)	0.1321(2)
	0.05	7.0(2)	3.779(5)	0.0198(3)	0.1321(2)
	0.04	6.8(2)	3.778(5)	0.0200(3)	0.1321(2)
	0.03	6.6(2)	3.778(5)	0.0203(3)	0.1321(2)
	0.02	6.3(2)	3.777(5)	0.0206(3)	0.1321(2)
	0.01	5.8(2)	3.776(5)	0.0211(4)	0.1322(2)
1.00	1.00	78 (3)	7.85(3)	0.0111(2)	0.0626(1)
	0.90	69 (3)	7.61(4)	0.0116(3)	0.0644(1)
	0.80	62 (4)	7.40(4)	0.0122(4)	0.0662(1)
	0.70	55 (5)	7.20(4)	0.0129(5)	0.0679(2)
	0.60	48 (6)	7.02(5)	0.0136(7)	0.0694(2)
	0.50	42 (7)	6.86(5)	0.0143(9)	0.0709(2)
	0.40	35 (5)	6.74(5)	0.015(1)	0.0722(2)
	0.30	29 (3)	6.65(5)	0.016(1)	0.0732(3)
	0.20	24 (2)	6.58(6)	0.017(1)	0.0741(4)
	0.10	18 (1)	6.54(6)	0.019(2)	0.0746(4)
	0.09	18 (1)	6.53(6)	0.019(2)	0.0746(4)
	0.08	17 (1)	6.53(6)	0.019(2)	0.0747(4)
	0.07	16 (1)	6.52(6)	0.020(2)	0.0747(4)
	0.06	15.7(9)	6.52(6)	0.020(2)	0.0747(4)
	0.05	15.0(8)	6.52(6)	0.020(2)	0.0747(4)
	0.04	14.1(8)	6.52(6)	0.021(2)	0.0748(4)
	0.03	13.2(7)	6.51(6)	0.021(2)	0.0748(4)
	0.02	12.1(6)	6.51(6)	0.022(2)	0.0748(4)
	0.01	10.6(4)	6.50(6)	0.023(2)	0.0748(4)

For completeness, the values from the high temperature series analysis in the range $1.0 \geq m_R \geq 0.5$ are also displayed.

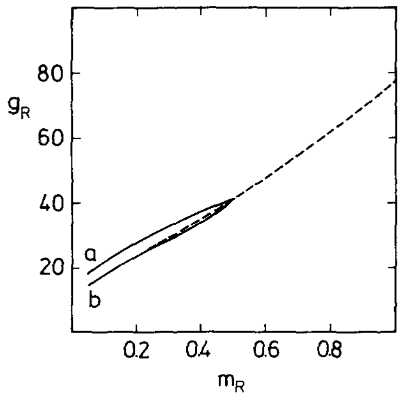

Fig. 4. Comparison of the high temperature expansion at $\bar{\lambda} = 1$ (dashed curve, $0.2 \leqslant m_R \leqslant 1.0$) with the solution of the renormalization group equation (4.26) using $m_R = 0.49$, $g_R = 41$ as initial data (curves a and b corresponding to the 2-loop and 3-loop approximation of the β-function).

For $m_R \to 0$, the values of κ obtained from the solution of the renormalization group equations must converge to κ_c. As is apparent from table 3, κ is indeed nearly constant within errors for $m_R \leqslant 0.05$ and it turns out that the asymptotic values are in complete agreement with our earlier estimation of κ_c (table 1), thus providing a good check on our calculations.

We finally come back to the discussion at the end of subsect. 4.1 of the possible non-invertibility of the mapping $(\kappa, \lambda) \to (m_R, g_R)$. As we have already noted in subsect. 6.4, the mapping is invertible in the high temperature region so that here we mainly comment on what may happen close to the critical line. A first observation is that one may always find acceptable lattice actions for which the mapping is not invertible somewhere in the plane of bare parameters. A simple example of such an action is obtained by adding an "irrelevant" term

$$\sum_x \frac{\lambda'}{6!} \phi(x)^6 \tag{8.1}$$

to the action (2.1) and choosing $\lambda' = \lambda'(\lambda)$ appropriately (in the space of parameters $\kappa, \lambda, \lambda'$, a two-dimensional submanifold $\lambda' = \lambda'(\lambda)$ can always be defined such that some of the lines with fixed m_R and g_R are cut more than once). Thus, for a given lattice action the invertibility of the mapping of bare to renormalized parameters is a rather accidental property which cannot be of fundamental importance. The crucial point to note is that if there are several regions in the κ, λ-plane with the same ranges of m_R, g_R, the corresponding renormalized vertex functions need not be different if we disregard scaling violations. In other words, a simple and natural possibility is that the theory in all these regions belongs to the same universality class and that the associated scaling laws are hence the same everywhere.

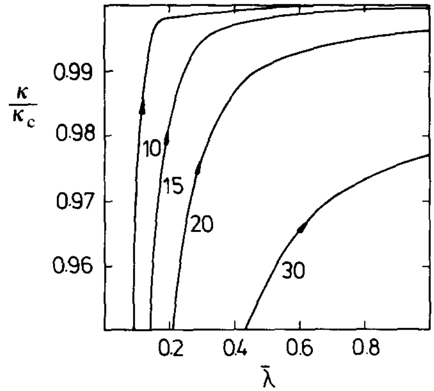

Fig. 5. Quantitative plot of the renormalization group trajectories in the plane of bare parameters. The curves are labelled by g_R and arrows are in the direction of decreasing m_R.

Of course, if the parameter mapping should not be invertible for the action (2.1), we cannot be absolutely sure that universality in the above sense holds. There are however good reasons to believe so, since the coupling g_R is already rather small for $\kappa > 0.95\kappa_c$ (and perturbation theory *is* universal [29]) and because our solution of the renormalization group equations agrees very well with the high temperature analysis and the available Monte Carlo data.

9. Renormalization group trajectories and the triviality bound

From our numerical solution of the renormalization group equations, it is now easy to determine the curves of constant coupling g_R in the scaling region $\kappa \geqslant 0.95\kappa_c$. As may be seen from fig. 5, these curves are first parallel to the free field line $\bar{\lambda} = 0$ (as one would expect from bare perturbation theory) and are then repelled by the "gaussian fixpoint" at $\bar{\lambda} = 0$, $\kappa = \kappa_c$. All the trajectories end at the Ising line $\bar{\lambda} = 1$ after having followed the critical curve at a small distance approximately given by

$$\delta\kappa \propto e^{-32\pi^2/3g_R}. \tag{9.1}$$

That all the interesting near continuum physics happens in such a narrow strip around the critical line is of course just a reflection of the fact that κ is associated to the "relevant" operator $\mathcal{O}$ in the action (κ is a "fine tuning" parameter in other words).

Without detailed calculations, the qualitative aspects of fig. 5 could have been anticipated from simple monotonicity arguments. As an example for this type of reasoning, we now prove that the minimal value of m_R at fixed g_R (the triviality bound in other words) is obtained in the Ising limit if the β-function is single

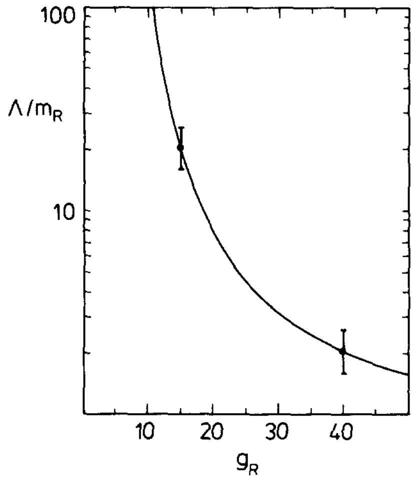

Fig. 6. Maximal value of the ultraviolet cutoff Λ in units of m_R for given coupling g_R. The size of the estimated errors as quoted in table 3 is indicated at two representative points.

valued, positive and continuous for $0 < m_R \leqslant 0.5$ and $g_R > 0$. Indeed, from the high temperature analysis we know that for $m_R = 0.5$, g_R is monotonically rising with $\bar{\lambda}$. Since this property is conserved by the differential equation (4.26), it holds for any $m_R < 0.5$, too. The curves in the phase diagram with a fixed value of m_R hence cross a given renormalization group trajectory at most once. In particular, m_R cannot assume a minimum on these trajectories in the interior of the region $0 \leqslant m_R \leqslant 0.5$, $0 \leqslant \bar{\lambda} \leqslant 1$. Finally, invoking the positivity of the β-function, all boundary points except those with $\bar{\lambda} = 1$ can easily be excluded thus proving our assertion.

As we have discussed in the preceding section, the true β-function is perhaps multiple valued somewhere close to the critical line and the above argument would then break down. However, assuming universality, the multiple valuedness of β is only present (if at all) on the level of scaling violations and we therefore expect that the true minimal value of m_R along a given renormalization group trajectory is always very close to the value of m_R in the Ising limit. In particular, there is no reason to doubt the validity of our approximate determination of the renormalization group trajectories, which was based on a single valued β-function and which hence yields the minimal m_R at $\bar{\lambda} = 1$.

To sum up, we have thus shown that the triviality bound (1.1) is (essentially) saturated by the Ising model. Using our numerical solution of the renormalization group equations (table 3), we hence obtain the curve shown in fig. 6, where we have reintroduced the ultraviolet cutoff Λ to conform with the notation of sect. 1 (by definition, $\Lambda = 1$ in lattice units). Fig. 6 reveals that as expected from eq. (1.1), the maximal value of the cutoff is very rapidly rising when g_R is made smaller. On the other hand, when g_R increases towards the tree level unitarity bound, Λ becomes of order m_R thus showing again that near continuum physics is apparently tied up with the applicability of renormalized perturbation theory. We finally note that at small g_R a more definite expression for the triviality bound, replacing the asymptotic formula (1.1), is given by

$$\ln(\Lambda/m_R) \leqslant \frac{1}{\beta_1 g_R} + \frac{\beta_2}{\beta_1^2}\ln(\beta_1 g_R) + C(g_R), \tag{9.2}$$

where $C(g_R)$ satisfies

$$-1.7 \leqslant C(g_R) \leqslant -1.3 \qquad \text{for } g_R \leqslant 10. \tag{9.3}$$

10. Conclusions

The results of this paper suggest the following remarkably consistent and simple picture of the lattice ϕ^4 theory in the symmetric phase.

(i) In the phase diagram there is a region Γ (the white area below the critical line in fig. 2), where the scaling violations in the scattering amplitude at low energies and in other physical quantities are small. Thus, in this region the theory effectively behaves like a continuum theory at low energies.

(ii) The maximal value of the renormalized coupling g_R in Γ is about 41, which is roughly $\frac{2}{3}$ of the tree level unitarity bound. In general, renormalized perturbation theory may therefore be applied to calculate the vertex functions at momenta well below the cutoff Λ.

(iii) In Γ and at fixed g_R, the cutoff Λ may assume any value between $2m_R$ and the triviality bound, which is given by fig. 6 for large g_R and by eqs. (9.2), (9.3) otherwise. This bound is (essentially) saturated in the Ising limit of the theory, i.e. for infinite bare coupling.

An important and perhaps surprising qualitative aspect of this picture is that a truly non-perturbative sector, where the coupling is well above the tree level unitarity bound and the cutoff is reasonably large (say $\Lambda = 10m_R$), does not exist. In other words, there is no strongly interacting lattice ϕ^4 theory, which could be regarded as an effective continuum theory at low energies.

We expect that our methods also apply to other lattice field theories including QED, the U(1) Higgs model and, of course, the n-component ϕ^4 theory. In the latter case, we have already worked out the large n limit and found that the situation there is qualitatively the same as in the one-component model. In the O(4)-symmetric theory, which is related to the physically interesting SU(2) Higgs model, we would therefore be surprised, if a very different picture would result [44]. The broken symmetry phase of the ϕ^4 theory is more difficult to treat because there is apparently no convenient expansion in this part of the phase diagram, which could play the rôle of the high temperature expansion in our analysis. However, perturbation theory and the Callan-Symanzik equation are still at our disposal so that at least the scaling laws and the qualitative shape of the renormalization group trajectories can be derived (cf. ref. [1]). One may perhaps also be able to carry over the results from the symmetric phase to a small region on the other side of the critical line using mass perturbation theory.

As we have already mentioned in the introduction, it is conceivable that the ultraviolet cutoff, which is needed to make a trivial theory interacting at low energies, may be provided by an embracing asymptotically free (or otherwise stabilized) theory. An interesting and also very difficult question then is, whether for a given trivial model such an embracing theory exists at all. Perhaps some restrictions could be obtained in this way on the possible scalar sectors in phenomenologically relevant theories, although one could always object that at energies as high as the Planck mass, nature is probably no longer describable by a quantum field theory and ultraviolet stability would then be provided by a different structure.

Appendix A

PERTURBATION EXPANSION OF THE CALLAN-SYMANZIK COEFFICIENTS β, γ AND δ

The renormalization group functions in the ϕ^4 theory have first been calculated up to three loops in the massless case using dimensional regularization and a momentum subtraction scheme [34]. With minimal subtraction, computations have later been performed through four loops (ref. [33] and references quoted therein) and more recently even to five loops [31, 32]. All these nice results are not immediately useful here, because our renormalization scheme is different. However, it is not too difficult to determine the relation between the various schemes up to two loops and this is then sufficient to obtain β, γ and δ as defined in this paper up to three loops. We do not present the details of this calculation here but merely quote the result in table 4, where we have used the notation

$$\beta(0, g_R) = g_R \sum_{\nu=1}^{\infty} \beta_\nu g_R^\nu, \tag{A.1}$$

$$\gamma(0, g_R) = \sum_{\nu=1}^{\infty} \gamma_\nu g_R^\nu, \tag{A.2}$$

$$\delta(0, g_R) = \sum_{\nu=1}^{\infty} \delta_\nu g_R^\nu \tag{A.3}$$

for the loop expansion.

As discussed in sect. 8, the scaling violations in the Callan-Symanzik coefficients are not always negligible and we have therefore also calculated β, γ, δ for arbitrary

TABLE 4
Perturbation expansion coefficients for β, γ and δ according to eqs. (A.1)–(A.3)

ν	$(16\pi^2)^\nu\beta_\nu$	$(16\pi^2)^\nu\gamma_\nu$	$(16\pi^2)^\nu\delta_\nu$
1	3	0	−1
2	−17/3	1/12	5/6
3	26.908403	0.14065121	−3.7708683

m_R up to one loop. Defining u_ν, v_ν, w_ν through

$$\beta(m_R, g_R) = g_R \sum_{\nu=0}^{\infty} u_\nu(m_R) g_R^\nu, \tag{A.4}$$

$$\gamma(m_R, g_R) = \sum_{\nu=0}^{\infty} v_\nu(m_R) g_R^\nu, \tag{A.5}$$

$$\delta(m_R, g_R) = \sum_{\nu=0}^{\infty} w_\nu(m_R) g_R^\nu, \tag{A.6}$$

we have

$$u_0 = \frac{4m_R^2}{8+m_R^2}, \tag{A.7}$$

$$u_1 = u_0\left\{\tfrac{3}{2}(8+m_R^2)J_3(m_R) - J_2(m_R) - \frac{J_1(m_R)}{2(8+m_R^2)}\right\}, \tag{A.8}$$

$$v_0 = \tfrac{1}{4}u_0, \tag{A.9}$$

$$v_1 = \tfrac{1}{8}u_0\left\{J_2(m_R) - \frac{J_1(m_R)}{8+m_R^2}\right\}, \tag{A.10}$$

$$w_0 = -\tfrac{1}{2}u_0, \tag{A.11}$$

$$w_1 = -2v_1 - \tfrac{1}{4}m_R^2\left\{(8+m_R^2)J_3(m_R) - 2J_2(m_R) + \frac{J_1(m_R)}{8+m_R^2}\right\}, \tag{A.12}$$

where the lattice 1-loop integrals $J_p(\mu)$ are defined by

$$J_p(\mu) = \int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}(\mu^2+\hat{k}^2)^{-p}, \qquad \hat{k}_\nu = 2\sin\tfrac{1}{2}k_\nu. \tag{A.13}$$

It may come as a surprise that the tree level coefficients u_0, v_0 and w_0 do not vanish, but since they are of order m_R^2, they are just scaling violation terms which disappear in the continuum limit. One may also easily show, using the asymptotic expansion of the integrals J_p given in appendix B, that the 1-loop coefficients u_1, v_1, w_1 agree with the universal coefficients $\beta_1, \gamma_1, \delta_1$ up to terms of order $m_R^2 \ln m_R$ as expected.

Appendix B

PROPERTIES OF THE INTEGRALS J_1, J_2, J_3

For $\mu \to 0$, the integrals $J_p(\mu)$ defined in appendix A have an asymptotic expansion of the form

$$J_p(\mu) = J_p^{\mathrm{as}}(\mu) + \mathrm{O}(\mu^{8-2p} \ln \mu^2), \tag{B.1}$$

$$J_1^{\mathrm{as}}(\mu) = r_0 + \mu^2(r_1 + s_1 \ln \mu^2) + \mu^4(r_2 + s_2 \ln \mu^2), \tag{B.2}$$

$$J_{p+1}^{\mathrm{as}}(\mu) = -\frac{1}{p}\frac{\partial}{\partial \mu^2} J_p^{\mathrm{as}}(\mu), \tag{B.3}$$

where the coefficients in (B.2) are given by

$$r_0 = 0.154\,933\,390, \tag{B.4}$$

$$r_1 = -0.030\,345\,755, \tag{B.5}$$

$$r_2 = 0.002\,775\,927, \tag{B.6}$$

$$s_1 = \frac{1}{16\pi^2}, \tag{B.7}$$

$$s_2 = -\frac{1}{128\pi^2}. \tag{B.8}$$

A useful numerical representation of J_1, J_2, J_3 in the range $0 < \mu \leqslant 1$ is

$$J_p(\mu) = J_p^{\mathrm{as}}(\mu) \sum_{n=0}^{N} c_{p,n} T_n(2\mu - 1) + \varepsilon, \tag{B.9}$$

where $T_n(x)$ denotes the Chebyshev polynomial defined by

$$T_n(\cos\theta) = \cos n\theta, \tag{B.10}$$

TABLE 5
Coefficients in the Chebyshev expansion (B.9) of the integrals J_1, J_2, J_3

n	$c_{1,n}$	$c_{2,n}$	$c_{3,n}$
0	0.999 533	1.009 218	0.966 548
1	−0.000 790	0.014 679	−0.049 015
2	−0.000 472	0.007 226	−0.017 210
3	−0.000 193	0.002 002	−0.000 822
4	−0.000 049	0.000 240	0.000 832
5	−0.000 006	0.000 006	−0.000 032
6		0.000 003	−0.000 032
7			0.000 010
8			−0.000 002

and for the error ε to satisfy

$$|\varepsilon/J_p(\mu)| < 10^{-6}, \tag{B.11}$$

it is sufficient to take $N = 5, 6, 8$ for $p = 1, 2, 3$ respectively. The corresponding coefficients $c_{p,n}$ are listed in table 5.

Appendix C

EXPANSIONS IN POWERS OF λ

Starting from the action (2.2), one derives in the usual way the perturbation expansion of the vertex functions $\Gamma^{(n,l)}$ in powers of g_0. Insertion of the relations (2.4) then yields the expansions in powers of λ, which we have worked out up to two loops for various quantities. The result of these calculations is given in table 6, where we use the notation

$$X = \lambda^p \sum_{\nu=0}^{\infty} X^{(\nu)} \lambda^\nu \tag{C.1}$$

for the expansion of X.

TABLE 6
Perturbation expansion coefficients for various quantities X according to eq. (C.1)

X	p	$X^{(0)}$	$X^{(1)}$	$X^{(2)}$
$\bar{\lambda}$	1	3	−33	576
κ_c	0	0.12500	0.2148	−2.351
$m_R(\kappa)$	0	0.64889	−1.106	22.67
$g_R(\kappa)$	1	4.2548×10^2	-8.815×10^3	2.421×10^5
$Z_R(\kappa)$	0	4.2105	−7.235	89.47
$Z_R^{\mathcal{O}}(\kappa)$	0	1.4844×10^{-2}	6.007×10^{-2}	−0.7464

m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ are evaluated along the line $\kappa = 0.95\kappa_c(\lambda)$.

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