ABSENCE OF ANOMALIES IN TWO-DIMENSIONAL NONABELIAN CHIRAL GAUGE THEORIES

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Using the appropriate quantization procedure for the gauge fields, we explicitly show that two-dimensional nonabelian gauge theories with chiral coupling to massless fermions are free of anomalies

1 *Introduction*

In the last year there arose a new development concerning anomalous gauge theories [1–4] It has been observed that integration over all gauge field configurations (also those which are related by gauge transformations) in the path integral automatically leads to a gauge-invariant, anomaly-free quantum theory [2–4] This prescription is justified by the fact that gauge fixing does not make sense in a theory without gauge invariance The procedure results in a theory which contains, compared to the old inconsistent treatment, additional bosonic degrees of freedom besides fermions and "transverse" gauge fields These boson fields cancel the anomalies of the fermionic sector This opens up the possibility to investigate a gauge theory with chiral coupling to fermions without encountering mortal defects from the very beginning

There is a formal proof of gauge invariance for the general case [4] Up to now, however, there is not much progress in showing this feature for specific models explicitly Only the chiral Schwinger model has been investigated in detail [4–13] In the present work we want to extend this to the case of a nonabelian theory where the absence of anomalies is not as simple as in the abelian case [7,8] However, we stick to two dimensions where the fermions can be integrated out explicitly In this way we hope to gain some experience with the mechanism of anomaly cancellation without being forced to enter perturbation theory (This would be necessary in four dimensions where also renormalization problems have to be faced) The nonabelian chiral gauge theory in two dimensions has also been treated in refs [7,14], however, in ref [7] the analysis was not completed, and ref [14], which has also been criticised in ref [7], used the so-called "anomalous" formulation [8]

2 *Nonabelian chiral gauge theory in two dimensions*

2 1 *Effective action* We consider an SU(N) gauge theory with chiral coupling to massless fermions,

$$S=\int d^2x\,\{-\tfrac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}+\bar{\psi}\gamma^{\mu}[i\partial_{\mu}+\tfrac{1}{2}gA_{\mu}(1+i\gamma_5)]\psi\}\,, \tag{1}$$

where $A_{\mu}=A^a_{\mu}\tau^a$ with τ^a being the generators of the gauge group, normalized according to $\mathrm{tr}\,\tau^a\tau^b=\tfrac{1}{2}\delta^{ab}$ Our notation is $\gamma_5=i\gamma^0\gamma^1$, $\eta_{00}=-\eta_{11}=1$, $\varepsilon_{01}=-\varepsilon^{01}=-1$ In the following we shall often use the light-cone representation $x_{\pm}=x_0\pm x_1$, $\eta^{+-}=\eta^{-+}=\tfrac{1}{2}$, $\varepsilon^{+-}=-\varepsilon^{-+}=-\tfrac{1}{2}$ In this representation the interaction reads (due to $i\gamma^{\mu}\gamma_5=\varepsilon^{\mu\nu}\gamma_{\nu}$)

$$S_I=\int d^2x\,\tfrac{1}{2}g\bar{\psi}\gamma_+A_-\psi\,, \tag{2}$$

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i.e., only A_- couples to the fermion fields. In two dimensions the fermions can be integrated out explicitly, the result for QCD_2 (pure vector coupling) reads [15,16] (we ignore topologically nontrivial gauge configurations [17])

$$\tilde{W}_{QCD}[A]=I[T_+]+I[T_-]+\frac{g^2}{4\pi}\int d^2x\, \mathrm{tr}\, A_+A_- , \tag{3}$$

where $\tilde{W}_{QCD}[A]$ is defined according to

$$\exp(i\tilde{W}_{QCD}[A])=\int d\psi\, d\bar{\psi}\, \exp\left(i\int d^2x\, \bar{\psi}\gamma^\mu(i\partial_\mu+gA_\mu)\psi\right) \tag{4}$$

The gauge field is represented as

$$A_\pm=-\frac{1}{g}(\partial_\pm T_\mp)T_\mp^{-1} , \tag{5}$$

and I is defined by ($\mathrm{Tr}=\int d^2x\,\mathrm{tr}$)

$$I[T]=\frac{1}{8\pi}\mathrm{Tr}[(\partial_+T)T^{-1}(\partial_-T)T^{-1}](x)+\frac{\varepsilon^{\mu\nu}}{4\pi}\mathrm{Tr}\int_0^1 dt\,[(\partial_tT)T^{-1}(\partial_\mu T)T^{-1}(\partial_\nu T)T^{-1}](x,t) \tag{6}$$

$T_\pm(x,t)$ is an interpolation between $\mathbb{1}$ and $T_\pm(x)$: $T_\pm(x,0)=\mathbb{1}$, $T_\pm(x,1)=T_\pm(x)$. In eq. (3), the regularization ambiguity has been resolved by requiring gauge invariance with respect to

$$A_\pm\to A_\pm^S=SA_\pm S^{-1}-(1/g)(\partial_\pm S)S^{-1} , \quad T_\mp\to T_\mp^S=ST_\mp \tag{7}$$

In the case of the chiral coupling as in eqs. (1) and (2), only T_+ couples such that $I[T_-]$ does not occur. Due to the regularization, however, the mass term has to be kept even though it contains also T_- [18]. In addition, since there is no symmetry principle to fix the relative strength of the local polynomial, there is an arbitrary parameter associated with the mass term. This leads for the chiral case to the effective action

$$\tilde{W}[A]=I[T_+]+(ag^2/8\pi)\,\mathrm{Tr}\,A_+A_- \tag{8}$$

This result, which is already indicated in ref. [19], can also be achieved by simply adding an arbitrary mass term to the light-cone gauge result of QCD_2 [20]. Another approach to derive the effective action is the direct evaluation of the fermion determinant using an appropriate regularization prescription [21].

Following now the procedure for quantizing chiral gauge theories as outlined in refs. [2–4], it has to be recognized that the relevant action is $\tilde{W}[A^{G^{-1}}]$ rather than $\tilde{W}[A]$ since the generating functional has the form

$$Z=\int dA\, dG\, \delta(f(A))\Delta_f[A]\, \exp(iW[A^{G^{-1}}]) , \tag{9}$$

where

$$W[A]=-\tfrac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}+\tilde{W}[A] \tag{10}$$

In eq. (9), $\delta(f(A))$ is the gauge fixing δ-function, $\Delta_f[A]$ the associated Faddeev–Popov determinant and dG the invariant group measure. In passing, we note that $W[A^{G^{-1}}]$ is gauge invariant with respect to $A\to A^S$, $G\to SG$, hence we have a gauge-invariant theory in spite of chiral fermions [4,7,8]. This gauge invariance is a formal hint for the absence of genuine anomalies, because the anomaly, which is defined as the covariant divergence of the current, is nothing else but the gauge variation of the effective action. Now we have to calculate

$$\tilde{W}[A^{G^{-1}}]=I[G^{-1}T]+\frac{ag^2}{8\pi}\,\mathrm{Tr}\,A_+^{G^{-1}}A_-^{G^{-1}} \tag{11}$$

The mass term gives

$$\frac{ag^2}{8\pi}\mathrm{Tr}\,A_+^{G^{-1}}A_-^{G^{-1}}=\frac{-a}{8\pi}\mathrm{Tr}\{(\partial_+T_-)T_-^{-1}(\partial_-T_+)T_+^{-1}+G(\partial_+G^{-1})G(\partial_-G^{-1})$$

$$+G(\partial_+G^{-1})(\partial_-T_+)T_+^{-1}+G(\partial_-G^{-1})(\partial_+T_-)T_-^{-1}\}\,, \tag{12}$$

and the Wess–Zumino action I changes according to

$$I[G^{-1}T_+]=I[G^{-1}]+I[T_+]+\frac{1}{4\pi}\mathrm{Tr}\,G(\partial_+G^{-1})(\partial_-T_+)T_+^{-1} \tag{13}$$

Altogether we find

$$\tilde{W}[A^{G^{-1}}]=I[T_+]-\frac{a}{8\pi}\mathrm{Tr}\,(\partial_+T_-)T_-^{-1}(\partial_-T_+)T_+^{-1}$$

$$+\frac{1}{4\pi}\mathrm{Tr}\Big(\tfrac{1}{2}(1-a)G(\partial_+G^{-1})G(\partial_-G^{-1})+(1-\tfrac{1}{2}a)G(\partial_+G^{-1})(\partial_-T_+)T_+^{-1}-\tfrac{1}{2}aG(\partial_-G^{-1})(\partial_+T_-)T_-^{-1}$$

$$+\int_0^1 \mathrm{d}t\,\varepsilon^{\mu\nu}G(\partial_tG^{-1})G(\partial_\mu G^{-1})G(\partial_\nu G^{-1})\Big) \tag{14}$$

2.2. *Currents and two-point functions.* In order to study possible anomalies, we have to construct currents. We are only interested in those currents which couple to the gauge field, these are defined as

$$\langle J_\pm\rangle^{uv}(x)=-\frac{1}{g}\frac{\delta}{\delta A_\mp^{vu}(x)}\exp(\mathrm{i}\tilde{W}[A^{G^{-1}}]) \tag{15}$$

where $\langle\ \rangle$ denotes an "average" over fermion fields only, the indices u and v specify elements of the SU(N) matrices. In order to perform the functional derivative, we replace $(\partial_\mp T_\pm)T_\pm^{-1}$ everywhere but in $I[T_+]$ by $A_\pm$. The $\langle J_-\rangle$ can be calculated directly with the result

$$\langle J_-\rangle=\frac{a}{8\pi}(gA_- -\mathrm{i}G\partial_-G^{-1})\exp(\mathrm{i}\tilde{W}[A^{G^{-1}}])=\frac{-\mathrm{i}a}{8\pi}[\partial_-T_+)T_+^{-1}+G\partial_-G^{-1}]\exp(\mathrm{i}\tilde{W}[A^{G^{-1}}]) \tag{16}$$

For $\langle J_+\rangle$ we need the derivative of $I[T_+]$ with respect to A_-. To this aim we use the formula

$$T_+^{(1)}=T_+^{(0)}\cdot T_+[T_+^{(0)-1}(A_-^{(1)}-A_-^{(0)})T_+^{(0)}]\,,\quad T_+^{(i)}=T_+[A_-^{(i)}]\,, \tag{17}$$

which is Lemma 4.2 of ref. [16]. The differential equation (5) for T_+ can be converted to an integral equation

$$T_+(x)=1+\mathrm{i}g\int \mathrm{d}^2y\,D_+(x-y)A_-(y)T_+(y)\,, \tag{18}$$

where

$$D_+(x)=\frac{-\mathrm{i}}{4\pi}\frac{1}{x^+-\mathrm{i}\varepsilon\,\mathrm{sgn}\,x^-}\Rightarrow\partial_-D_+(x)=\delta^2(x) \tag{19}$$

Applying ∂_- to eq. (18) thus leads to eq. (5). Eqs. (17) and (18) can be combined to give the first-order variation of T_+ with respect to A_-.

$$T_+[A_- + \delta A_-](x) = T_+[A_-](x)\left(1 + ig\int d^2y\, D_+(x-y)(T_+^{-1}[A_-]\delta A_- T_+[A_-])(y)\right) \tag{20}$$

This means for the variation of $I[T_+]$

$$I[T_+[A_- + \delta A_-]] = I[T_+[A_-]]\ \{1 + O(\delta A_-)\}$$

$$= I[T_+[A_-]] + I[1 + O(\delta A)] + \frac{ig}{4\pi}\,\mathrm{Tr}\,(\partial_+ T_+)T_+^{-1}\delta A_- \tag{21}$$

Since $I[1 + O(\delta A)] = O((\partial\delta A)^2)$ this can be neglected and we find

$$\frac{\delta I[T_+]}{\delta A_-^{\prime\prime\prime}(x)} = \frac{ig}{4\pi}[(\partial_+ T_+)T_+^{-1}]^{m}(x) \tag{22}$$

With this the + component of the current is calulated to be

$$\langle j_+\rangle = \frac{1}{4\pi}[(\partial_+ T_+)T_+^{-1} - \tfrac{1}{2}a(\partial_+ T_-)T_-^{-1} + (1-\tfrac{1}{2}a)G\partial_+ G^{-1}]\exp(i\tilde{W}[A^{G^{-1}}]) \tag{23}$$

Note that this coincides, as expected, with the result for the pure vector case [16], if we set $a=2$ Finally, we state the results for the time-ordered two-point functions of currents, which are the second derivatives of $\exp(i\tilde{W}[^{G^{-1}}])$ with respect to the gauge fields without explicit derivation

$$\langle T^* j_-^a(x) j_-^b(y)\rangle = \langle j_-^a(x)\rangle\langle j_-^b(y)\rangle\exp(-i\tilde{W}[A^{G^{-1}}]), \tag{24}$$

$$\langle T^* j_+^a(x) j_-^b(y)\rangle = \langle j_+^a(x)\rangle\langle j_-^b(y)\rangle\exp(-i\tilde{W}[A^{G^{-1}}]) - \frac{ia}{4\pi}\delta^{ab}\delta^2(x-y)\exp(i\tilde{W}[A^{G^{-1}}]), \tag{25}$$

$$\langle T^* j_+^a(x) j_+^b(y)\rangle = \langle j_+^a(x)\rangle\langle j_+^b(y)\rangle\exp(-i\tilde{W}[A^{G^{-1}}])$$

$$+\frac{1}{4\pi}\mathrm{tr}\{[T_+^{-1}\lambda^a T_+](x)D_+'(x-y)[T_+^{-1}\lambda^b T_+](y)\}\exp(i\tilde{W}[A^{G^{-1}}]), \tag{26}$$

where $\lambda^a = 2\tau^a$ are generalized Gell-Mann matrices and $D_+' = \partial_+ D_+$ Formally, eqs (25) and (26) have to coincide with the result of ref [16], if we set $a=2$, since the A-dependence of j_+ is the same as in this reference

2 3 Absence of anomalies The currents transform under gauge transformations according to $\langle j\rangle^S = S\langle j\rangle \times S^{-1}$, hence the covariant derivative of j reads

$$\mathscr{D}_\mu\langle j_\nu\rangle = \partial_\mu\langle j_\nu\rangle + ig[\langle j_\nu\rangle, A_\mu] \tag{27}$$

For a consistent dynamical theory of gauge fields it is necessary that the current which couples to the gauge field is covariantly conserved, i e ,

$$\mathscr{D}_\mu\langle j^\mu\rangle = \tfrac{1}{2}(\mathscr{D}_+\langle j_-\rangle + \mathscr{D}_-\langle j_+\rangle) = 0 \tag{28}$$

The left-hand side can be calculated to give

$$\mathscr{D}_\mu\langle j^\mu\rangle = \frac{1}{8\pi}\{\partial_+[(1-\tfrac{1}{2}a)(\partial_- T_+)T_+^{-1} - \tfrac{1}{2}aG\partial_- G^{-1}] + \partial_-[-\tfrac{1}{2}a(\partial_+ T_-)T_-^{-1} + (1-\tfrac{1}{2}a)G\partial_+ G^{-1}]$$

$$+(1-\tfrac{1}{2}a)[G(\partial_+ G^{-1}), (\partial_- T_+)T_+^{-1}] - \tfrac{1}{2}a[G(\partial_- G^{-1}), (\partial_+ T_-)T_-^{-1}]\}\exp(i\tilde{W}[A^{G^{-1}}]), \tag{29}$$

where we used

$$[(\partial_+ T_+)T_+^{-1}, (\partial_- T_+)T_+^{-1}] = \partial_+((\partial_- T_+)T_+^{-1}) - \partial_-((\partial_+ T_+)T_+^{-1}) \quad (30)$$

Experience with the abelian case where the additional boson field enforces current conservation [8] suggests to try whether the equation of motion for G has consequences for $D_\mu\langle J^\mu\rangle$ To this aim we write the effective action as

$$\tilde{W}[A^{G^{-1}}] = I[G^{-1}]$$

$$+\frac{1}{4\pi}\mathrm{Tr}\{-\tfrac{1}{2}aG(\partial_+ G^{-1})G(\partial_- G^{-1}) + (1-\tfrac{1}{2}a)G(\partial_+ G^{-1})(\partial_- T_+)T_+^{-1} - \tfrac{1}{2}aG(\partial_- G^{-1})(\partial_+ T_-)T_-^{-1}\}$$

$$+\text{terms independent of } G \quad (31)$$

For the variation of $I[G^{-1}]$ we write $(G+\delta G)^{-1} = G^{-1}(\mathbb{1} - \delta G\cdot G^{-1})$ and use the same procedure as that leading to eq (22), this yields

$$\frac{\delta I[G^{-1}]}{\delta G^{uv}(x)} = \frac{1}{4\pi}[G^{-1}\partial_-(G\partial_+ G^{-1})]^{vu}(x) \quad (32)$$

Furthermore we need

$$\frac{\delta\,\mathrm{Tr}\,G(\partial_\mu G^{-1})X}{\delta G^{ut}(x)} = (G^{-1})^{tr}(x)\{\partial_\mu X + [G(\partial_\mu G^{-1}), X]\}^{ru}(x)\,, \quad (33)$$

valid for any X With these results we can derive the equation of motion for G

$$0 = \frac{\delta\tilde{W}[A^{G^{-1}}]}{\delta G^{uv}}$$

$$= \frac{1}{4\pi}(G^{-1})^{vr}\{\partial_+[(1-\tfrac{1}{2}a)(\partial_- T_+)T_+^{-1} - \tfrac{1}{2}aG\partial_- G^{-1}] + \partial_-[-\tfrac{1}{2}a(\partial_+ T_-)T_-^{-1} + (1-\tfrac{1}{2}a)G\partial_+ G^{-1}]$$

$$+(1-\tfrac{1}{2}a)[G(\partial_+ G^{-1}), (\partial_- T_+)T_+^{-1}] - \tfrac{1}{2}a[G(\partial_- G^{-1}), (\partial_+ T_-)T_-^{-1}]\}^{ru}$$

$$= -2\mathrm{i}(G^{-1})^{vr}(\mathscr{D}_\mu\langle J^\mu\rangle)^{ru}\exp(-\mathrm{i}\tilde{W}[A^{G^{-1}}]) \quad (34)$$

Hence we learn that it is the dynamics of the G field which enforces current conservation and thus makes the theory consistent This also ensures a canonical $[J^0, J^0]$ equal-time commutator, i e, the absence of Schwinger terms The proof for this, being valid for arbitrary dimension n, goes as follows Let $\mathscr{A}^a$ be the anomaly, i e, the covariant divergence of the current and define

$$\mathrm{i}\frac{\delta}{\delta A_\nu^b(y)}\mathscr{A}^a(x) = I^{\mu\nu\,ab}\partial_\mu\delta^n(x-y) + I^{\nu\,ab}\delta^n(x-y) \quad (35$$

Then the 0–0 component of the equal-time current–current commutator can be expressed in terms of $I^{\mu0\,ab}$ and $I^{0\,ab}$ [22]

$$[\langle J^{0a}(x)\rangle, \langle J^{0b}(y)\rangle]_{\mathrm{ETC}}$$

$$= \mathrm{i}f^{abc}J^{0c}\delta^{n-1}(\boldsymbol{x}-\boldsymbol{y}) + (I^{0\,ab} - f^{acd}A_\mu^c I^{\mu0\,db})\delta^{n-1}(\boldsymbol{x}-\boldsymbol{y}) + (I^{i0\,ab} - I^{0i\,ab})\partial_i\delta^{n-1}(\boldsymbol{x}-\boldsymbol{y}) \quad (36)$$

This means that there are no Schwinger terms in $[J^0, J^0]_{\mathrm{ETC}}$ if the current J is conserved, since in this case $I^{\mu0\,ab}$ and $I^{0\,ab}$ vanish

3 Conclusion

In the present work we have explicitly shown that the chiral nonabelian gauge theory in two dimensions is gauge invariant, free of anomalies and that there are no Schwinger terms in the j^0–j^0 commutator The consistency of the model (at least up to this level) is dynamically ensured by the boson fields which stem from the appropriate quantization procedure for the gauge field In this way the theory seems to rescue itself as soon as there arise anomalies in the fermion sector, these boson fields become nontrivially coupled and cancel the fermionic anomalies, leaving behind a theory which appears to be consistent

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