THE PATH INTEGRAL ON THE POINCARÉ UPPER HALF PLANE AND FOR LIOUVILLE QUANTUM MECHANICS

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We present a rigorous path integral treatment of free motion on the Poincaré upper half plane. The Poincaré upper half plane, as a riemannian manifold, has recently become important in string theory and in the theory of quantum chaos. The calculation is done by a time-transformation and the use of the canonical method for determining quantum corrections to the classical lagrangian. Furthermore, we shall show that the same method also works for Liouville quantum mechanics. In both cases, the energy spectrum and the normalized wavefunctions are determined.

In this paper we present a complete path integral treatment for a particle moving freely on the Poincaré upper half plane $U\equiv\{z=x+iy\,|\,y>0\}$. Recently, this model for a non-euclidean geometry has become important in the theory of strings, in particular in the Polyakov approach for the bosonic string – see e.g. ref. [1], and in the theory of quantum chaos – see e.g. ref. [2–4]. In both cases one considers bounded domains in the upper half plane, which are fundamental regions of discrete subgroups of $PSL(2,\mathbb{R})$. We shall not consider the motion in bounded domains; our paper will deal with the free motion on the entire upper half plane.

The Poincaré upper half plane is analytically equivalent to three further riemannian spaces: the pseudosphere Λ^2, the Poincaré disc D and the hyperbolic strip S. For a review of classical and quantum mechanical motion (in bounded and unbounded domains) in these four riemannian spaces, see e.g. ref. [4].

The study of Liouville quantum mechanics and quantum field theory arises in many areas of mathematics and physics, recently also in string models – see e.g. refs. [5,6].

Classical mechanics on the Poincaré upper half plane is described by the classical lagrangian and hamiltonian, respectively:

$$\mathscr{L}_{\rm cl}=(m/2y^2)(\dot{x}^2+\dot{y}^2)\;,\qquad \mathscr{H}_{\rm cl}=(y^2/2m)(p_x^2+p_y^2)\;, \tag{1}$$

with $p_x=m\dot{x}/y^2$, $p_y=m\dot{y}/y^2$ and the metric $g_{ab}=(1/y^2)\delta_{ab}$. The Laplace–Beltrami operator or quantum hamiltonian reads ($\hbar=1$)

$$H=-(y^2/2m)(\partial^2/\partial x^2+\partial^2/\partial y^2)\;. \tag{2}$$

Notice that a necessary condition for wavefunctions $\psi\in L^2(U)\cap D(H)$ is $\psi(x,y)=0$ for $y=0$ $(x\in\mathbb{R})$. The scalar product for two functions f, g defined on U is given by

$$(f,g)=\int\limits_{-\infty}^{\infty}\mathrm{d}x\int\limits_0^{\infty}\frac{\mathrm{d}y}{y^2}f^*(x,y)g(x,y)\;.$$

In order to construct the path integral on U, we follow the canonical approach as described in our previous

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paper [7]. We want to express the hamiltonian (2) by hermitian momenta $p_a=-\mathrm{i}(\partial_a+\Gamma_a/2)$ $(a=x,y)$, where $\Gamma_a=\partial_a(\ln\sqrt{g})$ and g denotes the determinant of the metric tensor. The quantum correction ΔV to the classical lagrangian $\mathscr{L}_{\rm cl}$ follows then easily from the prescription given in ref. [7]. We have

$$\sqrt{g}=\frac{1}{y^2}\,,\quad \Gamma_x=0\,,\quad \Gamma_y=-\frac{2}{y}\,,\quad p_x=\frac{1}{\mathrm{i}}\frac{\partial}{\partial x}\,,\quad p_y=\frac{1}{\mathrm{i}}\left(\frac{\partial}{\partial y}-\frac{1}{y}\right),$$

$$\Delta_{\rm LB}\equiv\frac{1}{\sqrt{g}}\,\partial_a g^{ab}\sqrt{g}\,\partial_b=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),\qquad \Delta V\equiv\frac{1}{8m}\,[g^{ab}\Gamma_a\Gamma_b+2\partial_a(g^{ab}\Gamma_b)+g^{ab}{}_{,ab}]=\frac{1}{4m}\,, \tag{3}$$

and the hamiltonian (2) reads

$$H(x,p_x,y,p_y)=\frac{1}{8m}\,(g^{ab}p_ap_b+2p_ag^{ab}p_b+p_ap_bg^{ab})+\frac{1}{4m}\,. \tag{4}$$

Notice that the hamiltonian (4) is Weyl-ordered which implies the midpoint prescription (i.e. $\bar z_{(j)}=\frac{1}{2}(z_{(j)}+z_{(j-1)})$, all j) in the lattice definition of the path integral [8] [#1].

Now we can write down the hamiltonian path integral ($x(t')=x'$, $x(t'')=x''$, $y(t')=y'$, $y(t'')=t''$, $T=t''-t'$)

$$K(x'',y'',x',y';T)=C(g',g'')\int \mathrm{D}x(t)\mathrm{D}y(t)\mathrm{D}p_x(t)\mathrm{D}p_y(t)\,\exp\left(\mathrm{i}\int_{t'}^{t''}(p_x\dot x+p_y\dot y-\mathscr{H})\,\mathrm{d}t\right), \tag{5}$$

with

$$\mathrm{D}x(t)\mathrm{D}y(t)\mathrm{D}p_x(t)\mathrm{D}p_y(t)\to\prod_{j=1}^{N-1}\mathrm{d}x_{(j)}\mathrm{d}y_{(j)}\times\prod_{j=1}^{N}(2\pi)^{-2}\mathrm{d}p_{x(j)}\mathrm{d}p_{y(j)}\quad(N\to\infty)\,,$$

where $\mathscr{H}$ coincides with the classical version of the hamiltonian (4). Here C denotes the normalization (see also ref. [8])

$$C(g',g'')=(g'g'')^{-1/4}=y'y''\,, \tag{6}$$

where g' and g'' are the determinants of the metric tensor at initial and final points, respectively. Performing the momentum integrations we get ($\epsilon=T/N$):

$$K(x'',y'',x',y';T)=\int\frac{\mathrm{D}x(t)\mathrm{D}y(t)}{y^2}\exp\left(\frac{\mathrm{i}m}{2}\int_{t'}^{t''}\frac{1}{y^2}\,(\dot x^2+\dot y^2)\,\mathrm{d}t\right)$$

$$=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon}\right)^N\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\mathrm{d}x_{(1)}\mathrm{d}y_{(1)}}{y_{(1)}^2}\cdots\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\mathrm{d}x_{(N-1)}\mathrm{d}y_{(N-1)}}{y_{(N-1)}^2}\exp\left[\frac{\mathrm{i}m}{2\epsilon}\sum_{j=1}^{N}\frac{(x_{(j)}-x_{(j-1)})^2+(y_{(j)}-y_{(j-1)})^2}{y_{(j)}y_{(j-1)}}\right]. \tag{7}$$

In eq. (7) we replaced the midpoint expression $\bar y_{(j)}^2$ by $y_{(j)}y_{(j-1)}$, which yields additional terms of $\mathrm{O}(\epsilon)$, but which cancels exactly ΔV of eq. (3). Eq. (7) is the correct path integral on U. This can be verified by deriving the Schrödinger equation from the short time kernel of (7), see the appendix.

In order to make the path integral manageable we perform a time-transformation (see ref. [7]):

$$s(t)\equiv\int_{t'}^{t}\frac{1}{f(y(\sigma))}\,\mathrm{d}\sigma\,,\quad s''=s(t'')\,,\quad s(t')=0\,, \tag{8}$$

[#1] We wish to thank N.K. Falck for drawing our attention to ref. [8].

with $f(y)=1/y^2$. The variables x and y are transformed into

$$x(t)\to\xi(s) \quad \text{with} \quad \xi(s(t))=x(t)\,, \qquad y(t)\to\eta(s) \quad \text{with} \quad \eta(s(t))=y(t)\,, \tag{9}$$

with $\xi(0)=x'$, $\xi(s'')=x''$, $\eta(0)=y'$ and $\eta(s'')=y''$. Let us assume that the constraint

$$\int_0^{s''}\frac{\mathrm{d}s}{\eta^2(s)}=T \tag{10}$$

has for all admissible paths a unique solution $s''>0$. Of course, since T is fixed, the "time" s'' will be path-dependent. To incorporate the constraint (10) we use the identity

$$1=\frac{1}{y''^2}\int_0^\infty \mathrm{d}s''\,\delta\left(\int_0^{s''}\frac{\mathrm{d}s}{\eta^2(s)}-T\right)=\frac{1}{y''^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\,\mathrm{e}^{-\mathrm{i}TE}\int_0^\infty \mathrm{d}s''\,\exp\left(\mathrm{i}\int_0^{s''}\mathrm{d}s\,\frac{E}{\eta^2(s)}\right) \tag{11}$$

in the path integral (7). The only difference to the prescription given in ref. [7] is that we have now only a time- and not a space–time-transformation. This has the consequence that the additional factor in equation (IV.6) of ref. [7] is absent in the present case. Defining the energy-dependent Feynman kernel $G(E)$ via the Fourier transformation

$$K(x'',y'',x',y';T)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}TE}G(x'',y'',x',y';E)\,\mathrm{d}E\,, \tag{12}$$

we obtain the transformation formula

$$G(x'',y'',x',y';E)=\mathrm{i}\int_0^\infty \tilde{K}(\xi'',\eta'',\xi',\eta';s'')\,\mathrm{d}s''\,, \tag{13}$$

where the transformed path integral is given by

$$\begin{aligned}\tilde{K}(\xi'',\eta'',\xi',\eta';s'')&=\int \mathrm{D}\xi(s)\mu_\lambda[\eta]\mathrm{D}\eta(s)\,\exp\left(\frac{\mathrm{i}m}{2}\int_0^{s''}(\dot{\xi}^2+\dot{\eta}^2)\,\mathrm{d}s\right)\\ &=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\delta}\right)^N\int_{-\infty}^{\infty}\int_0^{\infty}\mathrm{d}\xi_{(1)}\mathrm{d}\eta_{(1)}\dots\int_{-\infty}^{\infty}\int_0^{\infty}\mathrm{d}\xi_{(N-1)}\mathrm{d}\eta_{(N-1)}\\ &\quad\times\mu_\lambda[\eta_{(j)}]\,\exp\left(\frac{\mathrm{i}m}{2\delta}\sum_{j=1}^{N}[(\xi_{(j)}-\xi_{(j-1)})^2+(\eta_{(j)}-\eta_{(j-1)})^2]\right)\,,\end{aligned} \tag{14}$$

with $\delta=s''/N$ and $\lambda=\sqrt{1/4-2mE}$. The functional measure is given by

$$\mu_\lambda[\eta]\to\prod_{j=1}^{N}\left[\sqrt{\frac{2\pi m}{\mathrm{i}\delta}\eta_{(j)}\eta_{(j-1)}}\,\exp\left(-\frac{m}{\mathrm{i}\delta}\eta_{(j)}\eta_{(j-1)}\right)I_\lambda\left(\frac{m}{\mathrm{i}\delta}\eta_{(j)}\eta_{(j-1)}\right)\right]\,. \tag{15}$$

I_λ denotes a modified Bessel function. Following our general theory [7], we have used

$$g_{ab}=\delta_{ab}\,,\quad \sqrt{g}=1\,,\quad \Gamma_\xi=0\,,\quad \Gamma_\eta=0\,,\quad \Delta V=0\,. \tag{16}$$

The path integral in (14) factorises into a path integral for a free particle in $\xi\in\mathbb{R}$, and into a radial path integral with "angular momentum" λ in the variable $\eta\in\mathbb{R}^+$. Using the well-known path integral identity

$$\int \mu_\lambda[r]\mathrm{D}r(t)\,\exp\left(\frac{im}{2}\int_{t'}^{t''}\dot{r}^2\,\mathrm{d}t\right)=\sqrt{r'r''}\,\frac{m}{\mathrm{i}T}\exp\left(\frac{im}{2T}(r'^2+r''^2)\right)I_\lambda\left(\frac{m}{\mathrm{i}T}r'r''\right) \tag{17}$$

(see ref. [9]) we can immediately write down the solution of (14):

$$\tilde{K}(\xi'',\eta'',\xi',\eta';s'')=\sqrt{\frac{\eta'\eta''}{2\pi}}\left(\frac{m}{\mathrm{i}s''}\right)^{3/2}\exp\left(-\frac{m}{2\mathrm{i}s''}[(\xi''-\xi')^2+\eta''^2+\eta'^2]\right)I_\lambda\left(\frac{m}{\mathrm{i}s''}\eta'\eta''\right). \tag{18}$$

Inserting (18) into eq. (13), the s''-integration can be carried out by first performing a Feynman–Wick rotation ($s''\to-\mathrm{i}\tau$, $\tau\in\mathbb{R}^+$), and then introducing the integration variable $z=my'y''/\tau$ and the Poincaré distance $\cosh d(z'',z')\equiv[(x''-x')^2+y'^2+y''^2]/2y'y''$. we then obtain (see p. 712 of ref. [10]):

$$G(x'',y'',x',y';E)=\frac{m}{\sqrt{2\pi}}\int_0^\infty \exp(-z\cosh d)I_{\mathrm{i}p}(z)\frac{\mathrm{d}z}{\sqrt{z}}=\frac{m}{\pi}\,\mathcal{Q}_{-1/2+\mathrm{i}p}(\cosh d)\,, \tag{19}$$

where we have introduced the momentum $p\equiv\sqrt{2mE-1/4}$. Eq. (19) gives a closed expression for the energy-dependent Green function (resolvent kernel) in terms of the Legendre function of the second kine $\mathcal{Q}_\nu$ [#2]. This result agrees with the one obtained by solving directly the Schrödinger equation (see e.g. ref. [2]). Using the integrals (see ref. [10] pp. 819, 732):

$$\mathcal{Q}_{\nu-1/2}\left(\frac{a^2+b^2+c^2}{2ab}\right)=\int_0^\infty \mathrm{d}p'\,\frac{p'\tanh\pi p'}{\nu^2+p'^2}\,\mathcal{P}_{\mathrm{i}p'-1/2}\left(\frac{a^2+b^2+c^2}{2ab}\right),$$

$$\mathcal{P}_{\nu-1/2}\left(\frac{a^2+b^2+c^2}{2ab}\right)=\frac{4\sqrt{ab}}{\pi^2}\cos\nu\pi\int_0^\infty \mathrm{d}k\,K_\nu(ak)K_\nu(bk)\cos ck\,, \tag{20}$$

eq. (19) can be rewritten as

$$G(x'',y'',x',y';E)=\frac{1}{\pi^3}\int_{-\infty}^{\infty}\mathrm{d}k\int_0^\infty \mathrm{d}p'\,\frac{p'\sinh\pi p'}{(p'^2+\frac14)/2m-E}\sqrt{y'y''}K_{\mathrm{i}p'}(|k|y')K_{\mathrm{i}p'}(|k|y'')\exp[\mathrm{i}k(x''-x')] \tag{21}$$

(K_ν denotes a modified Bessel function). The representation (21) shows clearly that $G(E)$ has a cut on the positive real axis in the complex energy plane with a branch point at $E=1/8m$. We thus infer that the quantum mechanical motion on the Poincaré upper half plane U has a continuous energy spectrum. From (21) we immediately read off the normalized wavefunctions

$$\psi_{p,k}(x,y)=\sqrt{\frac{p\sinh\pi p}{\pi^3}}\,\mathrm{e}^{\mathrm{i}kx}\sqrt{y}K_{\mathrm{i}p}(|k|y)\quad(x\in\mathbb{R},\,y>0)\,;\qquad E_p=\frac{1}{2m}(p^2+\tfrac14) \tag{22}$$

with $p>0$ and $k\in\mathbb{R}\setminus\{0\}$. these are the correct wavefunctions (see ref. [11]). The spectrum has a largest lower bound $E_0=1/8m$. A state with $p=0$ and $E_0=1/8m$ does not exist, because $\psi_{0,k}$ vanishes identically. One also has to exclude the case $k=0$, which is obvious from the asymptotic behaviour of the K_ν function for $z\to0$: $K_{\mathrm{i}p}(z)\to\frac12[\Gamma(\mathrm{i}p)(2/z)^{\mathrm{i}p}+\Gamma(-\mathrm{i}p)(2/z)^{-\mathrm{i}p}]$. It is nevertheless possible to define a function $\phi_p(y):=y^{\mathrm{i}p+1/2}$ which is an eigenstate of H, $H\phi_p=E_p\phi_p$, but this function is not normalizable in U. ϕ_p is only normalizable in a bounded domain.

[#2] We use $\mathcal{P}^\mu_\nu(z)$, $\mathcal{Q}^\mu_\nu(z)$ for $z\in\mathbb{C}\setminus[-1,1]$ and $P^\mu_\nu(x)$, $Q^\mu_\nu(x)$ for $x\in(-1,1)$ for the Legendre functions of the first and second kind, respectively.

As already mentioned in the introduction, U is equivalent to the pseudospere Λ^2 where the wavefunctions are given by ($p>0$, $k\in\mathbb{Z}\setminus 0$):

$$\psi_{p,k}(\tau,\phi)=\sqrt{\frac{p\sinh\pi p}{2\pi}}\,\Gamma(\mathrm{i}p+k+\tfrac{1}{2})\mathscr{P}^{-k}_{\mathrm{i}p-1/2}(\cosh\tau)\mathrm{e}^{\mathrm{i}k\phi} \tag{23}$$

in terms of the pseudospherical polar coordinates $\tau\geqslant 0$ and $\phi\in[0,2\pi]$ with $x=y\sinh\tau\cos\phi$, $y=(\cosh\tau+\sinh\tau\sin\phi)^{-1}$ [5,6,12,13]. The corresponding path integral expressions on U and Λ^2 can be transformed into each other as will be shown in our forthcoming paper [14].

Finally, we perform a Fourier transformation in (21) to get the time-dependent Feynman kernel

$$K(x'',y'',x',y';T)=\frac{1}{\pi^3}\int_{-\infty}^{\infty}\mathrm{d}k\int_0^{\infty}\mathrm{d}p\,p\sinh\pi p\exp[-\mathrm{i}T(p^2+\tfrac{1}{4})/2m]$$

$$\times\sqrt{y'y''}K_{\mathrm{i}p}(|k|y')K_{\mathrm{i}p}(|k|y'')\exp[\mathrm{i}k(x''-x')]\,. \tag{24}$$

The $\psi_{p,k}$ form an orthonormal basis,

$$N\equiv\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y^2}\psi^*_{p,k}(x,y)\psi_{p',k'}(x,y)=\delta(k-k')\delta(p-p')\,. \tag{25}$$

Proof. Inserting $\psi_{p,k}$ from eq. (22) and performing the x-integration yields

$$N=\delta(k-k')\,\frac{2\sqrt{pp'\sinh\pi p\sinh p'}}{\pi^2}\int_0^{\infty}\frac{1}{y}K_{\mathrm{i}p}(y)K_{\mathrm{i}p'}(y)\,\mathrm{d}y\,. \tag{26}$$

We now use the integral (ref. [10] p. 693):

$$\int_0^{\infty}y^{-\lambda}K_\mu(ay)K_\nu(by)\,\mathrm{d}y=\frac{a^{\lambda-\nu-1}b^\nu}{2^{2+\lambda}\Gamma(1-\lambda)}$$

$$\times\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)$$

$$\times F\left(\frac{1-\lambda+\mu+\nu}{2},\frac{1-\lambda-\mu+\nu}{2};1-\lambda;1-\frac{b^2}{a^2}\right). \tag{27}$$

Let $a=b=1$, $\lambda=1-2\epsilon$, $\mu=\mathrm{i}p$ and $\nu=\mathrm{i}p+2\mathrm{i}q$, $q=(p'-p)/2$, then

$$\int_0^{\infty}y^{2\epsilon-1}K_{\mathrm{i}p}(y)K_{\mathrm{i}p+2\mathrm{i}q}(y)\,\mathrm{d}y=\frac{\Gamma(\epsilon+\mathrm{i}p+\mathrm{i}q)\Gamma(\epsilon+\mathrm{i}q)\Gamma(\epsilon-\mathrm{i}q)\Gamma(\epsilon-\mathrm{i}p-\mathrm{i}q)}{\Gamma(2\epsilon)2^{3-2\epsilon}}\,. \tag{28}$$

The "good" terms yield in the limit $\epsilon\to 0$:

$$\lim_{\epsilon\to 0}\frac{\Gamma(\epsilon+\mathrm{i}p+\mathrm{i}q)\Gamma(\epsilon-\mathrm{i}p-\mathrm{i}q)}{2^{3-2\epsilon}}=\tfrac{1}{8}|\Gamma(\mathrm{i}p+\mathrm{i}q)|^2=\frac{\pi}{8(p+q)\sinh[\pi(p+q)]}\,, \tag{29}$$

where we have used a well-known property of the Γ-function. The remaining terms yield

$$\lim_{\epsilon\to 0}\frac{\Gamma(\epsilon+\mathrm{i}q)\Gamma(\epsilon-\mathrm{i}q)}{\Gamma(2\epsilon)}=2\pi\lim_{\epsilon\to 0}\frac{\epsilon}{\pi(\epsilon^2+q^2)}=4\pi\delta(p'-p)\,, \tag{30}$$

and eq. (25) is proved.

Vice versa, the $\psi_{p,k}$ form a complete set, i.e.

$$\int_{-\infty}^{\infty} dk \int_{0}^{\infty} dp\, \psi_{p,k}(x'', y'')\psi^{*}_{p,k}(x', y') = y'y''\delta(x''-x')\delta(y''-y') \tag{31}$$

(the factor $C=y'y''=(g'g'')^{-1/4}$ has to be included, see eq. (5), due to the riemannian structure of U).

Proof. Consider the integral ([10] p. 772):

$$\int_{0}^{\infty} dx\, K_{ix}(a)K_{ix}(b) \cosh[(\pi-\phi)x] = K_0(\sqrt{a^2+b^2-2ab\cos\phi}) . \tag{32}$$

Differentiation with respect to ϕ gives on the left-hand side

$$-\frac{\partial}{\partial\phi}\int_{0}^{\infty} dx\, K_{ix}(a)K_{ix}(b)\cosh[(\pi-\phi)x] = \int_{0}^{\infty} dx\, x \sinh[(\pi-\phi)x]K_{ix}(a)K_{ix}(b) , \tag{33}$$

while the right-hand side yields

$$-\frac{\partial}{\partial\phi}K_0(\sqrt{a^2+b^2-2ab\cos\phi}) = \frac{ab\sin\phi}{\sqrt{a^2+b^2-2ab\cos\phi}}K_1(\sqrt{a^2+b^2-2ab\cos\phi}) . \tag{34}$$

Here we have used some properties of the K_ν-function (see e.g. ref. [10], p. 510). Therefore we have in the limit $\phi\to 0$ and for $y'\neq y''$:

$$\int_{0}^{\infty} dp\, p \sinh \pi p\, K_{ip}(|k|y')K_{ip}(|k|y'') = 0 . \tag{35}$$

It remains to consider the case $y'\simeq y''$. Let us set $y=y'$, $y''=y+\delta$ with $|\delta|\ll 1$ and $\cos\phi\simeq 1-\phi^2/2$ for $|\phi|\ll 1$. Using $K_0\simeq -\ln(z/2)$ $(z\to 0)$ we get for the right-hand side of eq. (31):

$$\tfrac{1}{2}\pi K_0(|k|\sqrt{y'^2+y''^2-2y'y''\cos\phi}) \simeq \tfrac{1}{2}\pi[\ln\tfrac{1}{2}|k| + \tfrac{1}{2}\ln(\delta^2+y^2\phi^2)] \quad (|\delta|, |\phi|\ll 1) , \tag{36}$$

from which we get in the limit $\phi\to 0$:

$$\int_{0}^{\infty} dp\, p \sinh \pi p\, K_{ip}(|k|y')K_{ip}(|k|y'') = \tfrac{1}{2}\pi^2\sqrt{y'y''}\,\delta(y'-y'') . \tag{37}$$

Together with the well-known equation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \exp[ik(x''-x')] = \delta(x''-x')$$

the completeness relation (31) is proven

The same technique as for the path integral on the Poincaré upper half plane is also applicable to Liouville quantum mechanics. Let us consider the hamiltonian of Liouville quantum mechanics ($x\in\mathbb{R}$) [5,6,13]:

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{V_0^2}{2m}e^{2x} . \tag{38}$$

The path integral reads ($T=t''-t'$):

$$K(x'',x';T)=\int \mathrm{D}x(t)\,\exp\left(\mathrm{i}\int_{t'}^{t''}[\tfrac{1}{2}m\dot{x}^2-(V_0^2/2m)\mathrm{e}^{2x}]\,\mathrm{d}t\right). \tag{39}$$

In order to make the path integral manageable, we perform a space–time transformation. Following the general theory of section IV in ref. [7] we have to start with the Legendre transformed hamiltonian H_E:

$$H_E=-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{V_0^2}{2m}\mathrm{e}^{2x}-E. \tag{40}$$

We consider the transformation $q=\mathrm{e}^x$ and get a transformed hamiltonian $\hat{H}_E(\mathrm{d}/\mathrm{d}q,\,q)$. With $\tilde{H}(\mathrm{d}/\mathrm{d}q,\,q)\equiv(1/q^2)\hat{H}_E(\mathrm{d}/\mathrm{d}q,\,q)$ we obtain

$$\tilde{H}(\mathrm{d}/\mathrm{d}q,\,q)=-\frac{1}{2m}\left(\frac{\mathrm{d}^2}{\mathrm{d}q^2}+\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}q}\right)+\frac{V_0^2}{2m}-\frac{E}{q^2}. \tag{41}$$

The relevant expressions for calculating the quantum correction ΔV are

$$\Gamma=\frac{1}{q},\quad p_q=\frac{1}{\mathrm{i}}\left(\frac{\mathrm{d}}{\mathrm{d}q}+\frac{1}{2q}\right),\quad \Delta V(q)=-\frac{1}{8mq^2}. \tag{42}$$

Thus we arrive at the effective hamiltonian

$$H_{\mathrm{eff}}(p_q,q)=\frac{1}{2m}p_q^2+\frac{V_0^2}{2m}-\frac{2mE+\frac{1}{4}}{2mq^2}. \tag{43}$$

Notice that in this case one has a non-vanishing quantum correction. The path integral for Liouville quantum mechanics can now be calculated via the equations

$$K(x'',x';T)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}TE}G(x'',x';E)\,\mathrm{d}E, \tag{44}$$

where

$$G(x'',x';E)=\frac{\mathrm{i}}{\sqrt{q'q''}}\int_0^{\infty}\tilde{K}(q'',q';s'')\,\mathrm{d}s'', \tag{45}$$

and

$$\tilde{K}(q'',q';s'')=\exp(-\mathrm{i}s''V_0^2/2m)\int \mathrm{D}q(s)\mu_{\mathrm{i}\sqrt{2mE}}[q]\,\exp\left(\frac{\mathrm{i}m}{2}\int_0^{s''}\dot{q}^2\,\mathrm{d}s\right). \tag{46}$$

The functional measure is given by (15). With eq. (17) for radial path integrals we can write down the solution of (46) immediately, yielding

$$\tilde{K}(q'',q';s'')=\frac{m}{\mathrm{i}s''}\sqrt{q'q''}\,\exp\left(\frac{\mathrm{i}m}{2s''}(q'^2+q''^2)-\mathrm{i}\frac{V_0^2s''}{2m}\right)I_{\mathrm{i}\sqrt{2mE}}\left(\frac{m}{\mathrm{i}s''}q'q''\right). \tag{47}$$

For $G(E)$ we get

$$G(x'',x';E)=2mI_{\mathrm{i}\sqrt{2mE}}(V_0\mathrm{e}^{x'})K_{\mathrm{i}\sqrt{2mE}}(V_0\mathrm{e}^{x''}), \tag{48}$$

where we have used the integral (ref. [10] p. 719):

$$\int_0^\infty \exp(-a/x-bx)J_\nu(cx)\,\frac{dx}{x}=2J_\nu[\sqrt{2a(\sqrt{b^2+c^2}-b)}]K_\nu[\sqrt{2a(\sqrt{b^2+c^2}+b)}]\,. \tag{49}$$

We have assumed without loss of generality that $x''>x'$. Otherwise one has to interchange x'' and x'. We now use the integrals (20a) and (see ref. [15] p. 194):

$$I_\nu(ax)K_\nu(bx)=\frac{1}{\pi\sqrt{ab}}\int_0^\infty dt\,\mathcal{Q}_{\nu-1/2}\left(\frac{a^2+b^2+t^2}{2ab}\right)\cos xt\,,$$

$$K_\nu(ax)K_\nu(bx)=\frac{\pi}{2\sqrt{ab}}\int_0^\infty dt\,\mathcal{P}_{\nu-1/2}\left(\frac{a^2+b^2+t^2}{2ab}\right)\cos xt\,, \tag{50}$$

to obtain

$$G(x'',x';E)=\frac{2}{\pi^2}\int_0^\infty dp\,\frac{p\sinh\pi p}{p^2/2m-E}\,K_{ip}(V_0e^{x'})K_{ip}(V_0e^{x''})\,. \tag{51}$$

The resolvent kernel (51) has a cut on the positive real axis in the complex E-plane, and we immediately can read off the wavefunctions and the energy spectrum:

$$\psi_p(x)=(1/\pi)\sqrt{2p\sinh\pi p}\,K_{ip}(V_0e^x)\quad(p>0,\,x\in\mathbb{R})\,;\qquad E_p=p^2/2m\quad(p>0)\,. \tag{52}$$

This is the correct result – see refs. [5,13]. From eqs. (25) and (31) we infer that the wavefunctions have the correct normalization and form a complete set. The Feynman kernel $K(T)$ is given by

$$K(x'',x';T)=\frac{2}{\pi^2}\int_0^\infty dp\,p\sinh\pi p\,\exp(-iTp^2/2m)K_{ip}(V_0e^{x'})K_{ip}(V_0e^{x''})\,. \tag{53}$$

In this letter we have presented a complete path integral treatment of free motion on the entire Poincaré upper half plane. The calculation was based on the canonical method for calculating the quantum correction ΔV to the classical lagrangian and a time transformation in the lagrangian path integral.

The canonical method also works for Liouville quantum mechanics, where the path integral could be calculated via a space–time transformation.

In a forthcoming paper [14] we shall present a path integral treatment for the pseudosphere Λ^2, the Poincaré disc D and the hyperbolic strip S. Of special importance is also the d-dimensional pseudosphere Λ^{d-1}, where again the canonical method works very well, yielding the energy spectrum

$$E_p^{(d)}=(1/2mR^2)[p^2+\tfrac{1}{4}(d-2)^2]\quad(p>0) \tag{54}$$

with largest lower bound

$$E_0^{(d)}=(d-2)^2/8mR^2\,. \tag{55}$$

A recent path integral formulation due to Böhm and Junker [12] for the d-dimensional pseudosphere gives unfortunately a wrong result, because these authors missed the quantum correction ΔV, which is crucial and which is caused by the curvilinear nature of Λ^{d-1}. In our forthcoming paper we shall also show that the "mysterious phase factors" in Gutzwiller's semiclassical calculation [2] arise very naturally.

These new examples in path integral techniques show very clearly the great advantage of the canonical method [7] over other approaches, giving in a simple way the correct quantum corrections and thereby the correct path integrals.

Appendix

We want to prove that with the short time kernel of eq. (7):

$$K(\zeta, z; \epsilon)=\frac{m}{2\pi i\epsilon}\exp\left(\frac{im}{2\epsilon}\frac{(\xi-x)^2+(\eta-y)^2}{y\eta}\right) \tag{A.1}$$

and the time evolution equation

$$\psi(\zeta, t+\epsilon)=\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y^2}K(\zeta, z; \epsilon)\psi(z, t) \tag{A.2}$$

the Schrödinger equation follows:

$$i\frac{\partial\psi(\zeta, t)}{\partial t}=-\frac{\eta^2}{2m}\left(\frac{\partial^2\psi(\zeta, t)}{\partial\xi^2}+\frac{\partial^2\psi(\zeta, t)}{\partial\eta^2}\right). \tag{A.3}$$

(We have used the abbreviations $z=z_{(j)}$, $\zeta=z_{(j+1)}$, with $z=x+iy$, $\zeta=\xi+i\eta$, $x=x_{(j)}$, $\xi=x_{(j+1)}$, $y=y_{(j)}$ and $\eta=y_{(j+1)}$.) One has to perform a Taylor expansion in (A.2). We get ($\zeta_1=\xi$, $\zeta_2=\eta$):

$$\psi(\zeta, t)+\epsilon\frac{\partial\psi(\zeta, t)}{\partial t}=\frac{m}{2\pi i\epsilon}\Bigg(\psi(\zeta, t)B_0+\sum_{j=1,2}\frac{\partial\psi(\zeta, t)}{\partial\zeta_j}(B_{\zeta_j}-\zeta_j B_0)$$

$$+\tfrac{1}{2}\sum_{\substack{i,j=1,2\\ i\geqslant j}}\frac{\partial^2\psi(\zeta, t)}{\partial\zeta_i\partial\zeta_j}(B_{\zeta_i\zeta_j}-\zeta_i B_{\zeta_j}-\zeta_j B_{\zeta_i}+\zeta_i\zeta_j B_0)\Bigg) \tag{A.4}$$

with

$$B_0=\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y^2}\exp[i\epsilon\mathscr{L}^N(\zeta, z)]=2\left(\frac{2\pi i\epsilon}{m}\right)^{1/2}e^{m/i\epsilon}K_{-1/2}(m/i\epsilon)=\frac{2\pi i\epsilon}{m},$$

$$B_\xi=\int_{-\infty}^{\infty}x\,\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y^2}\exp[i\epsilon\mathscr{L}^N(\zeta, z)]=\xi B_0,$$

$$B_\eta=\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y}\exp[i\epsilon\mathscr{L}^N(\zeta, z)]=2\eta\left(\frac{\pi i\epsilon}{m}\right)^{1/2}e^{m/i\epsilon}K_{1/2}(m/i\epsilon)=\eta B_0,$$

$$B_{\xi\eta}=\int_{-\infty}^{\infty}x\,\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y}\exp[i\epsilon\mathscr{L}^N(\zeta, z)]=\xi\eta B_0,$$

$$B_{\xi^2}=\int_{-\infty}^{\infty}x^2\,\mathrm{d}x\int_0^{\infty}\frac{\mathrm{d}y}{y^2}\exp[i\epsilon\mathscr{L}^N(\zeta, z)]=\left(\xi^2+\frac{i\epsilon}{m}\eta^2\right)B_0,$$

$$B_{\eta^2}=\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\exp[i\epsilon\mathscr{L}^N(\zeta, z)]=\eta^2\left(1+\frac{i\epsilon}{m}\right)B_0. \tag{A.5}$$

Here

$$\mathcal{L}^N(\zeta, z) = \frac{m}{2\epsilon^2} \frac{(\xi - x)^2 + (\eta - y)^2}{y\eta} \tag{A.6}$$

denotes the lagrangian on the lattice. We shall only calculate the integral B_0. The remaining integrals are similar. We get

$$B_0 = \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \frac{\mathrm{d}y}{y^2} \exp\left(\frac{\mathrm{i}m}{\epsilon} \frac{(\xi - x)^2 + (\eta - y)^2}{y\eta}\right)$$

$$= \left(\frac{2\pi \mathrm{i}\epsilon}{m}\right)^{1/2} \sqrt{\eta}\, \mathrm{e}^{m/\mathrm{i}\epsilon} \int_0^{\infty} y^{-3/2} \exp\left(-\frac{m}{2\mathrm{i}\epsilon\eta} y - \frac{m\eta}{2\mathrm{i}\epsilon} \frac{1}{y}\right) \mathrm{d}y$$

$$= 2\left(\frac{2\pi \mathrm{i}\epsilon}{m}\right)^{1/2} \mathrm{e}^{m/\mathrm{i}\epsilon} K_{-1/2}(m/\mathrm{i}\epsilon) = \frac{2\pi \mathrm{i}\epsilon}{m}. \tag{A.7}$$

In the last step we have used the integral ([10] p. 340):

$$\int_0^{\infty} x^{\nu - 1} \exp(-\beta/x - \gamma x)\, \mathrm{d}x = 2(\beta/\gamma)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}) \tag{A.8}$$

and the expression $K_{\pm 1/2}(z) = \sqrt{\pi/2z}\, \mathrm{e}^{-z}$. Inserting the expressions (A.5) into (A.4) yields the Schrödinger equation (A.3).

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