Nuclear Physics B292 (1987) 443–460
North-Holland, Amsterdam

QUARK, LEPTON AND NEUTRINO MASSES IN GRAND UNIFIED THEORIES WITH LOCAL GENERATION GROUP

J. BIJNENS and C. WETTERICH

Deutsches Elektronen-Synchrotron DESY, Hamburg, FR Germany

Received 4 February 1987

We investigate unified models where all small quantities in the fermion mass matrices are given in terms of one small ratio of symmetry breaking scales. We describe for a U(1) generation symmetry how the size of masses and mixings is determined, including possible contributions from heavy mirror quarks and leptons. This can be used for computerized model scanning. We search for realistic mass patterns in anomaly-free $SU(5) \times U(1)_G$ models and find several examples. Interesting patterns for neutrino masses can be obtained.

1. Introduction

It has been proposed recently [1] that the orders of magnitude of all fermion masses and mixings can be understood in terms of symmetry and one small parameter λ which is the ratio of a symmetry breaking mass scale M_G divided by the overall mass scale M of the model. The main assumption is that all Yukawa couplings are of the same order as the gauge coupling g (as suggested in higher dimensional models) so that all small quantities in the fermion mass matrices should be related to symmetry [2]. This approach requires a symmetry G larger than $SU(3) \times SU(2) \times U(1)$. In the limit of unbroken G only the top quark (or fermions of a fourth generation) should be allowed to couple to the low-energy weak Higgs doublet φ and acquire a mass from weak symmetry breaking. Breaking of G at the scale M_G induces mass terms for the other fermions suppressed [3–6] by powers of $\lambda = M_G/M$. The various powers of λ appearing in the mass matrices determine their structure. They can be calculated by group theoretical methods [7].

We also assume here that all possible fermion bilinears are coupled to scalar fields. (All these scalar modes are typically present in compactified higher dimensional models, except when either the scalar fields or their couplings to a fermion bilinear are forbidden for topological reasons [2, 8].) The generic mass of these scalars (doublets under $SU(2)_L \times U(1)_Y$) is of the order M*, cubic couplings are

* We generally assume a high scale, say $M \approx 10^{17}$ GeV. Nevertheless, all our discussion is valid for lower M as long as $M_W/M \ll (M_G/M)^4$ (up to an appropriate rescaling of light neutrino masses).

$\sim gM$ and quartic couplings have strength g^2. If all super-heavy particles have a mass of the same order M the calculation of the structure of mass matrices is greatly simplified. It is sufficient to determine the G transformation properties of a fermion bilinear corresponding to a given mass matrix element. These then determine the power P needed to construct an invariant of the type $\psi\psi'\varphi\chi^P$. Here φ is the doublet whose vacuum expectation value (vev) breaks $SU(2)_L \times U(1)$, χ is an $SU(3) \times SU(2) \times U(1)$ singlet whose vev breaks the symmetry G. The corresponding element in the mass matrix will then be of the order $\lambda^P g\varphi_L$ [7], with $\varphi_L = 174$ GeV the scale of weak symmetry breaking.

However, not all heavy particles have always mass M. Sometimes G symmetry requires some of the superheavy masses to be of the order $\lambda^{\bar{P}}M$ instead of M. Effects from the exchange of these particles are enhanced and one has to account for this in the analysis of the structure of fermion mass matrices*. The most important case are fermions [6] which are chiral with respect to G but vectorlike with respect to the low energy symmetry $SU(3) \times SU(2) \times U(1)$. These fermions are massless in the limit of G symmetry. Once G breaks to $SU(3) \times SU(2) \times U(1)$ their mass is not protected anymore and masses appear from direct or indirect couplings to χ. They are of the order $\lambda^{\bar{P}}M$ where $\bar{P}$ can again be calculated by group theoretical methods. A second class of particles with mass $M_G = \lambda M$ are the gauge bosons corresponding to $G/SU(3) \times SU(2) \times U(1)$ if G contains local symmetries beyond the standard model. Due to Lorentz symmetry they give no direct contribution to mass terms in the tree approximation and play only a role in loops. We neglect them here**. Finally, the dynamics of spontaneous symmetry breaking requires that all scalars which belong to the same G multiplet as φ or χ has at most mass $M_G = \lambda M$. In this paper we are mainly concerned with the case $G = SU(3) \times SU(2) \times U(1) \times U(1)_G$ where exchange of such scalars in intermediate channels does not play a role for fermion mass matrices. We assume that the masses of χ and φ are M_G and M_W whereas all other scalars have mass M***.

In this paper we discuss the role of heavy mirror partners of quarks and leptons for the structure of fermion mass matrices. Pairs of mirrors and ordinary quarks (leptons) will acquire a heavy mass and disappear from the low energy spectrum, but their contribution to the low energy fermion mass matrices may be important. Mirror quarks and leptons appear for many compactifications of higher dimensional theories (including superstrings). Sometimes they are required for a cancellation of anomalies with respect to G. We are also interested in heavy $SU(3) \times SU(2) \times U(1)$ singlet fermions (right-handed neutrinos). They play an important role [11] for the masses and mixings of the (left-handed) low energy neutrinos.

* In ref. [1] we only considered the case where all relevant heavy particles have mass $\sim M$.

** A typical loop suppression $\sim \alpha/\pi$ is smaller than a realistic value $\lambda \approx \frac{1}{20} - \frac{1}{10}$. For fermion masses due to radiative corrections see ref. [9].

*** This assumption was used in refs. [3–6] and became later known as extended survival hypothesis [10]. For an example where scalars with mass M_G play an important role see refs. [3, 5].

Our method first determines the orders of magnitude of the full mass matrices for all fermions with mass smaller than M. This includes quark-mirror pairs and right handed neutrinos which are chiral with respect to G and therefore acquire mass $\sim \lambda^{\bar{P}}M$, $\bar{P} > 0$, as well as all light fermions, chiral with respect to $SU(3) \times SU(2) \times U(1)$, which have mass of the order M_W or smaller. We separate the heavy from the light modes and discuss the remaining low energy mass matrices. This is in some respect superior to a graphical method* with intermediate heavy fermions [5,6] since particular features as vanishing or small determinants of mass matrices can be easier detected. The mass matrices contain singlet terms $\sim \lambda^{\bar{P}}M$ as well as doublet terms $\sim \lambda^{P}M_W$ (or triplet terms $\sim \lambda^{\bar{P}}M_W^2/M$ for neutrinos). We give a simple algorithm how orders of magnitude of mass eigenvalues and mixings can be determined for such matrices. It can be implemented on a computer.

We concentrate on the case of an abelian local generation group, $G = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_G$. The structure of the mass matrices only depends on the $U(1)_G$ charges of the fermions and the scalars χ and φ. In sect. 2 we give the mass matrices in terms of the small parameter $\lambda = M_G/M$. In sect. 3 we show an example how realistic fermion masses and mixings can be obtained for a set of fermion charges without anomalies. We describe the general algorithm for finding masses and mixings for quarks and charged leptons in presence of mirror fermions.

In sect. 4 we turn to the neutrino sector. Light neutrino masses are of the order $\lambda^P M_W^2/M$. If there are no right-handed neutrinos ν^c the power P is necessarily positive (or zero) and neutrino masses come out small. (For a typical scale $M \approx 10^{17}$ GeV neutrino masses would be of the order 10^{-4} eV or smaller.) However, the power P can be negative due to the exchange of intermediate ν^c with mass smaller than M. We found examples with neutrino masses as high as ≈ 10 eV (although $M \approx 10^{17}$ GeV!) or examples where masses and mixings could account for solar neutrino oscillations. In general, the generation pattern observed for quarks and charged leptons is *not* repeated in the neutrino sector. The structure of neutrino mass matrices is given by the $U(1)_G$ charges of triplet, doublet and singlet operators in a way quite different than for quarks and charged leptons. (There are examples where the heaviest neutrino is the electron neutrino.) Neutrino mass patterns depend critically on the charges of right-handed neutrinos. For an anomaly free $U(1)_G$ symmetry the ν^c charges are related to quark and lepton charges by anomaly cancellation. Unfortunately this constraint is not strong enough to fix the ν^c charges completely.

In the last section we investigate conditions for a mass structure from local $U(1)_G$ generation symmetry compatible with (four-dimensional) grand unification. We perform a computerized scan for anomaly free models with $SU(5) \times U(1)_G$ symmetry and arbitrary charges for the fermion multiplets (within a certain range). We

* In principle, the graphical method is equivalent. For all our discussion we work in the tree approximation.

find several possible choices for the charges leading to realistic mass patterns where all small quantities are explained in terms of λ. No such solutions are found for models based on SO(10) $\times$ $U(1)_G$ or $E_6 \times U(1)_G$. Although our investigation should be extended to generation symmetries different from $U(1)_G$, we think that it will be rather difficult to obtain realistic mass matrices in terms of only one small parameter from a generation symmetry commuting with SO(10) or E_6. This suggests that possible unifications based on gauge groups containing SO(10) as a subgroup may be more attractive in higher dimensions, with a nontrivial breaking of SO(10) [2, 12] in the course of compactification.

2. The structure of mass matrices

We aim for a general discussion of fermion mass matrices in theories with a U(1) generation group broken at M_G somewhat below the characteristic scale M. Let us assume that the theory contains $n+m$ quarks (charged leptons) ψ_i and m mirror quarks (leptons) $\overline{\psi}_k$ of a given type $(\mathrm{q}, \mathrm{u^c}, \mathrm{d^c}, \ell, \mathrm{e^c})$. The "generation" charges of the quarks are Q_i and for the mirrors $\overline{Q}_k$. We only consider particles chiral with respect to $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_G$ and eliminate all pairs with $Q_i + \overline{Q}_k = 0$. (Different types of quarks and leptons may have different charges $Q_i, \overline{Q}_k$.) Quarks and mirrors have Yukawa couplings $\psi_i \overline{\psi}_k \chi_q$ (which we assume to be of the order of the gauge coupling g) to $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet scalars χ_q with charge $q = -(Q_i + \overline{Q}_k)$ $(\chi_q^* \equiv \chi_{-q})$. Once $U(1)_G$ is spontaneously broken, the χ_q acquire vacuum expectation values (vev's) and induce a mass matrix coupling quarks and mirrors

$$(M_M)_{ik} \approx g\langle\chi_q\rangle, \qquad q = -(Q_i + \overline{Q}_k). \tag{1}$$

Assume that $U(1)_G$ is broken by χ_1 at a scale M_G

$$\langle\chi_1\rangle \approx \frac{M_G}{g} \approx \lambda \frac{M}{g}. \tag{2}$$

The ratio $M_G/M = \lambda$ is the only natural small parameter appearing in the fermion mass matrices. We take $\lambda \approx \frac{1}{20} - \frac{1}{10}$. Interactions between the different χ_q induce non-leading vev's of the order [3–5]

$$\langle\chi_q\rangle \approx \lambda^{|q|} \frac{M}{g}. \tag{3}$$

Since the vev of any operator with charge q must be proportional to $\langle\chi_1\rangle^q$, one obtains for the quark-mirror mass matrix M_M

$$(M_M)_{ik} \approx \lambda^{P_{ik}} M,$$
$$P_{ik} = |Q_i + \overline{Q}_k|. \tag{4}$$

These mass terms will eliminate m quarks and mirrors from the spectrum of light particles and leave only n generations of quarks chiral with respect to $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Similarly, $SU(2)_L$ doublets φ_q will give contributions to the mass matrix for the light quarks. Assume that φ_{q_0} acquires a vev of the order $\varphi_L = 174$ GeV. By effects of $U(1)_G$ symmetry breaking a doublet φ_q will acquire an induced vev of the order [1] (note $\varphi_{-q} \neq \varphi_q^*$!)

$$\langle \varphi_q \rangle \approx \lambda^{|q-q_0|} \varphi_L . \tag{5}$$

Consider the up-quark matrix for u and u^c: Yukawa couplings $\sim g u_i^c u_j \varphi_q$, $q = -Q_i^{(u^c)} - Q_j^{(u)}$ * give a mass contribution

$$\left(M_F^{u^c u}\right)_{ij} \approx \lambda^{P_{ij}} g \varphi_L \approx \lambda^{P_{ij}} M_W ,$$

$$P_{ij} = |Q_i^{(u^c)} + Q_j^{(u)} + q_0| . \tag{6}$$

The total mass matrix including mirrors is

$$\mathscr{M}_U = \begin{array}{c|cc} & u & \bar{\bar{u}}^c \\ \hline u^c & M_F^{u^c u} & M_M^{u^c} \\ \bar{\bar{u}} & \tilde{M}_M^u & 0 \end{array} . \tag{7}$$

Here $\tilde{}$ denotes transposition and we have put the irrelevant $\bar{\bar{u}}^c \bar{\bar{u}}$ mass term to zero**. All orders of magnitude are determined by the calculable powers of λ, i.e. the P_{ij} in eqs. (4) and (6). For down-type quarks and charged leptons eq. (6) is replaced by $P_{ij} = |Q_i + Q_j - q_0|$ [1].

Neutrino mass matrices involve left-handed neutrinos ν, mirror neutrinos $\bar{\nu}$ and possible $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlets ν^c (which we may call right-handed neutrinos – or more exactly their antiparticles). The $U(1)_G$ charges are Q, $\bar{Q}$ and Q^c respectively. Up to irrelevant terms (which we put to zero) the mass matrix in the neutrino sector is

$$\mathscr{M}_\nu = \tilde{\mathscr{M}}_\nu = \begin{array}{c|ccc} & \nu & \bar{\bar{\nu}} & \nu^c \\ \hline \nu & M_T & M_M & M_D \\ \bar{\bar{\nu}} & \tilde{M}_M & 0 & 0 \\ \nu^c & \tilde{M}_D & 0 & M_R \end{array} . \tag{8}$$

* This is equivalent to sect. 1. The field φ_q could also be considered as composite $\varphi_q \sim \varphi_{q_0} \chi^{(q-q_0)}$.

** These terms only correct heavy masses by contributions of the order M_W.

The matrix M_{T} comes from Yukawa couplings to $\mathrm{SU(2)_L}$ triplets and is of the order [3,5]

$$(M_{\mathrm{T}})_{ij} \approx \lambda^{P_{ij}} \frac{M_{\mathrm{W}}^2}{M},$$
$$P_{ij} = |Q_i + Q_j + 2q_0|, \tag{9}$$

whereas the "Dirac mass" M_{D} is due to the doublets φ_q:

$$(M_{\mathrm{D}})_{im} \approx \lambda^{P_{im}} M_{\mathrm{W}},$$
$$P_{im} = |Q_i + Q_m^{\mathrm{c}} + q_0|. \tag{10}$$

The singlets χ_q couple both to M_{M} (see eq. (4)) and M_{R}*

$$(M_{\mathrm{R}})_{mn} \approx \lambda^{P_{mn}} M,$$
$$P_{mn} = |Q_m^{\mathrm{c}} + Q_n^{\mathrm{c}}|. \tag{11}$$

Again, orders of magnitude in $\mathscr{M}_\nu$ only depend on the charges Q, $\overline{Q}$ and Q^{c}.

3. Masses and mixings for the light quarks and leptons

We want to know which light quarks and leptons are left, what are their mass eigenvalues and flavour mixing angles. Before discussing the matrices $\mathscr{M}_{\mathrm{U}}$, $\mathscr{M}_{\mathrm{D}}$, $\mathscr{M}_{\mathrm{L}}$ and $\mathscr{M}_\nu$ more systematically we may understand the structure of the problem by studying an example. Consider an anomaly free set of $\mathrm{U(1)_G}$ quantum numbers, consistent with SU(5) without additional singlets (ν^{c}):

$$Q(\mathrm{u,d}) = Q(\mathrm{u^c}) = Q(\mathrm{e^c}) = (-1,0,1,1,4),$$
$$Q(\nu,\mathrm{e}) = Q(\mathrm{d^c}) = (-4,-4,2,2,3,3),$$
$$Q(\bar{\mathrm{u}},\bar{\mathrm{d}}) = Q(\bar{\mathrm{u}}^{\mathrm{c}}) = Q(\bar{\mathrm{e}}^{\mathrm{c}}) = (-3,-2),$$
$$Q(\bar{\nu},\bar{\mathrm{e}}) = Q(\bar{\mathrm{d}}^{\mathrm{c}}) = (-1,-1,0). \tag{12}$$

The model has three 5 and two $\overline{10}$ mirror representations of SU(5). (Of course one could add an arbitrary number of nonchiral quark-mirror pairs.) We assume that the leading part of the low energy doublet has $q_0 = -2$ and that $\lambda = M_{\mathrm{G}}/M$ is about $\frac{1}{20} - \frac{1}{10}$.

* If there is an additional $B-L$ symmetry broken at $M_{B-L} \ll M$ an additional factor $(M_{B-L}/M)^N$ appears in M_{R} [5].

We use a notation* for the mass matrices which only indicates the power of λ of the elements (P_{ij}). Consider first the singlet part of $\mathscr{M}_U$ coupling quarks to mirrors:

$$\frac{\tilde{M}^{u}_{M}}{M} = \begin{array}{c c} & \begin{array}{ccccc} -1 & 0 & 1 & 1 & 4 \end{array} \\ \begin{array}{c} -2 \\ -3 \end{array} & \begin{pmatrix} 3 & 2 & 1^* & \underline{1} & 2 \\ 4 & 3 & 2 & 2 & \underline{1} \end{pmatrix} \end{array} . \tag{13}$$

(A star means that the corresponding element, by a suitable choice of basis for the two u-quarks with charge one, will be one (or several) orders of magnitude smaller than the number indicated.) In leading order λ the two underlined elements will eliminate the two up quarks with charge 1 and 4 together with the two mirror quarks from the spectrum of light particles. Non leading orders of λ, however, lead to mixing between the Q eigenstates. For example, the light quark with $Q=1$ has an admixture of order λ of the quark with $Q=4$. We keep track of these mixings by indicating the power of λ of an admixture by a subscript. In this notation the charge of the remaining light u-quarks is

$$\hat{Q}(u) = \begin{cases} 1, & 4_1, \quad 0_1^*, \quad -1_2^* \\ 0, & 1_1, \quad 4_2, \quad -1_3 . \\ -1, & 1_2, \quad 0_3, \quad 4_3 \end{cases} \tag{14}$$

(We often omit the subscript zero and use the leading charge as a name for the corresponding light quark). The elimination of u^c is done in parallel and we obtain for the light up-quark matrix

$$\frac{\hat{M}_U}{M_W} = \begin{array}{c c} & \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{array}{c} -1 \\ 0 \\ 1 \end{array} & \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array} . \tag{15}$$

In this case the pattern (6) is not modified by the mixings and we have, as far as orders of magnitude are concerned, $\hat{M}_U = M_F^{(u^c u)}$ (restricted to the charges $(-1,0,1)$).

Concerning the down-quark mass matrix we eliminate the heavy $d-\bar{\bar{d}}$ pairs as described for u. The matrix determining the elimination of $d^c - \bar{\bar{d}}^c$ pairs is

$$\frac{\tilde{M}^{d^c}_{M}}{M} = \begin{array}{c c} & \begin{array}{cccccc} -4 & -4 & 2 & 2 & 3 & 3 \end{array} \\ \begin{array}{c} -1 \\ -1 \\ 0 \end{array} & \begin{pmatrix} 5 & 5 & \underline{1} & 1^* & 2 & 2 \\ 5 & 5 & 1^* & \underline{1} & 2 & 2 \\ 4 & 4^* & 2 & 2 & \underline{3} & 3^* \end{pmatrix} \end{array} . \tag{16}$$

* This is related to the notation of ref. [1] by $n_s = 4 - P$.

From the charge content of the light d^c

$$\hat{Q}(d^c) = \begin{cases} -4, & 2_2, \quad 3_1 \\ -4, & 2_2^*, \quad 3_1^*, \\ 3, & -4_1^*, \quad 2_1 \end{cases} \tag{17}$$

we derive the orders of magnitude of the mass matrix M_D for the light down-quarks

$$\frac{\hat{M}_D}{M_W} = \begin{array}{c} \\ 3 \\ -4 \\ -4 \end{array} \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} 4 & 3^* & 2^* \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} \end{array} . \tag{18}$$

(The elements $\hat{M}_{12}$ and $\hat{M}_{13}$ are 5 and 6 without the effects of charge mixing.) The lepton mass matrix $\hat{M}_L$ is of the order of $\hat{M}_D$ since 1 and d^c as well as e^c and d have the same $U(1)_G$ quantum numbers. As discussed earlier [1] the matrices (15), (18) reproduce correctly all orders of magnitude of mass eigenvalues and mixing angles.

Let us now give a more systematic discussion for the mass matrices of the light quarks and charged leptons. (We take three generations and up-type quarks as an example.) We denote the eigenstates of $U(1)_G$ charge by u_i^c and u_j with charge $Q_i^{(u^c)}$ and $Q_j^{(u)}$. As a consequence of $U(1)_G$ breaking and mass terms involving mirrors the mass eigenstates in the limit $\varphi_L = 0$ comprise three zero mass quarks u_α^c, u_β and a certain number of heavy states (coupling to the mirrors) $u_{H\alpha}^c$, $u_{H\beta}$. (The indices α, β have the same range as i, j and denote mass eigenstates in the limit $\varphi_L = 0$. Three values of these indices correspond to light quarks, the rest to heavy ones.) The light mass eigenstates u_α^c consist of a mixture of states with different charge

$$u_\alpha^c = \gamma_{\alpha i} u_i^c ,$$
$$\gamma_{\alpha i} \approx \lambda^{\kappa_{\alpha i}} , \tag{19}$$

and mixings involve various powers $\kappa_{\alpha i}$ of λ. Our short hand notation $\hat{Q}_\alpha = (Q_{i;\kappa_{\alpha i}})$ means, for example, that the light quark $(2_0, 1_1, 3_1, 5_4)$ has dominantly charge 2 with an order λ admixture of charge 3 and 1 and λ^4 admixture of charge 5. (For every α at least one $\kappa_{\alpha i}$ is zero.) The leading powers of λ appearing in the various elements of the mass matrix for the light quarks are given by

$$\hat{P}_{\alpha\beta} = \min_{i,j} \left\{ |Q_i^{(u^c)} + Q_j^{(u)} + q_0| + \kappa_{\alpha i} + \kappa_{\beta j} \right\}$$
$$= \min_{i,j} \left\{ P_{ij} + \kappa_{\alpha i} + \kappa_{\beta j} \right\} . \tag{20}$$

This is seen easily by writing $(M_\mathrm{F})_{ij}$ (eq. (6)) in the α, β basis and restricting it to the light quarks

$$u_i^\mathrm{c}(M_\mathrm{F})_{ij}u_j = u_\alpha^\mathrm{c}\gamma_{\alpha i}^*\gamma_{\beta j}^*(M_\mathrm{F})_{ij}u_\beta$$

$$+ \text{negligible terms involving } u_{\mathrm{H}\alpha}^\mathrm{c} \text{ and/or } u_{\mathrm{H}\beta}\,. \tag{21}$$

What remains is the determination of the light u_α^c and u_β and a calculation of the mixing coefficients $\kappa_{\alpha i}, \kappa_{\beta j}$. This depends on M_M (eq. (4)) and must in general be calculated separately for both u^c and u. We use a step by step procedure and start with the lowest power of λ, $P_{i1,k1}$, in M_M. (If there are several equal lowest P we may take an arbitrary one.) This dominant mass term will eliminate u_{i1}^c and $\bar{\bar{u}}_{k1}^\mathrm{c}$ from the spectrum of light particles. After this step the remaining $n+m-1$ quarks $u_\alpha^{\mathrm{c}(1)}$ have charges $\hat{Q}_\alpha^{(1)} = (Q_{i;\,\sigma_{\alpha i}^{(1)}})$ with mixing coefficients $\sigma^{(1)}$ (similar to the κ above) given by

$$\sigma_{\alpha i}^{(1)} = \begin{cases} 0 & \text{for } \alpha = i \\ \tilde{P}_{\alpha k_1}^{(1)} + \tilde{P}_{ik_1}^{(1)} - 2P_{i_1k_1} & \text{for } \alpha \neq i\,, \end{cases} \tag{22}$$

$$\tilde{P}_{ik}^{(1)} = \min_{k' \neq k}\left\{P_{ik};\left(P_{ik'} + P_{i_1k} + P_{i_1k'} - 2P_{i_1k_1}\right)\right\}. \tag{23}$$

(This accounts for mixing of u_α^c with $u_{i_1}^\mathrm{c}$, which in turns is mixed with u_i^c and for effects from $\bar{\bar{\mathrm{u}}}^\mathrm{c}$ mixing – see below.) If there are no mirrors left the $\kappa_{\alpha i}$ are given by $\sigma_{\alpha i}^{(1)}$. Otherwise we repeat the same procedure by looking for the lowest power of λ in the remaining matrix $M_\mathrm{M}^{(1)}$ for the $n+m-1$ quarks and $m-1$ mirrors left after elimination of u_{i1}^c and u_{k1}^c. We repeat this until all mirrors are eliminated and obtain

$$\kappa_{\alpha i} = \sigma_{\alpha i}^{(m)}\,. \tag{24}$$

For any given step the $\sigma^{(n)}$ can be calculated from $\sigma^{(n-1)}$ by

$$\sigma_{\alpha i}^{(n)} = \min\left\{\sigma_{\alpha i}^{(n-1)};\, \tilde{\sigma}_{\alpha i}^{(n)}\right\},$$

$$\tilde{\sigma}_{\alpha i}^{(n)} = \min_{\alpha'}\left\{\tilde{P}_{\alpha\gamma_n}^{(n)} + \tilde{P}_{\alpha'\gamma_n}^{(n)} - 2P_{\alpha_n\gamma_n}^{(n-1)} + \sigma_{\alpha' i}^{(n-1)}\right\}. \tag{25}$$

Here $P_{\alpha_n\gamma_n}^{(n-1)}$ is the leading element in $(M_\mathrm{M}^{(n-1)})_{\alpha\gamma}$ which will eliminate at step n the particles $u_{\alpha_n}^\mathrm{c}$ and $\bar{\bar{u}}_{\gamma_n}^\mathrm{c}$.

We have to specify how to calculate $(M_\mathrm{M}^{(n)})$ from $(M_\mathrm{M}^{(n-1)})$. This will also explain the formulae (22)–(25). The leading element at step n is $P_{\alpha_n\gamma_n}^{(n-1)}$ and we first rotate the $\bar{\bar{u}}_\gamma^\mathrm{c}$ so that the mass matrix has vanishing elements $(M_\mathrm{M})_{\alpha_n\gamma} = 0$ for $\gamma \neq \gamma_n$. After this, the powers of λ in M_{M} are

$$\tilde{P}^{(n)}_{\alpha\gamma} = \min_{\gamma' \neq \gamma} \left\{ P^{(n-1)}_{\alpha\gamma};\left(P^{(n-1)}_{\alpha\gamma'} + P^{(n-1)}_{\alpha_n\gamma} + P^{(n-1)}_{\alpha_n\gamma'} - 2P^{(n-1)}_{\alpha_n\gamma_n} \right) \right\}. \tag{26}$$

A subsequent rotation brings $(M_{\mathrm{M}})_{\alpha\gamma_n}$, $\alpha \neq \alpha_n$ to zero*. The mixing angles are given by powers of λ

$$\tilde{\sigma}^{(n)}_{\alpha\alpha'} = \begin{cases} 0 & \text{for } \alpha = \alpha' \\ \tilde{P}^{(n)}_{\alpha\gamma_n} + \tilde{P}^{(n)}_{\alpha'\gamma_n} - 2P^{(n-1)}_{\alpha_n\gamma_n} & \text{for } \alpha \neq \alpha' . \end{cases} \tag{27}$$

Expressing $u^{c}_{\alpha'}$ in terms of quantum number eigenstates u^{c}_{i} with the use of $\sigma^{(n-1)}_{\alpha' i}$ gives** (25). The final mass matrix after step n, eliminating $u^{c}_{\alpha_n}$ and $\bar{\bar{u}}^{c}_{\gamma_n}$, is given by

$$P^{(n)}_{\alpha\gamma} = \min_{\alpha' \neq \alpha} \left\{ \tilde{P}^{(n)}_{\alpha\gamma};\left(\tilde{P}^{(n)}_{\alpha'\gamma} + \tilde{P}^{(n)}_{\alpha'\gamma_n} + \tilde{P}^{(n)}_{\alpha\gamma_n} - 2P^{(n-1)}_{\alpha_n\gamma_n} \right) \right\}. \tag{28}$$

At this place we should note that the λ-powers in the mass matrices discussed so far are only the group theoretically allowed minimal values. There are certain cases where the actual powers of λ for some elements are higher even if there is no unnatural cancellation of contributions. For example, if two quarks have the same $\mathrm{U}(1)_{\mathrm{G}}$ charge one can always work with a basis where one appropriate element in the mass matrix for these particles is set to zero. (This corresponds to the star above.) Also, our procedure overestimates the contribution from mixing to the eigenvalues for matrices of the type $\begin{pmatrix} a & b & c \\ d & 0 & 0 \\ e & 0 & 0 \end{pmatrix}$ since it does not account for the vanishing determinant of this matrix (and similar if zeroes are replaced by small elements). Discarding these special cases (they are relatively rare and could in principle also be treated systematically) we have given an algorithm how to estimate orders of magnitudes of eigenvalues and mixings for matrices of the type (7). Of course, this algorithm does not depend on the specific assumption of a $\mathrm{U}(1)_{\mathrm{G}}$ generation group. The only input needed are the powers of λ, P_{ij}, and the separation of light and heavy mass scales M_{W} and M.

4. Neutrino masses

The mass matrix for the light neutrinos for the example (12) is

$$\frac{\hat{M}_\nu}{M_{\mathrm{W}}^2/M} = \begin{array}{c c} & \begin{array}{ccc} 3 & -4 & -4 \end{array} \\ \begin{array}{c} 3 \\ -4 \\ -4 \end{array} & \begin{pmatrix} 2 & 3 & 3^* \\ 3 & 4 & 4^* \\ 3^* & 4^* & 4^* \end{pmatrix} \end{array}. \tag{29}$$

* This induces again nonvanishing $(M_{\mathrm{M}})_{\alpha_n\gamma}$ and one has to check if repeating this procedure with these values would lead to lower $\sigma^{(n)}$ or $\tilde{P}^{(n)}$. For most cases this will not happen.

** Eq. (22) is obtained for $\sigma^{(0)}_{\alpha_i} = 0$ for $\alpha = i$ and ∞ otherwise.

In this case the electron neutrino would be the heaviest neutrino with mass $\sim \lambda^2 M_W^2/M \approx \lambda^2 \times 10^{-4}$ eV $\lesssim 10^{-6}$ eV. The second eigenvalue of the order λ^4 comes mainly from the order λ admixture of $Q=3$ to the $Q=-4$ neutrino (17). All neutrino masses are very small. This situation may change dramatically in models with right-handed neutrinos ν^c, where masses of the order of the cosmological bound ≈ 100 eV can be obtained naturally even if the unification scale M is high ($M \approx 10^{17}$ GeV). Neglecting mirror neutrinos $\bar{\bar{\nu}}$ for a moment, the mass matrix for the light neutrinos is [5]

$$\hat{M}_\nu = M_T + M_D M_R^{-1} \tilde{M}_D . \tag{30}$$

If all $SU(2)_L$ triplet scalars have mass M ★ the first contribution is always small (of the order λ^{P_T} 10^{-4} eV with $P_T \geqslant 0$). In contrast, the eigenvalues of M_R will be suppressed compared to M by one or several powers of λ if ν^c has nonvanishing $U(1)_G$ charge. This may result in light neutrino mass eigenvalues enhanced compared to M_W^2/M by inverse powers of λ. (The generation group $U(1)_G$ may therefore replace the role of $U(1)_{B-L}$ [5] for setting the scale of neutrino masses.)

As an example consider the following neutrino charges

$$\begin{aligned} Q(\nu) &= (-4,-4,3), \\ Q(\nu^c) &= (1,4). \end{aligned} \tag{31}$$

The ν-charges correspond to (12) neglecting those eliminated by coupling to mirrors and we assume again $q_0=-2$. The mass eigenvalues for ν^c are of the order $\lambda^2 M$ and $\lambda^8 M$. The Dirac mass term coupling the neutrino with $Q=-4$ to the ν^c with $Q=4$ is of the order λ^2 (compare (10)). A mixture of ν_τ and ν_μ ($Q=-4$) acquires therefore a mass of the order $\lambda^{-4} M_W^2/M \approx 1$–$20$ eV. For the remaining two light neutrinos the contribution of M_T is dominant (for the mass of ν_e the contribution from Dirac mass terms involving the ν^c with mass $\lambda^2 M$ is of the same size as the contribution from M_T). One finds $m_{\nu_e} \approx \lambda^2 M_W^2/M$ and the lowest neutrino mass is $\sim \lambda^4 M_W^2/M$. The neutrino mixing between the heaviest neutrino and ν_e is of the order λ^3.

An investigation of a few more examples with other $Q(\nu), Q(\nu^c)$ quickly shows that the spectrum of neutrino masses and mixings has often a quite unexpected structure, depending very sensitively on the $U(1)_G$ quantum numbers. In general, neutrino masses and mixings do *not* follow the usual generation pattern for charged quarks and leptons! There are many examples where ν_e is not the lightest neutrino and it may even have mass in the 10 eV range. There are also many examples where the neutrino mass pattern would be consistent with solar neutrino oscillations [13], where a linear combination of ν_μ and ν_τ has mass $\approx \lambda^{-2} M_W^2/M \approx 10^{-2}$ eV

★ See ref. [5] for cases where the triplet masses are required to be at a lower scale.

(whereas m_{ν_e} is smaller or roughly equal) and the mixing angle is not too small. It is not difficult to obtain neutrino masses in the range relevant for dark matter in cosmology ($m_\nu \approx \lambda^{-4} M_W^2/M$) – some choices for neutrino charges have even to be excluded because neutrino masses exceed the cosmological bound of about 50–100 eV. Mixing angles relevant for neutrino oscillations appear in various patterns, in general quite distinguished from the quark mixing pattern.

For a more systematic discussion of the neutrino mass matrix $\mathscr{M}_\nu$ (eq. (8)) we first eliminate the mirrors $\bar{\bar{\nu}}$ according to the procedure described above for the quarks. This will leave us with modified matrices $\hat{M}_T$ and $\hat{M}_D$ and a mixing $\sim \lambda^{\kappa_{\alpha i}}$ of Q eigenstates due to intermediate heavy states. The matrix element $(\hat{M}_\nu)_{\alpha\beta}$ has then maximal size $\lambda^{P_{\alpha\beta}^{(\nu)}} M_W^2/M$,

$$P_{\alpha\beta}^{(\nu)} = \min\left(\hat{P}_{\alpha\beta}, \hat{P}_{\alpha k}^{(D)} + \hat{P}_{\beta l}^{(D)} + R_{kl}\right), \tag{32}$$

with $\hat{P}_{\alpha\beta}$ and $\hat{P}_{\alpha k}$ the powers of λ in $\hat{M}_T$ and $\hat{M}_D$ and R_{kl} the power of λ in M_R^{-1}. Here R may be obtained from (11) by a simple matrix inversion algorithm and contains always some negative elements. Although (32) is useful for a quick inspection of the heaviest neutrino mass and for mixings it will often be misleading for a determination of the smaller neutrino mass eigenvalues. The determinant of the matrix $\hat{M}_D M_R^{-1} \tilde{\hat{M}}_D$ is always zero if the number of ν^c is smaller than the number of ν. Similarly, one very light ν^c gives a relatively large mass only to one of the light neutrinos. Instead of (32) (which only gives the maximal size of matrix elements consistent with $U(1)_G$ symmetry) we need a step-by-step procedure to extract eigenvalues and mixings from $\hat{M}_T$, $\hat{M}_D$ and M_R.

We first choose a basis for ν^c where M_R is diagonal

$$\nu_\mu^c \approx \lambda^{\rho_{\mu m}} \nu_m^c,$$

$$(M_R)_{\mu\nu} \approx \lambda^{P_\mu^{(R)}} M \delta_{\mu\nu} \tag{33}$$

by starting with the lowest P_{mn} in (11) and proceeding similar as for the mirror matrix M_M. In this basis $\hat{M}_D \approx \lambda^{\bar{P}^{(D)}} M_W$ with

$$\bar{P}_{\alpha,\mu}^{(D)} = \min_m \left\{ \hat{P}_{\alpha m}^{(D)} + \rho_{\mu m} \right\}. \tag{34}$$

The eigenvalues of $\hat{M}_D M_R^{-1} \tilde{\hat{M}}_D$ can now be obtained step by step, looking first for the lowest power of λ,

$$\min_{\alpha,\mu} \left\{ 2\bar{P}_{\alpha\mu}^{(D)} - P_\mu^{(R)} \right\}, \tag{35}$$

then eliminating ν_α and ν_μ^c (while accounting for mixing among ν_α) and proceeding in the same way until all ν_α or all ν_μ^c are eliminated. These eigenvalues have then to be compared with the corresponding relevant elements of $\hat{M}_T$ and the final neutrino mass eigenvalues are obtained by taking the dominant contribution either from $\hat{M}_T$ or from $\hat{M}_D \tilde{M}_R^{-1} \hat{M}_D$. Neutrino mixings are calculated by comparing off diagonal elements of the combined matrix $\hat{M}_\nu$ with the size of corresponding eigenvalues. The mixing angles relevant for neutrino oscillations are composed from these neutrino mixings and the mixings in the lepton mass matrix $\hat{M}_L$.

5. A scan of anomaly free SU(5) × U(1) models

In the preceding sections we have described how to calculate fermion masses and mixings for given quantum numbers with respect to the generation symmetry $U(1)_G$. Orders of magnitude only depend on $\lambda = M_G/M$ and on the leading charge q_0 of the low energy doublet. (We normalize the charge of χ to one.) For given $\lambda \approx \frac{1}{20} - \frac{1}{10}$ we can check if a certain choice of fermion charges and q_0 leads to an acceptable fermion mass spectrum. The algorithms for determining mass eigenvalues and mixings described above can be used for a computerized scan of which quantum numbers are realistic. In this paper we will only be concerned with quantum numbers consistent with (four-dimensional) grand unification, i.e. the $U(1)_G$ charges of u, d, u^c, e^c as well as d^c, ν, e must be equal. ($U(1)_G$ commutes with SU(5)). We also restrict our scan to models where $SU(5) \times U(1)_G$ is free of all anomalies. To start with, we first determine analytically for three generations all realistic $U(1)_G$ charges (not necessarily anomaly free) consistent with grand unification in the absence of mirror particles. We will use in this section a shorthand notation u_i instead of $Q(u_i)$ etc. For $u_i = d_i = u_i^c = e_i^c$ and $d_i^c = \nu_i = e_i$ the mass pattern (6) (and the corresponding formula for M_D, M_L) is left invariant under the following shift in quantum numbers [1]

$$\begin{aligned} u_i &\to u_i + \delta , \\ d_i^c &\to d_i^c - 3\delta , \\ q_0 &\to q_0 - 2\delta . \end{aligned} \tag{36}$$

We can therefore choose $t = b = t^c = \tau^c = 0$. A top quark mass of the order M_W requires $q_0 = 0$. From m_b, $m_\tau \approx \lambda M_W$ we conclude $b^c = \tau = 1$. (We have a freedom in the choice of the overall sign of $U(1)_G$ charges.) The only charge assignment compatible with m_c is $c = c^c = \pm 1$*. One obtains M_U of the type $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ with a mixing angle ϑ_{23} between the second and third generation of the order λ and a

* We have defined t^c, c^c and u^c so that they have the same Q than t, c and u, respectively. They are indeed also the mass partners. This is obvious for t since the largest element in M_U cannot be off diagonal. For the charm mass one easily finds that different assignments would lead to unacceptable values either for the up-quark mass or the mixing angle ϑ_{13}.

consistent order of magnitude for m_c (for a more detailed discussion of mass patterns compare ref. [1]). Now μ^c cannot be -1, otherwise M_L would have an element ($\sim \tau\mu^c$) of order M_W, and therefore $c = c^c = s = \mu^c = +1$. For the up-quark mass matrix there are two possible choices $u = u^c = d = e^c = +2$ or -4 with mass patterns

$$M_U = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \quad \text{or} \quad M_U = \begin{pmatrix} 8 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 1 & 0 \end{pmatrix}. \tag{37}$$

For the second choice we have a Fritzsch [14, 6]-type structure where the up-quark mass is generated by paired off diagonal elements of the order $\lambda^3 M_W$ (their size should be around 80 MeV). For both possibilities the fairly large Cabibbo angle ϑ_{12} is not obtained from M_U and must come dominantly from M_D. This requirement together with the values of m_s and m_μ fixes $s^c = \mu = +1$. (The charges $s^c = -3$ or $s^c = +2$ (limiting case) are consistent with m_s but do not lead to an acceptable Cabibbo angle.) From a typical value $m_e \approx \lambda^4 M_W$ we find for $u = 2$ the two possibilities $d^c = e = +2, -6$ and for $u = -4$ one gets $d^c = e = -4$. The M_D, M_L mass patterns for these three possibilities are

$$M_D = M_L^T = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 8 & 3 & 4 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}. \tag{38}$$

It is easily checked that for appropriate λ the mass patterns (37), (38) lead to correct mass eigenvalues and mixings up to a factor about three which is well within the uncertainty of our approach.

To summarize, we found the following realistic $U(1)_G$ charges consistent with SU(5)

$$\begin{array}{ccc|ccc} & 10 & & & \bar{5} & \\ \hline 0 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & -6 \\ 0 & 1 & -4 & 1 & 1 & -4 \end{array} \tag{39}$$

Other realistic charge assignments can be obtained by using the shift δ (eq. (36)). We note that for none of these examples the charges are equal for 10 and $\bar{5}$. It follows that a mass pattern from $U(1)_G$ symmetry breaking at $M_G = \lambda M$ is not realistic if $U(1)_G$ commutes with SO(10) or E_6 (unless additional small parameters are introduced). This feature persists in models with additional quark-mirror pairs. For $U(1)_G$ commuting with left-right symmetry every element in M_U generated by a doublet φ_q will be mapped by left-right symmetry onto a corresponding element in M_D generated by a doublet with opposite charge φ^*_{-q}. This does not imply that M_D and M_U must have the same structure (except for $q_0 = 0$) but this mapping is nevertheless the origin of the difficulties to construct realistic models based on $SO(10) \times U(1)_G$.

There are much more realistic charge assignments without mirrors if $U(1)_G$ only commutes with $SU(3) \times SU(2) \times U(1)$ but not necessarily with $SU(5)$. In this case there are three possible shifts similar to (36) which may be used to set $t = b = t^c = \tau^c = 0$ so that again $q_0 = 0$. One needs $b^c = 1$ and $\tau = 1$. (Both the sign of b^c and τ are convention since the overall sign of quark and lepton charges can be chosen separately.) One finds already 23 possibilities for the assignment of (c, s), c^c and s^c (compared to only one for the SU(5) case). A few simple conditions for realistic mass patterns are

$$\begin{aligned}
&(c,s) = 1 \quad \text{or} \quad 2 \quad \text{or} \quad -3 \quad \text{or} \quad -4,\\
&|c + c^c| = 1 \quad \text{or} \quad c = c^c = 1,\\
&|s + s^c| = 2 \quad \text{or} \quad s = 1, \quad s^c = 2 \quad \text{or} \quad s = -3, \quad s^c = -1, -2,\\
&|\mu + \mu^c| = 2 \quad \text{or} \quad \mu = 2, \quad |\mu^c + \tau| = 1, 2.
\end{aligned} \tag{40}$$

None of the $SU(5) \times U(1)_G$ models with realistic charge patterns (without mirrors and without singlets ν^c) is anomaly free. If we denote the $U(1)_G$ charges of the SU(5) representations $10, \overline{10}, \overline{5}, 5, 1$ by a_i, A_i, b_i, B_i, N_i the absence of all anomalies of $SU(5) \times U(1)_G$ (including mixed gravitational anomalies) requires

$$\begin{aligned}
3\sum a_i + 3\sum A_i + \sum b_i + \sum B_i &= 0,\\
2\sum a_i + 2\sum A_i + \sum b_i + \sum B_i + \tfrac{1}{5}\sum N_i &= 0,\\
2\sum a_i^3 + 2\sum A_i^3 + \sum b_i^3 + \sum B_i^3 + \tfrac{1}{5}\sum N_i^3 &= 0.
\end{aligned} \tag{41}$$

If we interpret $U(1)_G$ as a local gauge symmetry the cancellation of anomalies implies relations between singlet (ν^c) charges and quark charges. As a consequence the quark and lepton mass matrices (7) and the neutrino mass matrix (8) are not independent anymore. For given quark and lepton charges the choice of N_i and therefore the possible neutrino mass patterns are restricted. First of all, the sums ΣN_i and ΣN_i^3 must be divisible by five (in a normalization where a_i and b_i are integer). We give in table 1 the charges of up to three neutrinos with $|N_i| \leqslant 5$ fulfilling this condition. (This includes the ν^c charges (31) discussed in the preceding section.) Table 2 shows the number of different (linear + cubic) anomaly contributions for up to ten ν^c. (Nonchiral pairs of ν^c with opposite N or $N = 0$ are always discarded. We note that different sets $\{N_i\}$ may give the same value for ΣN_i and ΣN_i^3.)

The vanishing of mixed $SU(5)^2 \times U(1)_G$ anomalies (the first equation in (41)) is independent of N_i. We found that it cannot be accomplished for realistic charges ((39) or those obtained from (39) by δ-shifts). Additional quark-mirror pairs, which are chiral with respect to $U(1)_G$, are therefore needed for any anomaly free $SU(5) \times U(1)_G$ model with realistic mass patterns involving only one small parame-

TABLE 1
Possible sets of right-handed neutrino (ν^c) quantum numbers with up to three ν^c of $U(1)_G$ charges smaller than 5 and anomalies divisible by 5

n_ν	$\frac{1}{5}\Sigma N_i$	$\frac{1}{5}\Sigma N_i^3$	N_1	N_2	N_3
1	−1	−25	−5		
2	−2	−50	−5	−5	
2	−1	−13	−4	−1	
2	−1	− 7	−3	−2	
3	−3	−75	−5	−5	−5
3	−2	−38	−5	−4	−1
3	−2	−32	−5	−3	−2
3	0	−12	−5	1	4
3	0	−18	−5	2	3

The sets with opposite signs are also possible.

TABLE 2
The total number of different sets of anomalies divisible by 5 for n_ν right-handed neutrinos and quantum numbers $|N_i| \leqslant \bar{N}$

$\bar{N}$ \ n_ν	1	2	3	4	5	6
4	1	5	5	13	29	37
5	3	9	19	37	69	105
6	3	13	29	63	129	199
10	5	37	135	411		

ter λ. Using the general algorithm of sect. 3 one may perform a computerized scan for anomaly free $SU(5) \times U(1)$ models with mirrors which lead to realistic mass patterns. As a first step we have chosen the following simplified (incomplete) procedure: We take the "realistic" quantum numbers obtained from (39) and add all possible chiral quark-mirror pairs with charges such that all anomalies are cancelled for appropriate ν^c quantum numbers. We then evaluate the mirror mass matrices and ask if the leading quantum numbers of the light quarks and leptons correspond to (39) – i.e. if the "right" quarks are eliminated by couplings to the mirrors. The number of "realistic" solutions fulfilling these criteria* is shown in tables 3 and 4, where we have considered up to six quark-mirror pairs with $|Q| \leqslant 4$. Finally we checked explicitly (by hand) for some of these solutions if they lead indeed to realistic mass patterns, including all mixing effects from mirrors as described in sect. 3.

* Instead of (26), (28) we used a simplified algorithm $P_{\alpha\gamma}^{(n)} = \min\{P_{\alpha\gamma}^{(n-1)}; (P_{\alpha\gamma_n}^{(n-1)} + P_{\alpha_n\gamma}^{(n-1)} - P_{\alpha_n\gamma_n}^{(n-1)})\}$ which for most (but not all!) cases lead to identical results.

TABLE 3
The leading quantum numbers of the $\bar{5}$ and 10's for the light SU(5) families and the number of totally anomaly free "realistic" solutions with at most six added $\bar{5}$ or 10's and six 5 or $\overline{10}$'s with $U(1)_G$ quantum numbers less than or equal to 4 (absolute values)

$\bar{5}$ charges			10 charges			# sol. without ν^c	# sol. with one ν^c $N = \pm 5$
1	1	−4	0	1	−4	40	40
4	4	−1	−1	0	−5	0	13
−1	−1	−2	0	−1	−2	0	0
2	2	1	−1	−2	−3	2	27
5	5	4	−2	−3	−4	0	0
−4	−4	5	1	0	−1	1632	2175
−4	−4	3	1	0	−1	16	952
−1	−1	6	0	−1	−2	0	0

We give the number of solutions without right-handed neutrinos ν^c and with one right-handed neutrino of charge ± 5.

TABLE 4
A more detailed search, including more neutrino patterns with up to six extra quark-mirror pairs with $U(1)_G$ quantum numbers $|Q| \leqslant 4$ and up to three SU(5) singlets with $U(1)_G$ quantum numbers $|N| \leqslant 5$

n_5	n_{10}	n_{sol}	n_5	n_{10}	n_{sol}
0	0	0	4	0	0
0	1	0	0	5	0
1	0	0	1	4	14
0	2	0	2	3	38
1	1	0	3	2	45
2	0	0	4	1	0
0	3	0	5	0	0
1	2	3	0	6	0
2	1	0	1	5	28
3	0	0	2	4	109
0	4	0	3	3	145
1	3	8	4	2	105
2	2	9	5	1	0
3	1	0	6	0	0

Here n_5 is the number of 5 and $\bar{5}$'s added and n_{10} is the number of additional $10 + \overline{10}$'s. The charges of the light $\bar{5}$'s are 2 2 1 and those of the light 10's -1 -2 -3. There are a total of 504 solutions.

Among the "realistic" solutions we found various interesting neutrino mass patterns as described in sect. 4. The explicit check showed that realistic anomaly free $SU(5) \times U(1)_G$ models indeed exist which explain all masses and mixings in terms of a single small parameter λ (compare the example in sect. 3). However, for none of these examples the $U(1)_G$ quantum numbers look particularly attractive. Often three or more quarks have the same $U(1)_G$ charge. Without additional criteria on the "allowed" $U(1)_G$ quantum numbers it seems difficult to single out one specific model. On the other hand, the additional requirement that a given charge a_i appears at most twice (similar for b_i etc.) already leads to a drastic reduction of the number of solutions. If the general form of the charge spectrum is given – as may be expected for higher dimensional compactification – the answer about the existence of realistic mass patterns may be unique. We have looked at a six dimensional example [2,7] where $U(1)_G$ is embedded into a generation group $SU(2) \times U(1)_q$ with $q = \pm\frac{1}{2}$ for all fermions and with $SU(2)_G$ representations given by monopole numbers from spontaneous compactification. There is no realistic mass pattern at all if $U(1)_G$ commutes with SU(5), independently of the various possible embeddings of $U(1)_G$ into $SU(2) \times U(1)_q$.

References

[1] J. Bijnens and C.Wetterich, Nucl. Phys. B283 (1987) 237; Phys. Lett. 176B (1986) 431
[2] C. Wetterich, Nucl. Phys. B260 (1985) 402, B261 (1985) 461
[3] M. Magg and C. Wetterich, Phys. Lett. 94B (1980) 61
[4] G. Lazarides, Q. Shafi and C. Wetterich, Nucl. Phys. B181 (1981) 343
[5] C. Wetterich, Nucl. Phys. B187 (1981) 343
[6] S. Dimopoulos, Phys. Lett. 129B (1983) 417;
J. Bagger, S. Dimopoulos, H. Georgi and S. Raby, 5th Workshop on Grand Unification, Providence, *in* Providence Grand Unif. (1984) p. 95
[7] C. Wetterich, Nucl. Phys. B279 (1987) 711
[8] A. Strominger and E. Witten, Comm. Math. Phys. 101 (1985) 341
[9] S.M. Barr, Phys. Rev. D21 (1979) 1424, D24 (1981) 1895;
R. Barbieri and D.V. Nanopoulos, Phys. Lett. 95B (1980) 43;
M.J. Bowick and P. Ramond, Phys. Lett. 103B (1981) 338
[10] S. Dimopoulos and H. Georgi, Phys. Lett. 140B (1984) 67
[11] M. Gell-Mann, P. Ramond and R. Slansky, *in* Supergravity, eds. D. Freedman and P. van Nieuwenhuizen (North-Holland, 1980);
T. Yanagida, Proc. Workshop on the Unified theory and the baryon number in the universe (KEK, 1979)
[12] E. Witten, Nucl. Phys. B258 (1985) 75
[13] J.K. Rowley, B.T. Cleveland and R. Davis, AIP Conf. Proc. 126 (1985) 1;
B. Pontecorvo, Zh. Eks. Teor. Fiz 34 (1958) 247, 53 (1967) 1725;
J.N. Bahcall and S. Frautschi, Phys. Lett. 29B (1969) 623;
S.P. Mikheyev and A.Y. Smirnov, Moscow INR preprint (1985);
L. Wolfenstein, Phys. Rev. D17 (1978) 2369;
H.A. Bethe, Phys. Rev. Lett. 56 (1986) 1305;
P. Langacker, S.T. Petcov, G. Steigman and S. Toskev, Nucl. Phys. B282 (1987) 589
[14] H. Fritzsch, Nucl. Phys. B155 (1979) 189