Chiral anomaly of antisymmetric tensor fields

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For antisymmetric tensor gauge fields of rank $2n-1$ coupled to gravity in $4n$ dimensions it is shown that the symmetry under duality rotations is broken by quantum effects. The anomaly is related to a local version of the signature index theorem. The ζ-function technique, Fujikawa's method, and the stochastic regularization scheme are discussed.

I. INTRODUCTION

As was first pointed out by Alvarez-Gaumé and Witten,[1] (anti-)self-dual antisymmetric tensor fields in $4n-2$ dimensions have anomalies in their coupling to gravity. Similar to the case of fermions, these gravitational anomalies are related to chiral anomalies in $4n$ dimensions.[2] For the antisymmetric tensor fields these chiral transformations are realized as duality rotations of a tensor field of rank $2n$. On the other hand, considering this field

$$F_{\mu_1\cdots\mu_{2n}}\equiv 2n\,\partial_{[\mu_1}A_{\mu_2\cdots\mu_{2n}]}$$

as being the field strength of a gauge potential $A_{\mu_1}\cdots_{\mu_{2n-1}}$ one is led to study the "duality anomaly" of a U(1) gauge theory where the gauge fields are antisymmetric tensor fields of rank $2n-1$. For $n=1$, say, this means that there is an anomaly associated with duality transformations of the field strength $F_{\mu\nu}$ belonging to the photon A_μ.

Along a different line of investigation, Dolgov, Khriplovich, and Zakharov[3], recently used dispersion-relation techniques to show that if the photon field is quantized in a four-dimensional curved space-time (with Minkowski signature) for which the pseudoscalar $\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta\rho\sigma}$ does not vanish, the vacuum expectation value of the Pauli-Ljubanski vector

$$K^\mu=g^{-1/2}\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma \tag{1.1}$$

is not conserved. This means that the pseudoscalar $F_{\mu\nu}{}^*F^{\mu\nu}$ acquires a vacuum expectation value:

$$\nabla_\mu\langle K^\mu\rangle\equiv\tfrac{1}{2}\langle F_{\mu\nu}{}^*F^{\mu\nu}\rangle = \frac{1}{192\pi^2}g^{-1/2}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta\rho\sigma} . \tag{1.2}$$

As we shall see below, this equation expresses the fact that quantum effects spoil the invariance of the classical theory under duality transformations. For a free electromagnetic field this symmetry causes the difference of the numbers of right and left circularly polarized photons to be conserved.[4] Hence, if the right-hand side (RHS) of (1.2) is nonzero, the gravitational field continuously produces (chiral) photons from the vacuum. This is analogous to the anomalous fermion pair creation by Yang-Mills fields as expressed by the famous relation

$$\Delta Q_5=\frac{1}{8\pi^2}\int d^4x\,\mathrm{tr}(F_{\mu\nu}{}^*F^{\mu\nu}) , \tag{1.3}$$

where ΔQ_5 is the change of the chiral charge. In the present case the chiral charge density $\psi^\dagger\gamma_5\psi$ is replaced by K^0 whose space integral is $+1$ (-1) for right- (left-) handed photons.

The purpose of this paper is to show how the result of Dolgov, Khriplvich, and Zakharov, is related to an anomalous breaking of the duality symmetry. Furthermore, we shall see that a similar effect exists in all $4n$-dimensional theories containing antisymmetric tensor gauge fields of rank $2n-1$ coupled to gravity. Working in Euclidean space, we will relate the generalization of (1.2) to a local version of the signature index theorem so that for all n the anomaly can be expressed by the Hirzebruch L polynomial.[5,6] In Sec. II this result is derived using the ζ-function method for regularizing infinite-dimensional determinants. Then, in Sec. III, we show that, similar to the fermionic case, the chiral anomaly of antisymmetric tensor fields is associated with a nontrivial Jacobian of the path-integral measure.[7] Finally, in Sec. IV it is briefly described how the anomaly is obtained in the framework of stochastic quantization.[8]

II. ζ-FUNCTION REGULARIZATION

We are considering a $4n$-dimensional oriented Riemannian manifold $\mathcal{M}$ of Euclidean signature which has no boundary: $\partial\mathcal{M}=\varnothing$. We define totally antisymmetric tensor fields $A_{\mu_1\cdots\mu_{2n-1}}(x)$ on $\mathcal{M}$, which we frequently will write as differential forms:

$$A(x)=\frac{1}{(2n-1)!}A_{\mu_1\cdots\mu_{2n-1}}(x)dx^{\mu_1}\cdots dx^{\mu_{2n-1}} . \tag{2.1}$$

We associate to A a field-strength $2n$-form in the usual way:[9]

$$F=dA , \tag{2.2a}$$

where

$$F_{\mu_1\cdots\mu_{2n}}=2n\,\partial_{[\mu_1}A_{\mu_2\cdots\mu_{2n}]} . \tag{2.2b}$$

Obviously F is invariant under gauge transformations

$$A \rightarrow A + d\chi \quad (2.3)$$

for any $(2n-2)$-form χ, so that for $n=1$ ordinary (Euclidean) electrodynamics is recovered. For our purposes it is convenient to introduce the scalar product

$$(\alpha,\beta) \equiv *[\alpha \wedge *\beta] = \frac{1}{p!}\alpha_{\mu_1 \cdots \mu_p}\beta^{\mu_1 \cdots \mu_p} \quad (2.4)$$

for all p-forms α and β. Here we used the Hodge operator $*$ defined as

$$*dx^{\mu_1} \cdots dx^{\mu_p} = \frac{\sqrt{g}}{(4n-p)!}\epsilon^{\mu_1 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_{4n}} \times dx^{\mu_{p+1}} \cdots dx^{\mu_{4n}} . \quad (2.5)$$

The action for A is the following generalization of the Maxwell action (ϵ denotes the volume form $\sqrt{g}\,dx^1 \cdots dx^{4n}$):

$$S = \tfrac{1}{2}\int (F,F)\epsilon = \tfrac{1}{2}\int (A,\delta d A)\epsilon . \quad (2.6)$$

The second equality follows from the fact that the coderivative δ is the adjoint of d. We will perform all calculations in a generalized Lorentz gauge defined by

$$\delta A = 0 . \quad (2.7)$$

For $n=1$ this reduces to the ordinary Lorentz condition $\nabla_\mu A^\mu = 0$, where ∇_μ is the covariant derivative constructed from the metric of $\mathcal{M}$. For fields satisfying (2.7) the action can be written as

$$S = \tfrac{1}{2}\int (A,\Delta A)\epsilon , \quad (2.8)$$

where $\Delta = d\delta + \delta d$ is the Laplacian. In its original form (2.6), S is invariant under global duality rotations (or "chiral transformations") of the form

$$F \rightarrow F\cos\alpha + *F\sin\alpha , \quad (2.9)$$

or, infinitesimally,

$$\delta_\alpha F = \alpha * F . \quad (2.10)$$

Now we turn to our main task: namely, the calculation of the vacuum expectation value of

$$\nabla_\mu K^\mu \equiv (F, *F) = \frac{1}{(2n)!}F_{\mu_1 \cdots \mu_{2n}} {}^*F^{\mu_1 \cdots \mu_{2n}} \quad (2.11)$$

for the prescribed background gravitational field given by the metric $g_{\mu\nu}(x)$ of $\mathcal{M}$. [The generalization of (1.1) for arbitrary n can be read off from $(F,*F) = \delta K$, where $K = *(A\,dA)$.] We define the generating functional

$$Z[\eta] = \int [\mathcal{D}A]_{\rm LG} \exp\left[-\tfrac{1}{2}\int (A,\Delta A)\epsilon + \int (dA, \eta * dA) \right] \quad (2.12)$$

for any real, scalar function η because then the expectation value of (2.11) reads

$$\langle (F(x), *F(x)) \rangle \equiv \frac{\langle 0 | (F(x), *F(x)) | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{\delta}{\delta\eta(x)} \ln Z[\eta] \Big|_{\eta=0} . \quad (2.13)$$

The subscript LG at the path-integral measure is to indicate that the integration has to be performed only over fields obeying the Lorentz gauge condition (2.7). We are not going to exponentiate this constraint since it is much simpler to explicitly take it into account in doing the Gaussian integration (see below). In particular, any complication due to the necessity of introducing ghosts for the ghosts is avoided.[10]

In the representation (2.13) we can evaluate $\langle (F, *F) \rangle$ using the same technique as developed in Ref. 11 for fermionic chiral anomalies and subsequently used for the evaluation of various other types of anomalies.[12] Performing the integral for Z one formally obtains

$$\langle (F(x), *F(x)) \rangle = -\frac{1}{2}\frac{\delta}{\delta\eta(x)} \ln \det\Omega \Big|_{\eta=0} , \quad (2.14)$$

where the operator Ω is given by

$$\Omega = \Delta + \mu^2 + 2*(d\eta)\wedge d . \quad (2.15)$$

To control the usual IR divergences associated with massless fields we have introduced a small mass parameter μ. For $\eta(x) \equiv 0$ the operator Ω is positive and Hermitian. Because we may consider η infinitesimal, this is sufficient for the ζ-function method to be applicable; i.e., we can define the determinant as $\exp[-\zeta'(\Omega\,|\,0)]$, where $\zeta(\Omega\,|\,s)$ is the ζ function associated with Ω (Ref. 13). Hence one has

$$\langle (F(x), *F(x)) \rangle = \frac{1}{2}\frac{\delta}{\delta\eta(x)}\frac{d}{ds}\Big|_0 \zeta(\Omega\,|\,s) . \quad (2.16)$$

To evaluate the RHS of (2.16) we introduce a complete set of normalized eigenfunctions of the operator $\Delta + \mu^2$ acting on $(2n-1)$-forms:

$$(\Delta + \mu^2)a_i(x) = \lambda_i a_i(x), \quad \lambda_i > 0 , \quad (2.17)$$

$$\int (a_i, a_j)\epsilon = \delta_{ij} . \quad (2.18)$$

It has been shown in Ref. 11, for instance, that in terms of the a_i's the functional derivative of the ζ function can be written as

$$\frac{\delta\zeta(\Omega\,|\,s)}{\delta\eta(x)} = -s \sum_i \lambda_i^{-(1+s)} \int \left[a_i, \frac{\delta\Omega}{\delta\eta(x)} a_i \right] \epsilon . \quad (2.19)$$

As was already mentioned, the path integration (2.12) has to be performed only over fields satisfying $\delta A = 0$. Therefore, it is only their eigenvalues which contribute to the determinant in (2.14). Consequently, in (2.19) the sum runs only over eigenvectors a_i satisfying $\delta a_i = 0$. Inserting (2.15) yields

$$\langle (F(x),*F(x))\rangle = \frac{d}{ds}\bigg|_0 s\sum_i \lambda_i^{-(1+s)}(da_i(x),*da_i(x)) \; . \tag{2.20}$$

At this point it is advantageous to change the normalization of the basis fields; we introduce

$$\alpha_i(x)=\lambda_i^{-1/2}a_i(x) \; . \tag{2.21}$$

For $\mu\to 0$ the α_i are normalized according to

$$\int (d\alpha_i,d\alpha_j)=\delta_{ij} \; . \tag{2.22}$$

Exploiting the identity

$$x^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-xt}, \quad x>0 \tag{2.23}$$

we obtain

$$\langle (F(x),*F(x))\rangle = \frac{d}{ds}\bigg|_0 \frac{s}{\Gamma(s)} \times \int_0^\infty dt\, t^{s-1}e^{-\mu^2 t} \times \sum_i (d\alpha_i(x),*d\alpha_i(x))e^{-\lambda_i t} \; . \tag{2.24}$$

In this representation we can relate the expectation value of $(F,*F)$ to the index theorem for the signature complex.[5,6] Let us recall that the signature τ of a $4n$-dimensional manifold $\mathcal{M}$, i.e., the index of the signature complex is defined by

$$\tau(\mathcal{M})=\mathrm{Tr}_+(e^{-t\Delta})-\mathrm{Tr}_-(e^{-t\Delta}) \; . \tag{2.25}$$

The traces $\mathrm{Tr}_\pm$ refer to the space of self-dual and anti-self-dual $2n$-forms, respectively. Note that because the projectors on these spaces are $P_\pm=\frac{1}{2}(1\pm *)$ this also could be written as

$$\tau(\mathcal{M})=\mathrm{Tr}(*e^{-t\Delta}) \; , \tag{2.26}$$

where the trace is over all $2n$-forms now. [One even could perform the trace with respect to the whole exterior algebra; all additional contributions would cancel between the p- and the $(2n-p)$-forms.] A standard argument shows that only the zero modes of Δ contribute to the signature. Hence $\tau(\mathcal{M})$ is the difference of the number of self-dual and anti-self-dual zero modes of the Laplacian. The signature can be explicitly calculated from the asymptotic expansion of the relevant heat kernels:[5,14,15]

$$K_\pm(x;t)\equiv \mathrm{tr}\langle x\,|\,e^{-\Delta_\pm t}\,|\,x\rangle = \frac{1}{t^{2n}}\sum_{k=0}^\infty B_{2k}^\pm(x)t^k \; . \tag{2.27}$$

The trace tr refers to the tensor indices only and $\Delta_\pm=\Delta P_\pm$ is the Laplacian restricted to the space of (anti)self-dual fields. One finds

$$\tau(\mathcal{M})=\int d^{4n}x\sqrt{g}\,B_{4n}(x) \; , \tag{2.28}$$

where $B_{4n}=B_{4n}^+-B_{4n}^-$. One possible strategy to evaluate the coefficients $B_{2k}(x)$ is to use a method similar to Fujikawa's computation of spinorial chiral anomalies.[7] This has recently been done by Endo and Takao.[2] They add to the kernel (2.27) additional tensor fields (cf. the remarks above) to form a Dirac-Kähler fermion;[1,16] the computation is then similar to the evaluation of the anomaly for the Rarita-Schwinger field.[17] In accordance with the mathematical literature their result can be represented as

$$\tau(\mathcal{M})=\int \det\left[\frac{\Omega/2\pi}{\tanh(\Omega/2\pi)}\right]^{1/2} \; . \tag{2.29}$$

The integrand is the Hirzebruch L polynomial for the curvature two-forms,

$$\Omega^\mu{}_\nu=\tfrac{1}{2}R^\mu{}_{\nu\rho\sigma}dx^\rho dx^\sigma \; , \tag{2.30}$$

constructed from the metric of $\mathcal{M}$. This notation means that we have to expand the integrand in a power series in Ω and to keep only those terms which have the correct $4n$-dimensional volume form.

To make contact with Eq. (2.24) we consider the kernel

$$K(x;t)=K_+(x;t)-K_-(x;t) = \sum_i (f_i(x)*f_i(x))e^{-\lambda_i t} \; . \tag{2.31}$$

The sum runs over a complete set of $2n$-forms f_i with

$$\Delta f_i=\lambda_i f_i, \quad \lambda_i\geq 0 \; , \tag{2.32}$$

and

$$\int (f_i,f_j)\epsilon=\delta_{ij} \; . \tag{2.33}$$

According to the Hodge decomposition theorem, our $2n$-forms f can be uniquely decomposed as a sum of an exact form [the derivative of a $(2n-1)$-form α], a co-exact form [the co-derivative of a $(2n+1)$-form β], and a harmonic form:

$$f=d\alpha+\delta\beta+\phi, \quad \Delta\phi=0 \; . \tag{2.34}$$

The three pieces are mutually orthogonal and satisfy the eigenvalue equation (2.32) separately. This means that we can divide the f_i's into three classes: $f_i^{(1)}=d\alpha_i$, $f_i^{(2)}=\delta\beta_i$, and $f_i^{(3)}=\phi_i$. Hence (2.31) decomposes according to

$$K(x;t)=\sum_{\alpha}(d\alpha_i,*d\alpha_i)e^{-\lambda_i t}+\sum_{\beta}(\delta\beta_i*\delta\beta_i)e^{-\lambda_i t}+\sum_i(\phi_i,*\phi_i)\ . \tag{2.35}$$

It is important to note that the first two sums in (2.35) are equal. This follows from the identity

$$(\delta\beta_i,*\delta\beta_i)=(d[*\beta_i],*d[*\beta_i])$$

and the fact that the Hodge operator provides an isomorphism between the space of the α's and the β's. Furthermore, this sum coincides with the one appearing in Eq. (2.24), since for $\lambda\neq 0$ the spectrum of Δ, acting on α and $d\alpha$ coincides and the normalization (2.22) is the same as in (2.33).

Returning to (2.24) we may write

$$\langle (F(x),*F(x))\rangle=\frac{1}{2}\frac{d}{ds}\left|_0 \frac{s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-\mu^2 t}\left[K(x;t)-\sum_i(\phi_i(x),*\phi_i(x))\right]\ . \right. \tag{2.36}$$

Inserting the expansion (2.27) one easily finds the μ-independent relation

$$\nabla_\mu\langle K^\mu\rangle=\langle (F(x),*F(x))\rangle=\tfrac{1}{2}B_{4n}(x)-\tfrac{1}{2}\sum_i(\phi_i(x),*\phi_i(x))\ . \tag{2.37}$$

This is the desired result. It states that apart from the $2n$-form zero modes of the Laplacian the vacuum expectation value is given by the well-known Seeley coefficients $B_{4n}(x)$ (Refs. 5, 6 and 14). For $n=1$ and 2 they are explicitly given by

$$\begin{aligned}B_4(x)&=\frac{1}{96\pi^2}g^{-1/2}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta\rho\sigma}\ ,\\ B_8(x)&=\frac{1}{4608\pi^4}g^{-1/2}\epsilon^{\mu_1\cdots\mu_8}R^{\nu_1}{}_{\nu_2\mu_1\mu_2}R^{\nu_2}{}_{\nu_1\mu_3\mu_4}R^{\rho_1}{}_{\rho_2\mu_5\mu_6}R^{\rho_2}{}_{\rho_1\mu_7\mu_8}\\ &\quad-\frac{7}{11\,520\pi^4}g^{-1/2}\epsilon^{\mu_1\cdots\mu_8}R^{\nu_1}{}_{\nu_2\mu_1\mu_2}R^{\nu_2}{}_{\nu_3\mu_3\mu_4}R^{\nu_3}{}_{\nu_4\mu_5\mu_6}R^{\nu_4}{}_{\nu_1\mu_7\mu_8}\ .\end{aligned} \tag{2.38}$$

If we integrate (2.37) over a manifold which has no boundary so that the $\nabla_\mu\langle K^\mu\rangle$ term does not contribute, we recover the signature index theorem

$$n_+-n_-=\int d^{4n}x\sqrt{g}B_{4n}(x)\ , \tag{2.39}$$

where n_+ and n_- denote the number of self-dual and anti-self-dual zero modes. For $n=1$ we have found

$$\begin{aligned}\nabla_\mu\langle K^\mu\rangle&=\tfrac{1}{2}\langle F_{\mu\nu}{}^*F^{\mu\nu}\rangle\\ &=\frac{1}{192\pi^2}g^{-1/2}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta\rho\sigma}\\ &\quad-\tfrac{1}{2}\sum_i(\phi_i,*\phi_i)\ .\end{aligned} \tag{2.40}$$

This differs from the result (1.2) for Minkowski space by the zero-mode term. The situation is similar to the case of the fermionic anomaly. In Euclidean space the complete form of the axial-vector divergence reads[18]

$$\nabla_\mu\langle j_5^\mu\rangle=-\frac{1}{384\pi^2}g^{-1/2}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta\rho\sigma}+\frac{1}{8\pi^2}\mathrm{tr}(F_{\mu\nu}{}^*F^{\mu\nu})+2\sum_i\psi_{0i}^\dagger\gamma_5\psi_{0i}\ , \tag{2.41}$$

where the ψ_{0i}'s are the zero modes of the Dirac operator $\not{D}$. Integrating equation (2.41) over a manifold with $\partial\mathcal{M}=\varnothing$ one reproduces the index theorem for the spin complex. In analogy we therefore expect that in Minkowski space $\nabla_\mu\langle K^\mu\rangle$ is given by $\frac{1}{2}B_{4n}(x)$ for all n.

III. THE ANOMALOUS JACOBIAN

In this section we show that the anomaly (2.40) can be understood as arising from a nontrivial Jacobian of the path-integral measure. The problem one encounters in this approach is that the relevant transformations, the dual rotations (2.9) or (2.10), are defined in terms of F rather than in terms of the integration variable A. One possibility to avoid this complication is to use the first-order formalism.[19] Here we use a different approach. We define a chiral transformation of A by the requirement

$$d(\delta_\alpha A)=\alpha*dA\ . \tag{3.1}$$

This transformation law guarantees that (2.10) is satisfied by $F=dA$. As we shall see below, for our purposes it is not necessary to solve this equation for $\delta_\alpha A$. Following Fujikawa,[7] the derivation of the anomaly proceeds as follows. Consider the path integral

$$Z\equiv\int[\mathcal{D}A]_{\mathrm{LG}}\exp\left[-\tfrac{1}{2}\int(dA,dA)\epsilon\right]\ , \tag{3.2}$$

and change the integration variable from A to $A'=A+\delta_\alpha A$, where the parameter α is allowed to depend on the space-time point x. Then the classical action changes by $\int\alpha(F,*F)\epsilon$. For the Jacobian we make the ansatz

$$[\mathcal{D}A]'_{\rm LG}=\exp\left[\int d^{4n}x\sqrt{g}\,\alpha(x)\mathcal{A}_{4n}(x)\right][\mathcal{D}A]_{\rm LG}\ . \quad (3.3)$$

Because Z is independent of α, one finds the "Ward identity"

$$\int[\mathcal{D}A]_{\rm LG}[(F(x),*F(x))-\mathcal{A}_{4n}(x)]e^{-S}=0 \quad (3.4a)$$

or

$$\langle (F(x),*F(x))\rangle=\mathcal{A}_{4n}(x)\ . \quad (3.4b)$$

From (3.4) we learn that $\langle (F,*F)\rangle\neq 0$ is equivalent to $\mathcal{A}_{4n}\neq 0$, i.e., to a quantum-mechanical breaking of the symmetry (2.10). To compute $\mathcal{A}_{4n}$, we first expand the gauge field as

$$A(x)=\sum_i c_i a_i(x)\ , \quad (3.5)$$

where the $(2n-1)$-forms a_i are a complete set of basis vectors which are orthonormalized as in (2.18). For convenience we choose them as eigenvectors of the Laplacian. Furthermore, because the integral (3.2) is only over fields satisfying the Lorentz gauge condition, they are constrained to satisfy $\delta a_i=0$. Thus one has

$$[\mathcal{D}A]_{\rm LG}=\prod_i dc_i\ . \quad (3.6)$$

Under $A\rightarrow A'$ the coefficients c_i change according to

$$c_i\rightarrow c_i'=\sum_j\int(a_i,[1+\delta_\alpha]a_j)\epsilon c_j$$
$$\equiv\sum_j(\delta_{ij}+M_{ij})c_j\ ; \quad (3.7)$$

i.e., the Jacobian for an infinitesimal transformation is given by

$$\det M=\exp[\mathrm{Tr}\ln(1+M)]$$
$$=\exp\left[\sum_i\int(a_i,\delta_\alpha a_i)\epsilon\right]\ . \quad (3.8)$$

To be able to apply (3.1), we exploit the fact that the a_i's are eigenfunctions of Δ (with eigenvalues λ_i) and satisfy $\delta a_i=0$ to write

$$a_i=\lambda_i^{-1}(d\delta+\delta d)a_i=\lambda_i^{-1}\delta d a_i\ .$$

Inserting this in (3.8), integrating by parts, and using (3.1) one obtains

$$\sum_i\int(a_i,\delta_\alpha a_i)\epsilon=\sum_i\lambda_i^{-1}\int(\delta d a_i,\delta_\alpha a_i)\epsilon$$
$$=\sum_i\lambda_i^{-1}\int(da_i,d\delta_\alpha a_i)\epsilon$$
$$=\sum_i\lambda_i^{-1}\int(da_i,\alpha * da_i)\epsilon\ . \quad (3.9)$$

Again introducing the differently normalized function α_i of (2.21), we find, by comparison with (3.3),

$$\mathcal{A}_{4n}(x)=\sum_i(d\alpha_i(x),*d\alpha_i(x))\ . \quad (3.10)$$

Because of the completeness of the α_i's this is an ill-defined quantity of the form $0\times\infty$. (This is analogous to the sum $\sum_i\varphi_i^\dagger\gamma_5\varphi_i$ for the fermionic case.) We regularize it by introducing a Gaussian cutoff:

$$\mathcal{A}_{4n}(x)=\lim_{M\to\infty}\sum_i(d\alpha_i,*e^{-\lambda_i/M^2}d\alpha_i)\ . \quad (3.11)$$

According to the discussion which led to (2.36) this is the same as

$$\mathcal{A}_{4n}(x)=\lim_{M\to\infty}\tfrac{1}{2}\left[K(x;1/M^2)-\sum_i(\phi_i(x),*\phi_i(x))\right]$$
$$=\tfrac{1}{2}B_{4n}(x)-\tfrac{1}{2}\sum_i(\phi_i(x),*\phi_i(x))\ . \quad (3.12)$$

This shows that, contrary to the claims in Ref. 2, Fujikawa's method is very well applicable even in a first-order formulation. The result coincides with that of the ζ-function method and the calculation clearly displays the analogy with the fermionic anomaly.

IV. STOCHASTIC QUANTIZATION

Recently stochastic quantization[8] received much attention as an alternative to the usual canonical or path-integral quantization. One of the reasons might have been that this scheme provides a new type of invariant regularization which was hoped to respect simultaneously all symmetries of the field-theory model under consideration. Later on it turned out, however, that the anomalies associated with continuous symmetries also appear within stochastic quantization.[20,21] It is only in the case of the parity-violating anomaly in $2n+1$ dimensions that an ambiguity has been observed.[22] In this section we are going to explicitly show that the (continuous) duality anomalies, too, are unambiguously reproduced in the framework of stochastic quantization. Our essential tool will be the stochastic regulator function introduced by Breit, Gupta, and Zaks.[23]

The basic ingredients to start with are the Langevin equation derived from the action (2.6),

$$\frac{\partial}{\partial\tau}A(x,\tau)=-\delta d A(x,\tau)+\eta(x,\tau)\ , \quad (4.1)$$

and the correlation function for the random source $\eta(x,\tau)$:

$$\langle\eta_{\mu_1\cdots\mu_{2n-1}}(x,\tau)\eta^{\nu_1\cdots\nu_{2n-1}}(x',\tau')\rangle_\eta=2g^{-1/2}(x)g^{\nu_1}_{[\mu_1}g^{\nu_2}_{\mu_2}\cdots g^{\nu_{2n-1}}_{\mu_{2n-1}]}\delta(x-x')a_\Lambda(\tau-\tau')\ . \quad (4.2)$$

Being a source for $A(x,\tau)$, the noise $\eta(x,\tau)$ is a $(2n-1)$-form, also. In (4.2) we introduced the Breit-Gupta-Zaks regulator $a_\Lambda(\tau-\tau')$ defined by the properties[23]

$$\lim_{\Lambda\to\infty} a_\Lambda(\tau-\tau')=\delta(\tau-\tau') ,$$
$$a_\Lambda(\tau-\tau')=a_\Lambda(\tau'-\tau) , \tag{4.3}$$
$$\int d\tau' a_\Lambda(\tau-\tau')=1 .$$

The limit $\Lambda\to\infty$ will be performed after all calculations have been done. As is well known, the stochastic quantization of gauge theories does not necessarily require the fixing of a gauge,[24] but, nevertheless, it can be computationally advantageous to fix a gauge. In our case we would like to do this in a way so that the Langevin equation contains the complete Laplacian rather than the δd operator only. To achieve this we perform a gauge transformation which depends on the stochastic time τ:

$$A'(x,\tau)=A(x,\tau)+d\mathcal{X}(x,\tau) . \tag{4.4}$$

This transformation changes the form of the Langevin equation, but it does not change any gauge-invariant expectation value calculated from it.[8,24] Choosing $\mathcal{X}$ to satisfy

$$\frac{\partial}{\partial\tau}d\mathcal{X}+d\delta(A+d\mathcal{X})=0 , \tag{4.5}$$

the new Langevin equation has the desired form

$$\frac{\partial}{\partial\tau}A'(x,\tau)=-\Delta A'(x,\tau)+\eta(x,\tau) . \tag{4.6}$$

A possible solution of (4.5) is

$$\mathcal{X}(x,\tau)=-\int_0^\tau d\tau' e^{-\Delta(\tau-\tau')}\delta A(x,\tau') . \tag{4.7}$$

For notational simplicity we do the following calculations for $n=1$; the generalization will be obvious. To solve Eq. (4.6), we define

$$G_\mu^\nu(x,x';\tau)=\langle \mu,x\,|\,e^{-\tau\Delta}\,|\,\nu,x'\rangle . \tag{4.8}$$

The solution of (4.6) with the initial condition $A(x,0)=0$ then reads

$$A_\mu^\eta(x,\tau)=\int_0^\tau d\tau' \int d^4x'\sqrt{g}\,G_{\mu\nu}(x,x';\tau-\tau')\eta^\nu(x',\tau') . \tag{4.9}$$

(We omit the prime from A again.) Employing this solution, we can compute

$$\langle (F(x),*F(x))\rangle=\tfrac{1}{4}g^{-1/2}\epsilon^{\mu\nu\rho\sigma}\times\lim_{x'\to x}\frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x^\rho}\times\langle A_\nu(x')A_\sigma(x)\rangle \tag{4.10}$$

by straightforwardly evaluating the expectation value:

$$\begin{aligned}\langle A_\nu(x')A_\sigma(x)\rangle &= \lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\langle A_\nu^\eta(x',\tau)A_\sigma^\eta(x,\tau)\rangle_\eta\\ &= \lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\int_0^\tau d\tau_1\int_0^\tau d\tau_2\int d^4y_1\sqrt{g(y_1)}\int d^4y_2\sqrt{g(y_2)}G_{\nu\alpha}(x',y_1;\tau-\tau_1)G_{\sigma\beta}(x,y_z;\tau-\tau_2)\\ &\qquad\times\langle \eta^\alpha(y_1,\tau_1)\eta^\beta(y_2,\tau_2)\rangle_\eta\\ &= \lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\int_0^\tau d\tau_1\int_0^\tau d\tau_2 2a_\Lambda(\tau_1-\tau_2)G_{\nu\sigma}(x',x;2\tau-\tau_1-\tau_2) .\end{aligned} \tag{4.11}$$

Inserting into (4.11) yields

$$\langle (F(x),*F(x))\rangle=\lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\int_0^\tau d\tau_1\int_0^\tau d\tau_2 2a_\Lambda(\tau_1-\tau_2)\sum_i (da_i(x),*da_i(x))e^{-\lambda_i(2\tau-\tau_1-\tau_2)} . \tag{4.12}$$

Here the heat kernel (4.8) has been represented in terms of a complete set of orthonormalized eigenfunctions of the Laplacian:

$$G^{\mu\nu}(x,x';\tau)=\sum_i a_i^\mu(x)a_i^\nu(x')e^{-\lambda_i\tau} . \tag{4.13}$$

Equation (4.12) holds for all values of $n\geq 1$ again. The following steps parallel Tzani's[21] treatment of the fermionic case. Introducing

$$t=\tau_1-\tau_2,\quad T=\tfrac{1}{2}(\tau_1+\tau_2) ,$$

we have

$$\langle (F,*F)\rangle=\lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\left[\int_0^{\tau/2}dT\int_{-2T}^{+2T}dt+\int_{\tau/2}^{\tau}dT\int_{-2(\tau-T)}^{+2(\tau-T)}dt\right]2a_\Lambda(t)\sum_i (da_i,*da_i)e^{-\lambda_i(2\tau-\tau_1-\tau_2)} . \tag{4.14}$$

The properties (4.3) imply, for the regulator functions,

$$\int_{-2T}^{2T}dt\,a_\Lambda(t)=\Theta\left[T-\frac{1}{2\Lambda^2}\right]+O\left[\frac{1}{\Lambda^2}\right],\quad \int_{-2(\tau-T)}^{+2(\tau-T)}dt\,a_\Lambda(t)=\Theta\left[\tau-\frac{1}{2\Lambda^2}-T\right]+O\left[\frac{1}{\Lambda^2}\right] . \tag{4.15}$$

Inserting this into (4.15) and performing the T integration yields

$$\langle (F,*F)\rangle = \lim_{\Lambda\to\infty}\lim_{\tau\to\infty}\sum_i \lambda_i^{-1}(da_i,*da_i)(e^{\lambda_i(2\tau-1/\Lambda^2)}-e^{\lambda_i/\Lambda^2})e^{-2\lambda_i\tau}\ . \tag{4.16}$$

At this point the τ limit may be performed:

$$\langle (F,*F)\rangle = \lim_{\Lambda\to\infty}\sum_i \lambda_i^{-1}(da_i,*da_i)e^{-\lambda_i/\Lambda^2}\ . \tag{4.17}$$

(To obtain strictly positive eigenvalues λ_i one could introduce a small "photon" mass as in Sec. II.) Finally, we can change the normalization of the a_i's according to (2.21) and (2.22). What we find is

$$\langle (F,*F)\rangle = \lim_{\Lambda\to\infty}\sum_i (d\alpha_i,*e^{-\lambda_i/\Lambda^2}d\alpha_i)\ , \tag{4.18}$$

i.e., the same expression as in Fujikawa's approach, cf. Eq. (3.11). Obviously, Fujikawa's prescription for cutting off the large eigenvalues is equivalent to using the Breit-Gupta-Zaks regularized version of the noise-noise correlation function. Hence the stochastic quantization scheme leads to the same anomaly as the path-integral or ζ-function method. This gives further support to the assumption that the stochastic method correctly reproduces all anomalies associated with continuous symmetries.

V. CONCLUSIONS

Using various independent methods, we have established the Euclidean analogue of relation (1.2) for any arbitrary $4n$-dimensional space-time. The result (2.37) differs from the Minkowski-space version of the anomaly equation by the zero-mode terms. Their presence allows the anomalous divergence equation to be interpreted as a local version of the signature index theorem. In the path-integral formalism the anomaly manifests itself in a nontrivial Jacobian for the duality transformations. This result has been established within the second-order formulation of the quantum theory, where duality transformations are defined by (3.1). We also found that the anomaly is correctly reproduced by the stochastic quantization procedure and that the stochastic regulator a_Λ is equivalent to Fujikawa's prescription. All these properties of the duality, or chiral anomalies of antisymmetric tensor fields are very similar to those of spinor fields.

To close, let us ask for possible physical applications of (1.2) or its generalizations. An example of a metric for which the pseudoscalar $\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta\rho\sigma}$ does not vanish is the metric of a rotating mass distribution, a rotating star, say. This means that such a star permanently creates photons and thereby reduces its angular momentum in very much the same way as a dyon produces fermion pairs via (1.3) and thereby reduces its electric charge.[3] It is important to note that this radiation has nothing to do with the familiar Hawking radiation or the gravitational particle creation in expanding universes.[25] These phenomena have their origin in a nontrivial Bogoliubov transformation between the creation and annihilation operators used by different observers: the vacuum of one observer can correspond to a state with a nonzero particle number for another observer. On the other hand, the (pseudo)scalar $\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta\rho\sigma}$ which is responsible for the anomalous photon creation, cannot be arranged to vanish by employing a particular vacuum state; it is the same for any observer.

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