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Parity-violating anomalies and the stationarity of stochastic averages

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Within the framework of stochastic quantization the parity-violating anomalies in odd space-time dimensions are derived from the asymptotic stationarity of the stochastic average of a certain fermion bilinear. Contrary to earlier attempts, this method yields the correct anomalies for both massive and massless fermions.

Recently the derivation of anomalies within the framework of stochastic quantization[1] has been studied intensively. In the case of chiral anomalies it has been shown that even by using the stochastic regularization scheme[2] it is not possible to maintain both chiral symmetry and gauge invariance as intact symmetries at the quantum level and that the anomalies known from standard field theory are also present in stochastic quantization.[3] The situation is less clear in the case of the parity-violating anomalies in odd dimensions which were first discussed by Redlich.[4–6] He demonstrated that similar to the conflict between gauge and chiral invariance in even dimensions there is a conflict between parity and gauge invariance in theories of fermions interacting with gauge fields in odd dimensions. Using a gauge-invariant regularization scheme, say, the vacuum current induced by an external Yang-Mills field contains a parity-breaking piece, which is responsible for the (possibly fractional) vacuum charge and for the quantum Hall effect of the vacuum.[7] This is true for both massive and massless fermions. The effective action corresponding to the anomalous part of the current is given by the Chern-Simons term of the respective dimensionality.[5,6] In Ref. 8, henceforth referred to as I, we showed that for massive fermions this anomaly is unambiguously reproduced by the stochastic quantization procedure. This was proven by explicitly solving the fermionic Langevin equation and then calculating the vacuum charge

$$Q_{2n+1}=\int d^{2n}x\,\langle 0\,|\,\bar{\psi}(x)\gamma^0\psi(x)\,|\,0\rangle \tag{1}$$

by expressing the field-theory expectation value in the usual way in terms of an average over the white noise. (The dimensionality of space-time is assumed to be $2n+1$.) Doing the same for massless fermions an inconsistency is found. It turns out that the dimensional stochastic time τ acts as an additional IR cutoff which has no analogue in standard field theory. This allows us to regularize the theory in a way so as to obtain no anomaly for massless fermions. On the other hand, using Pauli-Villars regularization and computing the contribution of the regulator field to the effective action by stochastic quantization too, one finds the same anomaly as in the massive case. This is the same type of inconsistency which also has been found by Nissimov and Pacheva[9] using a different approach.

Recently Namiki, Ohba, Tanaka, and Yanga[10] proposed an interesting new method to obtain the ordinary (chiral) anomalies from stochastic quantization. It is the purpose of this paper to apply their strategy to the parity-violating anomalies in odd dimensions. Let us briefly recall the main ingredients of this approach which is based on Ito's stochastic differential calculus.[11] For a theory of fermion fields $\psi(x,\tau)$ and $\bar{\psi}(x,\tau)$ which is defined by an action S, the basic stochastic differential equations read

$$d\psi_\alpha(x,\tau)=-\frac{\delta S}{\delta\bar{\psi}_\alpha(x,\tau)}d\tau+d\theta_\alpha(x,\tau)\ , \tag{2a}$$

$$d\bar{\psi}_\alpha(x,\tau)=\frac{\delta S}{\delta\psi_\alpha(x,\tau)}d\tau+d\bar{\theta}_\alpha(x,\tau)\ , \tag{2b}$$

where α is a spinor index. The differentials of the Grassmann random sources $\theta(x,\tau)$ and $\bar{\theta}(x,\tau)$, which are a fermionic analogue of the Wiener process, have the averages

$$\langle d\theta_\alpha\rangle=\langle d\bar{\theta}_\alpha\rangle=0\ , \tag{3a}$$

$$\langle d\theta_\alpha d\theta_\beta\rangle=\langle d\bar{\theta}_\alpha d\bar{\theta}_\beta\rangle=0\ , \tag{3b}$$

$$\langle d\theta_\alpha(x,\tau)d\bar{\theta}_\beta(x',\tau)\rangle=2\delta_{\alpha\beta}\delta(x-x')d\tau\ . \tag{3c}$$

It is important to note that any functional F of the variables ψ and $\bar{\psi}$ satisfies

$$\langle F[\psi(\tau),\bar{\psi}(\tau)]d\theta_\alpha(x,\tau)\rangle =\langle F[\psi(\tau),\bar{\psi}(\tau)]d\bar{\theta}_\alpha(x,\tau)\rangle=0\ . \tag{4}$$

This follows from (3a) together with the fact that $F[\psi(\tau),\bar{\psi}(\tau)]$ is a nonanticipating function of τ (Ref. 11). Applying the rules of Ito's calculus, it is easy to derive a stochastic differential equation for the functional F itself:[12]

$$dF(\tau)=\int dx\left[-\frac{\delta S(\tau)}{\delta\bar\psi_\alpha(x,\tau)}\frac{\delta F(\tau)}{\delta\psi_\alpha(x,\tau)}+\frac{\delta S(\tau)}{\delta\psi_\alpha(x,\tau)}\frac{\delta F(\tau)}{\delta\bar\psi_\alpha(x,\tau)}\right]d\tau$$
$$+\int dx\left[d\theta_\alpha(x,\tau)\frac{\delta F(\tau)}{\delta\psi_\alpha(x,\tau)}+d\bar\theta_\alpha(x,\tau)\frac{\delta F(\tau)}{\delta\bar\psi_\alpha(x,\tau)}\right]$$
$$+\int dx\,dy\left[d\theta_\alpha(x,\tau)d\bar\theta_\beta(y,\tau)\frac{\delta^2F(\tau)}{\delta\bar\psi_\beta(y,\tau)\delta\psi_\alpha(x,\tau)}+\tfrac{1}{2}d\theta_\alpha(x,\tau)d\theta_\beta(y,\tau)\frac{\delta^2F(\tau)}{\delta\psi_\beta(y,\tau)\delta\psi_\alpha(x,\tau)}\right.$$
$$\left.+\tfrac{1}{2}d\bar\theta_\alpha(x,\tau)d\bar\theta_\beta(y,\tau)\frac{\delta^2F(\tau)}{\delta\bar\psi_\beta(y,\tau)\delta\bar\psi_\alpha(x,\tau)}\right]. \quad (5)$$

(The functional derivatives are understood to be left derivatives.) A further point which will be important is that in the equilibrium state for $\tau\to\infty$ the average of F will become stationary, i.e.,

$$\lim_{\tau\to\infty}\langle dF(\tau)\rangle=0\ . \quad (6)$$

A formal proof can be found in Ref. 10. The averaged version of equation (5) was used by Namiki, Ohba, Tanaka, and Yanga to derive the chiral anomaly. Their choice for the functional F was

$$F=\bar\psi(x,\tau)\gamma_5\psi(x,\tau)\ .$$

Inserting this on the right-hand side (RHS) of (5) yields for the first integral precisely the axial-vector divergence $\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi)$, the second one vanishes according to (4) when the average is taken, and the third integral, finally, turned out to be the anomaly term. Because for $\tau\to\infty$ the LHS of the averaged equation vanishes, one thus recovers the usual anomalous divergence relation of the axial-vector current from the stationarity property of the pseudoscalar $\langle\bar\psi\gamma_5\psi\rangle$.

Now let us turn to the parity-violating anomalies in $2n+1$ dimensions. We first consider massive (Euclidean) Dirac fermions interacting with a topologically nontrivial background Yang-Mills field $A_\mu=A_\mu^aT^a$, where T^a are the gauge group generators. The action is given by

$$S=\int d^{2n+1}x\ \bar\psi(i\not{D}-m)\psi\ , \quad (7)$$

where $\not{D}=\gamma^\mu(\partial_\mu+iA_\mu)$. [We use the same conventions as in I; in particular, we write $x^\mu=(x^0,x^k)=(x^0,\mathbf{x})$ with $i,j,k,\ldots=1,\ldots,2n$ and $\mu,\nu,\rho,\ldots=0,\ldots,2n$.] To detect the anomalous term in the vacuum current, it is sufficient to calculate the induced charge (1) for static magnetic background fields, since it is known[5,6] that the parity-even part of the current does not contribute to the vacuum charge. Therefore we may set $A^0=0$ and $A^k=A^k(x^i)$. For the functional F whose stationarity property is to be exploited we make the ansatz

$$F(\tau)=\int d^{2n+1}x\ \bar\psi(x,\tau)G\psi(x,\tau)\ . \quad (8)$$

The operator G is implicitly defined by the relation

$$G(i\not{D}-m)+(i\not{D}-m)G=\gamma^0\ . \quad (9)$$

This equation could be solved by introducing appropriate Green's functions. It turns out, however, that the explicit solution will not be needed. The reason for this definition of F is that, when inserted into the first integral on the RHS of the stochastic differential equation (5) for the action (7), it essentially yields the vacuum charge (1):

$$\langle dF(\tau)\rangle=-\int d^{2n+1}x\langle\bar\psi(x,\tau)\gamma^0\psi(x,\tau)\rangle d\tau$$
$$-\int d^{2n+1}x\langle d\bar\theta(x,\tau)G\,d\theta(x,\tau)\rangle$$
$$\equiv -d\tau\int dx^0Q_{2n+1}(\tau)+\mathcal{A}\ . \quad (10)$$

Note that because of (4) the terms linear in $d\theta$ and $d\bar\theta$ vanish upon taking the average of Eq. (5). Since the LHS of (10) vanishes for $\tau\to\infty$, we suspect the anomaly to be contained in the quantity

$$\mathcal{A}\equiv-\int d^{2n+1}x\langle d\bar\theta(x,\tau)G\,d\theta(x,\tau)\rangle\ . \quad (11)$$

(Since all expectation values are time independent, we restrict the x^0 integration to a finite interval.) As we shall see below, in the present form $\mathcal{A}$ does not yet have a well-defined meaning and hence must be regularized. We do this by smearing out the δ function[10] of the correlation function (3c):

$$\langle d\theta_\alpha(x,\tau)d\bar\theta_\beta(x',\tau)\rangle$$
$$=2\,d\tau\lim_{\Lambda\to\infty}\sum_i\phi_{i\beta}^\dagger(x')e^{-\lambda_i^2/\Lambda^2}\phi_{i\alpha}(x)\ . \quad (12)$$

This corresponds to an *a priori* regularization of the continuum regularization program developed by Bern, Chan, and Halpern.[3] Here $\{\phi_i\}$ denotes a complete set of orthonormalized eigenfunctions of the Dirac operator: $\not{D}\phi_i=\lambda_i\phi_i$. Hence (11) yields

$$\mathcal{A}=2\,d\tau \lim_{\Lambda\to\infty} \sum_i e^{-\lambda_i^2/\Lambda^2} \int d^{2n+1}x\, \phi_i^\dagger(x) G \phi_i(x)$$

$$=d\tau \lim_{\Lambda\to\infty} \sum_i e^{-\lambda_i^2/\Lambda^2} \frac{1}{i\lambda_i - m} \int d^{2n+1}x\, \phi_i^\dagger(x)\gamma^0\phi_i(x)$$

$$=-d\tau \lim_{\Lambda\to\infty} \int d^{2n+1}x \left\langle x \left| \mathrm{tr}\gamma^0 \{ i\not{D} + m \} \frac{1}{\not{D}^2+m^2} e^{-\not{D}^2/\Lambda^2} \right| x \right\rangle . \tag{13}$$

To obtain the second line of (13), relation (9) and the eigenvalue equation for $\not{D}$ have been used. We note that the $\not{D}=\gamma^0\partial_0+\gamma^k D_k$ part in the curly brackets in the last line of (13) does not contribute to $\mathcal{A}$. The first term vanishes because, when going to momentum space, it becomes odd in k_0 and the second one gives no contribution since it anticommutes with γ^0. Using an integral representation for the inverse of $\not{D}^2+m^2$, we thus have found

$$\mathcal{A}=-m\, d\tau \lim_{\Lambda\to\infty} \int d^{2n+1}x \int_0^\infty dw \langle x \,|\, \mathrm{tr}\gamma^0 e^{-(\not{D}^2+m^2)(w+\Lambda^{-2})} \,|\, x \rangle . \tag{14}$$

If we note that $\not{D}^2=-\partial_0^2+\not{D}_{2n}^2$ where

$$\not{D}_{2n}=\gamma^k[\partial_k+iA_k(x^j)]$$

is independent of x^0 and that, upon introducing $s=w+\Lambda^{-2}$,

$$\langle x^0 \,|\, e^{\partial_0^2 s} \,|\, x^0 \rangle = (4\pi s)^{-1/2} ,$$

we may write

$$\mathcal{A}=-m(4\pi)^{-1/2}d\tau \times \lim_{\Lambda\to\infty} \int_{\Lambda^{-2}}^\infty ds\, e^{-m^2 s} s^{-1/2} \times \int d^{2n+1}x \langle \mathbf{x} \,|\, \mathrm{tr}\gamma^0 e^{-\not{D}_{2n}^2 s} \,|\, \mathbf{x} \rangle . \tag{15}$$

Next we exploit that, since γ^0 anticommutes with $\not{D}_{2n}$,

$$\int d^{2n}x \langle \mathbf{x} \,|\, \mathrm{tr}\gamma^0 e^{-\not{D}_{2n}^2 s} \,|\, \mathbf{x} \rangle \equiv \mathrm{Tr}(\gamma^0 e^{-\not{D}_{2n}^2 s}) \tag{16}$$

is the index of the Dirac operator $\not{D}_{2n}$, which is given by the Chern character[13] of the gauge field $A_i(x^k)$ with the field-strength form $F=(i/2)F_{ij}dx^i dx^j$:

$$\mathrm{index}\, \not{D}_{2n} = \int \exp\left[\frac{i}{2\pi}F\right] . \tag{17}$$

(We use the standard differential form notation.[13]) In I this was shown explicitly by applying Fujikawa's method[14] to Eq. (15). Thus (10), (15), and (16) imply

$$Q_{2n+1}(\tau)=-m(4\pi)^{-1/2} \lim_{\Lambda\to\infty} \int_{\Lambda^{-2}}^\infty ds\, e^{-m^2 s} s^{-1/2} \times \mathrm{index}\, \not{D}_{2n} - \frac{d}{d\tau}\langle \tilde{F}(\tau) \rangle , \tag{18}$$

where $\tilde{F}$ is obtained from F by omitting the x^0 integration. Equation (18) describes the τ evolution of the vacuum charge. Obviously, the anomaly does not depend on the stochastic time; the only τ dependence arises through the time derivative of $\langle \tilde{F}(\tau) \rangle$ which vanishes in the infinite-τ limit. After a trivial integration we therefore arrive at

$$Q_{2n+1} = \lim_{\tau\to\infty} Q_{2n+1}(\tau) = -\frac{1}{2}\frac{m}{|m|} \times \mathrm{index}\, \not{D}_{2n} . \tag{19}$$

This is the correct result for massive fermions. Here we concentrated our discussion on the vacuum charge; however, it is well known that $Q_{2n+1}\neq 0$ is equivalent to the presence of the parity-odd part in the vacuum current and of a Chern-Simons term in the Heisenberg-Euler effective action.[5,6] Hence we may conclude that the present approach correctly reproduces the anomaly in the case of massive fermions.

We now come to the discussion of massless fermions. It is here that we will find a crucial difference to the more standard approach of I where the Langevin equations were solved explicitly and the solution was inserted into

$$Q_{2n+1}(\tau)=\int d^{2n}x \langle \bar{\psi}(x,\tau)\gamma^0\psi(x,\tau) \rangle . \tag{20}$$

The stochastic evolution of Q_{2n+1} was found to be given by

$$Q_{2n+1}(\tau)=-m(4\pi)^{-1/2} \lim_{\Lambda\to\infty} \int_{\Lambda^{-2}}^{2\tau-\Lambda^{-2}} ds\, e^{-m^2 s} s^{-1/2} \times \mathrm{index}\, \not{D}_{2n} . \tag{21}$$

For $m\neq 0$ this equation yields the correct result (19). On the other hand, in the derivation of (21) it was nowhere used that $m\neq 0$. Hence the anomaly of massless fermions is obtained by setting $m=0$ in Eq. (21). For finite values of τ, the integral exists even without the exponential factor, so that we have $Q_{2n+1}(\tau)=0$ for all τ. Taking the limit $\tau\rightarrow\infty$, we would conclude that there is no anomaly. This is clearly inconsistent, because, as we discussed in I, even within stochastic quantization there is a regularization scheme with $Q_{2n+1}\neq 0$. We can introduce a Pauli-Villars regulator field of mass M and regularize the effective action (calculated via stochastic quantization) by subtracting its contribution for $M\rightarrow\infty$ (Ref. 4). What one finds is (19) with m replaced by M. In I we attributed this inconsistency to the fact that the dimensionful stochastic time τ acts as an IR cutoff for the proper-time integral in (21), which has no analogue in standard field theory. There, in all comparable regularization schemes (ζ function, heat kernel), the upper limit is equal to infinity from the outset, so that we are not allowed to set $m=0$ in the integrand and thus are forced to use Pauli-Villars regularization for $m=0$.

Let us now come back to the approach based on the asymptotic stationarity of $\langle F(\tau)\rangle$. The important difference between (18) and (21) is that in the present formulation the anomaly is τ independent. The proper-time integral in (15) or (18) ranges to infinity even for finite stochastic time and therefore it is not possible to set $m=0$ in (21) (Ref. 15). Hence massless fermions cannot be treated in the way described above. This means that we are now *forced* to use a different regularization, such as a Pauli-Villars regulator, say, to allow for a well-defined determination of the vacuum charge. Then our calculation applies to this massive regulator field (cf. Redlich[4]) and hence the anomaly is recovered. This shows that, contrary to the naive approach to stochastic quantization used in I, the present method unambiguously reproduces the parity-violating anomaly for both massive and massless fermions.

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[15] This can be traced back to Eq. (13) where the transition from the first to the second line is not allowed even if we could invent a regularization prescription which yields a well-defined solution of (9) for $m=0$.