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SCALING LAWS AND TRIVIALITY BOUNDS IN THE LATTICE ϕ^4 THEORY
(II). One-component model in the phase with spontaneous symmetry breaking

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Our earlier analysis of the lattice ϕ^4 theory in four dimensions is extended to a neighborhood of the critical line in the broken symmetry phase, which includes the "scaling region" characterized by $am_R \leqslant 0.5$ (a: lattice spacing, m_R: renormalized particle mass). As in the symmetric phase, the renormalized coupling g_R at zero momentum is bounded by a function of the cutoff $\Lambda = 1/a$, which decreases logarithmically as $\Lambda \to \infty$. The maximal possible value of g_R in the scaling region is found to be about $\frac{2}{3}$ of the tree level unitarity bound, i.e. the coupling is never really strong and bound state particles, if they exist at all, are expected to have a small binding energy. In terms of the renormalized vacuum expectation value v_R of the field ϕ, the upper bound on the coupling corresponds to $m_R \leqslant 3.2 v_R$.

1. Introduction

Although the lattice ϕ^4 theory in four dimensions is a free field theory in the continuum limit, the particle interactions at low energies need not a priori be weak when the ultra-violet cutoff $\Lambda = 1/a$ (a: lattice spacing) is finite and reasonably large compared to the renormalized mass m_R. Still, for the one-component model in the symmetric phase, we have recently been able to show [1] that the renormalized coupling g_R at zero momentum is always smaller than about $\frac{2}{3}$ of the tree level unitarity bound whenever $\Lambda \geqslant 2m_R$. Thus, the interactions are never really strong in this model and renormalized perturbation theory may be applied to calculate the scattering matrix at low energies, for example. Using a combination of conventional analytic methods, many further results on the model in the symmetric phase were

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obtained in ref. [1] and these could be directly compared with data produced by a large scale numerical simulation [2] at infinite bare coupling (which is the most difficult case for the analytic approach). A complete matching was observed and this then leaves little doubt about the validity of the (weak) qualitative assumptions on which our analysis was based.

In this paper, we consider the one component ϕ^4 theory in the phase, where the reflection symmetry $\phi \to -\phi$ is spontaneously broken. The motivation and goals are the same as for our earlier work on the symmetric phase, in particular, we shall determine the renormalization group trajectories (curves of constant coupling g_R) in the plane of bare parameters and the maximum value of Λ/m_R for given g_R (the triviality bound). Also we have again taken care to present the results in a form which can be readily compared with data from future Monte Carlo simulations.

At weak coupling, the ϕ^4 theory in the broken symmetry phase describes a single massive particle with local 3- and 4-body interactions. Through one-particle exchange processes, these interactions give rise to an attractive force between the particles, which has a range equal to m_R^{-1}. This is quite different from the symmetric phase, where the interactions are repulsive and have only half this range (at least two particles must be exchanged in this case). At strong coupling, the low energy properties of the model are therefore likely to be more complicated in the broken symmetry phase than in the symmetric phase, in particular, it is conceivable that bound state particles form.

In the Ising model limit of the theory such bound states do in fact occur as the following simple consideration shows. Deep in the broken symmetry phase (at very low "temperatures" in other words), a one particle state may be described as a single spin, which is flipped relative to the ground state magnetization. Similarly, a two-particle state is obtained by flipping two spins. When the flipped spins are separated by a distance greater than one, there is, to a first approximation, no interaction between them (the Ising hamiltonian only couples nearest neighbors) and the two particles are hence unbound. The situation is different if the spins are at a distance one, because the number of opposite nearest neighbor spin pairs is reduced by 2 in this case. Such states therefore describe two-particle bound states with a binding energy equal to 2 spin interaction units.

The above example shows that bound states exist in a limit, where the cutoff Λ is very low. For larger values of Λ, the triviality bound sets an upper limit on the coupling g_R and the attraction between particles may then no longer be sufficiently strong for the formation of stable bound states. On grounds of the results obtained in this paper, we shall in fact argue that for $\Lambda \gtrsim 10 m_R$ such states are unlikely to be present. In the intermediate range $2m_R \leqslant \Lambda \lesssim 10 m_R$, it is more difficult to make a safe prediction, but if bound states exist for sufficiently large bare coupling, their binding energy should be rather small.

Our analysis of the broken symmetry phase relies heavily on the results and methods of ref. [1]. We shall therefore assume from now on that the reader is familiar with that paper, in particular, unless otherwise stated, the notation introduced in ref. [1] will be taken over without further notice*.

As we have already pointed out in ref. [1], the broken symmetry phase appears to be more difficult to treat than the symmetric phase, because for general bare coupling λ, there is no known (practical) expansion for $\kappa \to \infty$, which could play the rôle the high temperature expansion did in our analysis of the symmetric phase. An exceptional case is the Ising limit $\lambda = \infty$, where a "low temperature" expansion can be derived [3] which could be worked out to high orders without great difficulty. However for $\lambda < \infty$, the situation seems to be less favorable even though the correlation length is small for large κ and one might therefore hope to be able to perform a kind of cluster expansion. Anyhow, we did not pursue these matters any further, because we realized that our goals could also be reached by a different strategy, which does not require the theory to be solved for large κ.

In outline the new strategy is as follows. As we have discussed in subsect. 4.4 of [1], the renormalized coupling g_R in the symmetric phase scales to zero as one approaches the critical line $\kappa = \kappa_c(\lambda)$ in such a way that the limit

$$C_1(\lambda) = \lim_{\kappa \to \kappa_c} am_R(\beta_1 g_R)^{\beta_2/\beta_1^2} e^{1/\beta_1 g_R} \tag{1.1}$$

exists. Similarly, a constant $C_1'(\lambda)$ may be defined by approaching κ_c from the broken symmetry phase. Both constants are defined at the critical line and it is therefore not surprising that they can be given an interpretation in terms of the critical (massless) theory. It then becomes apparent that $C_1'(\lambda)$ is actually proportional to $C_1(\lambda)$ with a proportionality constant, which is exactly given by

$$C_1'(\lambda) = e^{1/6} C_1(\lambda) \tag{1.2}$$

for our choice of renormalization conditions.

$C_1(\lambda)$ can be calculated for all λ to a reasonable estimated accuracy from our solution of the model in the symmetric phase. Thus, $C_1'(\lambda)$ is known and may be used as initial datum for the integration of the renormalization group equations along the lines $\lambda = \text{constant}$ in the broken symmetry phase (see fig. 1). Since the Callan-Symanzik coefficients β, γ, δ (and ε) are only known in perturbation theory, the integration must be stopped when the coupling g_R becomes large (point D in fig. 1). Thus, in this way the theory can only be "solved" in a narrow band around the critical line in the broken symmetry phase, but as we shall see, this band includes the whole scaling region $\Lambda \geqslant 2m_R$. The shaded area in fig. 1, where the theory remains unsolved, is therefore not a very interesting region since there, similarly to the high temperature region in the symmetric phase, the physics at scales of m_R is strongly influenced by non-universal cutoff effects.

* Equations in ref. [1] will be referred to by equation numbers prefixed by I such as for example (I.4.30).

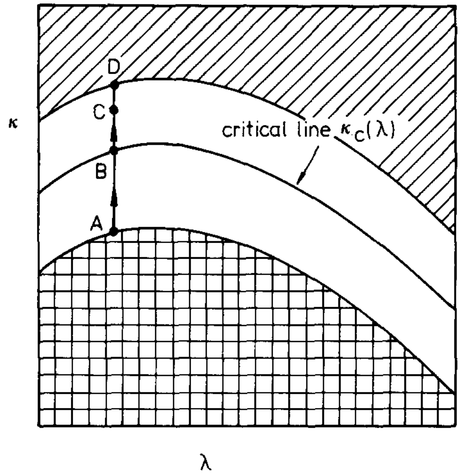

Fig. 1. Qualitative plot of the phase diagram of the lattice ϕ^4 theory. The integration of the renormalization group equations is started at e.g. point A at the boundary of the high temperature region (cross-hatched area) and follows the line $\lambda = \text{constant}$ towards point B, where the constants C_1 and C_1' are determined. The integration can then be continued away from the critical line to (say) point C in the broken symmetry phase.

The crucial step in the procedure just described is the proof of eq. (1.2). It is based on the definition of a renormalized massless theory and the reconstruction of the massive theory on both sides of the critical line using mass perturbation theory and mass independent counterterms. This scheme has been advocated by Weinberg [4] (although not for scalar theories) and was explained most clearly by Brézin, Le Guillou and Zinn-Justin [5] in their review of the field theoretical approach to critical phenomena*.

The organization of this paper is as follows. In sect. 2, we discuss the renormalization of the theory in the broken symmetry phase and fix the renormalization scheme. Scaling laws, analogous to those in the symmetric phase, are derived subsequently (sect. 3) and the relation between the integration constants C_1 and C_1' alluded to above is established in sect. 4. After that, the renormalization group equations in the broken symmetry phase can be integrated and the main results, a quantitative plot of the renormalization group trajectories and the triviality bound, are obtained (sect. 5). Conclusions are drawn in sect. 6 and the perturbation expansion coefficients for the Callan-Symanzik functions β, γ, δ in the broken symmetry phase are listed up to 3 loops in appendix A.

In many respects, sect. 4 is the heart of our paper and the steps taken there are therefore explained in great detail. Otherwise we shall be rather brief, because the

* It is also implied if one uses dimensional regularisation and minimal subtraction [6].

methods used and the argumentation are practically the same as in our earlier paper on the symmetric phase [1].

2. Renormalization in the broken symmetry phase

In this section we define the bare and renormalized vertex functions of the fundamental field ϕ and the composite field $\mathcal{O}$ (eq. (I.2.5)) in the broken symmetry phase $\kappa > \kappa_c(\lambda)$. In our discussion of renormalization, we shall essentially follow Symanzik's analysis [7].

2.1. BARE VERTEX FUNCTIONS

As in the symmetric phase (sect. 2 of ref. [1]), we start off by defining the generating functional $W(H, K)$ for the connected correlation functions of ϕ and $\mathcal{O}$. Because of the spontaneous symmetry breaking, some care must be paid to obtain the correlation functions in a pure state. Thus we first introduce a positive "magnetic" field h and then define $W(H, K)$ as a limit for $h \to 0$:

$$e^{W} = \lim_{h \to 0} \frac{1}{\mathcal{Z}_h} \int \prod_x d\phi(x) \exp\left\{ -S_h + \sum_x \left(H(x)\phi(x) + K(x)\mathcal{O}(x) \right) \right\}, \quad (2.1)$$

$$S_h = S - h \sum_x \phi(x), \quad (2.2)$$

$$\mathcal{Z}_h = \int \prod_x d\phi(x)\, e^{-S_h} \quad (2.3)$$

(S is the action (I.2.1) and the source fields $H(x)$, $K(x)$ are assumed to decay rapidly for large x). With these conventions $W(0,0) = 0$ and the spontaneous vacuum expectation value of ϕ, given by

$$v = \left. \frac{\partial W}{\partial H(x)} \right|_{H=K=0}, \quad (2.4)$$

is positive.

The bare vertex functions $\Gamma^{(n,l)}$ are again defined via the Legendre transform $\Gamma(M, K)$ of $W(H, K)$. However, because of the non-zero expectation value of ϕ, it is more natural to introduce the local magnetization $M(x)$ through

$$M(x) = \frac{\partial W}{\partial H(x)} - v \quad (2.5)$$

instead of (I.2.7) and the definition of $\Gamma(M, K)$ is accordingly changed to

$$\Gamma = W - \sum_x H(x)\left(M(x) + v \right). \quad (2.6)$$

Then the vertex functions $\Gamma^{(n,l)}(p_1,\ldots,p_n;q_1,\ldots,q_l)$ are generated by expanding $\Gamma(M,K)$ in powers of M and K as in eq. (I.2.9).

By definition, $\Gamma^{(0,0)}$ and $\Gamma^{(1,0)}$ vanish. Except for $\Gamma^{(2,0)}(p,-p)$, which is equal to the negative inverse propagator of ϕ, all other vertex functions are the one-particle irreducible parts of the connected $n+l$-point functions with full propagator amputated external ϕ-legs. We emphasize that one-particle irreducibility here only refers to channels with *non-empty* sets of external momenta on both sides, in particular, in a Feynman diagram expansion of $\Gamma^{(n,l)}$, tadpole subgraphs are not excluded.

We finally note that for $(n,l)\neq(0,0)$, we have

$$\frac{\partial}{\partial\kappa}\Gamma^{(n,l)}=\frac{\partial v}{\partial\kappa}\Gamma^{(n+1,l)}|_{p_{n+1}=0}+\Gamma^{(n,l+1)}|_{q_{l+1}=0}\,. \tag{2.7}$$

This relation will later be used to derive the Callan-Symanzik equation. Compared to the analogous eq. (I.2.11) in the symmetric phase, it involves an additional term, which arises from the fact that the Legendre transform (2.5), (2.6) depends on κ through v.

2.2. PERTURBATION THEORY

The weak coupling perturbation expansion of the vertex functions $\Gamma^{(n,l)}$ is obtained by expanding about the absolute minima of the action S. For $\kappa>\kappa_c(\lambda)$ and small λ, S has exactly two degenerate absolute minima at

$$\phi(x)=\pm s_0\,, \tag{2.8}$$

$$s_0^2=\frac{1}{2\lambda}(8\kappa-1+2\lambda)\,,\qquad s_0>0\,. \tag{2.9}$$

According to the definition (2.1) of $W(H,K)$, the functional integral is to be evaluated with a small magnetic field h, which is turned off at the end of the calculation. In perturbation theory, an equivalent procedure is to expand only about the positive minimum $\phi(x)=+s_0$. Thus, we introduce a fluctuation field $\varphi_0(x)$ and an associated composite field $\mathcal{O}_0(x)$ through

$$\varphi_0(x)=\sqrt{2\kappa}\,(\phi(x)-s_0)\,, \tag{2.10}$$

$$\mathcal{O}_0(x)=\sum_{\mu=0}^{3}\{\varphi_0(x)\varphi_0(x+\hat\mu)+\varphi_0(x)\varphi_0(x-\hat\mu)\}\,. \tag{2.11}$$

The action then becomes

$$S=\sum_x\left\{\tfrac{1}{2}\sum_{\mu=0}^{3}\left(\partial_\mu\varphi_0(x)\right)^2+\tfrac{1}{2}m_0^2\varphi_0(x)^2+\frac{1}{3!}\sqrt{3g_0}\,m_0\varphi_0(x)^3+\frac{g_0}{4!}\varphi_0(x)^4\right\}+\text{const}, \tag{2.12}$$

where the bare mass $m_0>0$ and the coupling g_0 are given by*

$$m_0^2=\frac{2}{\kappa}(8\kappa-1+2\lambda), \tag{2.13}$$

$$g_0=\frac{6\lambda}{\kappa^2}. \tag{2.14}$$

The Feynman rules for the calculation of the correlation functions of φ_0 and $\mathcal{O}_0$ in powers of g_0 may now be derived from (2.12) in the usual way. The corresponding vertex functions are sums of one-particle irreducible Feynman diagrams (in the sense explained above) and will be denoted by $\Gamma_0^{(n,l)}$.

There is a simple relation between the vertex functions $\Gamma^{(n,l)}$ and $\Gamma_0^{(n,l)}$, which can be easily established using the associated generating functionals. In general we have

$$\Gamma^{(n,l)}=(2\kappa)^{n/2-l}\Gamma_0^{(n,l)}, \tag{2.15}$$

the exceptions being

$$\Gamma^{(0,1)}=(2\kappa)^{-1}\Gamma_0^{(0,1)}+(2\kappa)^{-1/2}16v_0s_0+8s_0^2, \tag{2.16}$$

$$\Gamma^{(1,1)}(p;q)=(2\kappa)^{-1/2}\Gamma_0^{(1,1)}(p;q)+2s_0\sum_{\mu=0}^{3}\left(1+\cos q_\mu\right). \tag{2.17}$$

In (2.16), v_0 denotes the vacuum expectation value of φ_0, which is related to the vacuum expectation value of ϕ through

$$v=(2\kappa)^{-1/2}v_0+s_0. \tag{2.18}$$

In perturbation theory, v_0 is given by the sum of all connected tadpole graphs.

Through eqs. (2.15)–(2.18) the Feynman diagram expansion of v_0 and $\Gamma_0^{(n,l)}$ immediately leads to the desired perturbation expansion of v and $\Gamma^{(n,l)}$ in powers

* Note that the definition (I.2.4) of the bare mass m_0 in the symmetric phase is different.

of g_0. In the lowest order (tree level) approximation, we have

$$v=\left(\frac{3m_0^2}{2\kappa g_0}\right)^{1/2} \tag{2.19}$$

and the non-zero vertex functions are given by

$$\Gamma^{(2,0)}(p,-p)=-2\kappa\left(m_0^2+\hat{p}^2\right), \tag{2.20}$$

$$\Gamma^{(3,0)}=-(2\kappa)^{3/2}\sqrt{3g_0}\,m_0, \tag{2.21}$$

$$\Gamma^{(4,0)}=-(2\kappa)^2 g_0, \tag{2.22}$$

$$\Gamma^{(0,1)}=8v^2, \tag{2.23}$$

$$\Gamma^{(1,1)}(p;q)=v\left(16-\hat{q}^2\right), \tag{2.24}$$

$$\Gamma^{(2,1)}(p_1,p_2;q)=16-\hat{p}_1^2-\hat{p}_2^2, \tag{2.25}$$

where $\hat{p}^2$ is defined by

$$\hat{p}^2=4\sum_{\mu=0}^{3}\sin^2\frac{p_\mu}{2}. \tag{2.26}$$

For small coupling, the model thus describes a single particle of mass m_0 with weak 3- and 4-body interactions, which come in a certain proportion as dictated by the spontaneously broken symmetry.

2.3. RENORMALIZATION

From the work of Symanzik [7], one knows that the renormalization of the theory in the broken symmetry phase does not require any more counterterms than are already needed in the symmetric phase. Thus, apart from additive subtractions of $\Gamma^{(0,1)}$ and $\Gamma^{(0,2)}$, it is sufficient to express the bare parameters κ and λ through a renormalized mass m_R and coupling g_R and to multiply the fields ϕ and $\mathcal{O}$ by appropriate wave function renormalization constants Z_R and $Z_R^{\mathcal{O}}$. To the extent that finite renormalizations can be made, the precise definitions of m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ are of course arbitrary. Here we adopt a scheme which is easy to work with in perturbation theory and leads to some simplifications later on when we discuss the Callan-Symanzik equation.

As in the symmetric phase, the renormalized mass m_R and the wave function renormalization constant Z_R are determined through

$$\Gamma^{(2,0)}(p,-p)=-Z_R^{-1}\left\{m_R^2+p^2+\mathrm{O}(p^4)\right\}, \qquad (p\to 0). \tag{2.27}$$

The renormalized vacuum expectation value v_R of ϕ is then given by

$$v_R = v(Z_R)^{-1/2}. \tag{2.28}$$

With these conventions, the tree level formula (2.19) can be rephrased in the form

$$3m_R^2/v_R^2 = g_0 + O(g_0^2), \tag{2.29}$$

and a natural definition of the renormalized coupling g_R therefore reads

$$g_R = 3m_R^2/v_R^2. \tag{2.30}$$

In our scheme, the "mass formula"

$$m_R^2 = \tfrac{1}{3} g_R v_R^2 \tag{2.31}$$

is hence an identity, whereas the four-point coupling at zero momentum is a non-trivial function of g_R.

We finally fix the wave function renormalization constant $Z_R^{\mathcal{O}}$ of $\mathcal{O}$ through

$$\Gamma^{(1,1)}(0;0) = v\left(Z_R Z_R^{\mathcal{O}}\right)^{-1}. \tag{2.32}$$

In view of eqs. (2.24) and (2.25), this definition is equivalent to eq. (I.4.4) at tree level, but in higher orders the two definitions would differ by a finite factor.

Renormalized vertex functions $\Gamma_R^{(n,l)}$ may now be defined in the usual way by

$$\Gamma_R^{(n,l)} = 0 \qquad \text{for } n = 0,\ l \leqslant 1, \tag{2.33}$$

$$\Gamma_R^{(0,2)}(q,-q) = \left(Z_R^{\mathcal{O}}\right)^2 \left\{\Gamma^{(0,2)}(q,-q) - \Gamma^{(0,2)}(0,0)\right\}, \tag{2.34}$$

$$\Gamma_R^{(n,l)} = (Z_R)^{n/2}\left(Z_R^{\mathcal{O}}\right)^l \Gamma^{(n,l)} \qquad \text{for all other } n, l. \tag{2.35}$$

We shall always consider $\Gamma_R^{(n,l)}$ to be a function of m_R, g_R and the momenta (through eq. (2.30), v_R is also considered a dependent quantity). When one inserts the bare perturbation expansion derived in the preceding subsection into the above equations and eliminates m_0 and g_0 in favor of m_R and g_R, one obtains the renormalized perturbation expansion of $\Gamma_R^{(n,l)}$ in powers of g_R. By Symanzik's analysis, the coefficients in this expansion are ultra-violet finite, i.e. their continuum limit exists and what we have said about scaling in subsect. 4.2 of [1], carries over literally.

The renormalized vertex functions fulfill the normalization conditions

$$\Gamma_{\mathrm{R}}^{(2,0)}(p,-p) = -\{m_{\mathrm{R}}^2 + p^2 + \mathrm{O}(p^4)\}, \qquad (p \to 0), \tag{2.36}$$

$$\Gamma_{\mathrm{R}}^{(0,2)}(0,0) = 0, \tag{2.37}$$

$$\Gamma_{\mathrm{R}}^{(1,1)}(0;0) = v_{\mathrm{R}} \tag{2.38}$$

(and eq. (2.33)). To completely specify the normalization of the vertex functions, a further condition must be added, which insures that v_{R} is in fact the renormalized vacuum expectation value of ϕ. There is no simple expression of the latter in terms of the renormalized vertex functions, but as discussed by Symanzik [7], it is nevertheless well determined by them, because it enters the Ward identities implied by the symmetry $\phi \to -\phi$ of the action in a non-trivial way. While it is not necessary to recall any details of this construction, it is important to know that our renormalization scheme can be unambiguously characterized without recourse to the bare correlation functions. In particular, if we had used (say) dimensional regularisation instead of a lattice and followed the steps (2.27)–(2.35) to define the renormalized theory, the resulting renormalized vertex functions would have been the same (up to scaling violation terms, of course).

3. Scaling behavior in the critical region

The scaling properties of the ϕ^4 theory in the broken symmetry phase are essentially the same as in the symmetric phase, so that here we only summarize the basic results and refer the reader to sect. 4 of [1] for a more detailed discussion.

3.1. THE CALLAN-SYMANZIK EQUATION

The composite field $\mathcal{O}(x)$ has been introduced in such a way that the basic equation

$$\frac{\partial S}{\partial \kappa} = -\sum_x \mathcal{O}(x) \tag{3.1}$$

holds. In terms of the bare vertex functions, this property is expressed through eq. (2.7), which in turn implies the Callan-Symanzik equation*

$$\left\{ m_{\mathrm{R}} \frac{\partial}{\partial m_{\mathrm{R}}} + \beta \frac{\partial}{\partial g_{\mathrm{R}}} - n\gamma - l\delta \right\} \Gamma_{\mathrm{R}}^{(n,l)}$$
$$= \vartheta m_{\mathrm{R}} \left\{ \Gamma_{\mathrm{R}}^{(n+1,l)}\big|_{p_{n+1}=0} - \delta_{n0}\delta_{l2}\Gamma_{\mathrm{R}}^{(1,2)}(0;0,0) \right\}$$
$$+ \varepsilon m_{\mathrm{R}}^2 \left\{ \Gamma_{\mathrm{R}}^{(n,l+1)}\big|_{q_{l+1}=0} - \delta_{n0}\delta_{l2}\Gamma_{\mathrm{R}}^{(0,3)}(0,0,0) \right\}. \tag{3.2}$$

* Eq. (3.2) is valid for all n, l except $n = 0$, $l \leqslant 1$.

The coefficients β, γ, δ and ε in this equation are given by eqs. (I.4.21)–(I.4.24), while the new coefficient ϑ is related to the vacuum expectation value of ϕ through

$$\vartheta = Z_R^{-1/2} \frac{\partial v}{\partial \kappa} \bigg/ \frac{\partial m_R}{\partial \kappa} \tag{3.3}$$

(derivatives with respect to κ are at fixed λ).

In our renormalization scheme, the coefficients ϑ and ε can be algebraically expressed through β and γ. Indeed, from eqs. (2.28), (2.31) and the definition of β, γ, ϑ one easily shows that

$$\vartheta = \sqrt{\frac{3}{g_R}} \left(1 + \gamma - \frac{\beta}{2g_R}\right). \tag{3.4}$$

Next, choosing $n = 1$, $l = 0$ in the Callan-Symanzik equation and using the normalization conditions (2.36), (2.38) one obtains $\varepsilon = \vartheta m_R / v_R$ and hence

$$\varepsilon = 1 + \gamma - \frac{\beta}{2g_R}. \tag{3.5}$$

We emphasize that these relations are a special feature of our renormalization scheme, in particular, if we had adopted the normalization (I.4.8) of the operator $\mathcal{O}$ we chose in the symmetric phase, there would have been no formula for ε in terms of β, γ, δ.

We have computed β, γ, δ in renormalized perturbation theory up to three loops for $m_R = 0$ and up to one loop for general m_R (appendix A). The results show that the universal coefficients β_1, β_2, γ_1 and δ_1 are the same as in the symmetric phase. A curious although not very important fact is that the β-function becomes negative for $g_R \to 0$ and fixed m_R (the tree level coefficient u_0 is negative). Sufficiently close to the critical line, where scaling violation terms can be neglected, β is however always positive so that the critical behavior of the theory is not influenced by this effect in any relevant way.

3.2. SCALING LAWS FOR m_R, g_R, Z_R AND $Z_R^{\mathcal{O}}$

As we have discussed in [1], the ϕ^4 theory becomes a free field theory when one approaches the critical line $\kappa = \kappa_c(\lambda)$ from below. In sect. 4 we shall argue that the same happens also in the broken symmetry phase, in particular, the coupling g_R is expected to be small close to the critical line. In this regime, perturbation theory may therefore be used to evaluate the β-function and the precise asymptotic behaviour of g_R for $m_R \to 0$ and fixed λ then follows from the renormalization group equation (I.4.26). In an implicit form, this scaling law reads

$$m_R = C_1'(\beta_1 g_R)^{-\beta_2/\beta_1^2} e^{-1/\beta_1 g_R} \{1 + O(g_R)\}, \tag{3.6}$$

where C_1' is an integration constant depending on λ and β_1, β_2 are the one- and two-loop coefficients of the β-function.

Similarly, the scaling laws

$$Z_R = C_2'\{1 + O(g_R)\}, \tag{3.7}$$

$$Z_R^{\phi} = C_3' g_R^{-1/3}\{1 + O(g_R)\}, \tag{3.8}$$

follow from the renormalization group equations (I.4.27), (I.4.28), while the relation

$$\kappa - \kappa_c = \tfrac{1}{2} C_3' m_R^2 g_R^{-1/3}\{1 + O(g_R)\}, \tag{3.9}$$

derives from

$$m_R \frac{\partial \kappa}{\partial m_R} = m_R^2 \varepsilon Z_R^{\phi}, \tag{3.10}$$

with ε given by (3.5). Note that compared to the corresponding formula in the symmetric phase, there is an extra factor of $-\frac{1}{2}$ in (3.9), which can be traced back to the fact that at tree level, the coefficient ε is different in the two phases. Otherwise the scaling laws are identical except, of course, that the integration constants C_i and C_i' need not be the same.

We finally note that the spontaneous vacuum expectation value of ϕ scales to zero according to [5]

$$v \underset{\tau \to 0}{\propto} (-\tau)^{1/2} |\ln(-\tau)|^{1/3}, \qquad \tau = 1 - \frac{\kappa}{\kappa_c}, \tag{3.11}$$

as one may easily show from eqs. (2.28), (2.30) and the scaling laws above. On the other hand, the dimensionless ratio v_R/m_R diverges logarithmically, i.e. in physical units, the renormalized vacuum expectation value is infinite in the continuum limit.

4. Properties of the integration constants $C_i(\lambda)$ and $C_i'(\lambda)$

Our goal in this section is mainly to establish a relation between the constants C_i and C_i', which occur in the scaling laws in the symmetric and broken symmetry phase, respectively. As we have already indicated in the introduction, this will be achieved by defining a massless renormalized theory along the critical line from which the massive theories on both sides of the critical line can be reconstructed using mass perturbation theory. In this framework, the scaling behaviour of the model in the two phases can be mapped onto a single property of the critical theory and this then implies that the constants C_i and C_i' must be related.

4.1. RENORMALIZATION OF THE CRITICAL THEORY

Along the critical line $\kappa = \kappa_c(\lambda)$, the bare vertex functions $\Gamma^{(n,l)}$ are in general only well-defined for non-exceptional external momentum configurations★. We therefore fix the renormalization constants at some (arbitrary) momentum scale $\mu > 0$, which is to be scaled as a mass parameter in the continuum limit. Specifically, the wave function renormalization constant Z_c is defined by

$$\frac{\partial}{\partial p_0}\Gamma^{(2,0)}(p,-p)\big|_{p=n_1} = -2\mu Z_c^{-1}, \qquad n_1 = (\mu,0,0,0), \tag{4.1}$$

and the renormalized coupling g_c is determined through

$$\Gamma^{(4,0)}(n_1,n_2,n_3,n_4) = -Z_c^{-2}g_c, \tag{4.2}$$

$$n_i \cdot n_j = \tfrac{1}{3}\mu^2(4\delta_{ij}-1). \tag{4.3}$$

The vectors n_i, $i=1,\dots,4$, are the corners of a regular tetrahedron, whose position relative to the lattice axes should also be specified for g_c to be defined unambiguously. We assume that this has been done in some way, but do not actually write down the coordinates of n_i, since they will never be needed. We finally fix the wave function renormalization constant $Z_c^{\mathcal{O}}$ of the operator $\mathcal{O}$ through

$$\Gamma^{(2,1)}(n_3,n_4;n_5) = \left(Z_c Z_c^{\mathcal{O}}\right)^{-1}, \qquad n_5 = -n_3 - n_4. \tag{4.4}$$

The quantities g_c, Z_c and $Z_c^{\mathcal{O}}$ introduced in this way are free of infrared divergences and are hence well-defined functions of λ and μ.

We now proceed to define renormalized vertex functions $\Gamma_c^{(n,l)}$ by

$$\Gamma_c^{(n,l)} = 0 \qquad \text{for } n=0, \qquad l \leqslant 1, \tag{4.5}$$

$$\Gamma_c^{(0,2)}(q,-q) = \left(Z_c^{\mathcal{O}}\right)^2\left\{\Gamma^{(0,2)}(q,-q) - \Gamma^{(0,2)}(n_5,-n_5)\right\}, \tag{4.6}$$

$$\Gamma_c^{(n,l)} = (Z_c)^{n/2}\left(Z_c^{\mathcal{O}}\right)^l\Gamma^{(n,l)} \qquad \text{for all other } n, l. \tag{4.7}$$

In these equations, $\kappa = \kappa_c(\lambda)$ as before and λ is to be eliminated in favor of g_c, i.e. $\Gamma_c^{(n,l)}$ is considered to be a function of μ, g_c and the momenta.

The scaling properties of the critical renormalized vertex functions $\Gamma_c^{(n,l)}$ are entirely analogous to those of the massive renormalized vertex functions $\Gamma_R^{(n,l)}$ with μ playing the rôle of m_R (cf. sect. 4.2 of [1]). Furthermore, because the bare vertex

★ $\Gamma^{(n,l)}$ is defined as in the symmetric phase. Note also that $\Gamma^{(0,1)}$ is infrared finite even though the external momentum vanishes in this case.

functions are independent of μ, the renormalization group equation

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta_{\mathrm{c}}\frac{\partial}{\partial g_{\mathrm{c}}}-n\gamma_{\mathrm{c}}-l\delta_{\mathrm{c}}\right\}\Gamma_{\mathrm{c}}^{(n,l)}=\delta_{n0}\delta_{l2}B \tag{4.8}$$

holds, where the coefficients β_{c}, γ_{c}, δ_{c} and B are given by

$$\beta_{\mathrm{c}}=\mu\frac{\partial}{\partial\mu}g_{\mathrm{c}}, \tag{4.9}$$

$$\gamma_{\mathrm{c}}=\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_{\mathrm{c}}, \tag{4.10}$$

$$\delta_{\mathrm{c}}=\mu\frac{\partial}{\partial\mu}\ln Z_{\mathrm{c}}^{\mathcal{O}}, \tag{4.11}$$

$$B=-\left(Z_{\mathrm{c}}^{\mathcal{O}}\right)^2\mu\frac{\partial}{\partial\mu}\Gamma^{(0,2)}(n_5,-n_5) \tag{4.12}$$

(derivatives with respect to μ are at fixed λ).

β_{c}, γ_{c} and δ_{c} have been computed in powers of g_{c} up to three loops [5]. Neglecting scaling violations, one finds that the leading coefficients β_1, β_2, γ_1 and δ_1 are the same as in the massive phases. From eqs. (4.9)–(4.11), we can thus derive scaling laws for g_{c}, Z_{c} and $Z_{\mathrm{c}}^{\mathcal{O}}$ in the usual way, provided we assume that for some μ, the coupling g_{c} is sufficiently small to be attracted by the trivial zero of β_{c}. We shall later argue that this is indeed the case (subsect. 4.3) and hence conclude that for $\mu\to 0$ and fixed λ, the scaling laws

$$\mu=C_1^{\mathrm{c}}(\beta_1 g_{\mathrm{c}})^{-\beta_2/\beta_1^2}\mathrm{e}^{-1/\beta_1 g_{\mathrm{c}}}\{1+\mathrm{O}(g_{\mathrm{c}})\}, \tag{4.13}$$

$$Z_{\mathrm{c}}=C_2^{\mathrm{c}}\{1+\mathrm{O}(g_{\mathrm{c}})\}, \tag{4.14}$$

$$Z_{\mathrm{c}}^{\mathcal{O}}=C_3^{\mathrm{c}}g_{\mathrm{c}}^{-1/3}\{1+\mathrm{O}(g_{\mathrm{c}})\}, \tag{4.15}$$

hold, where the C_i^{c} denote yet another set of integration constants depending on λ.

4.2. RECONSTRUCTION OF THE MASSIVE THEORY

Following Brézin et al. [5], we here show how the vertex functions $\Gamma_{\mathrm{R}}^{(n,l)}$ defined for $\kappa\neq\kappa_{\mathrm{c}}$ can be obtained from the critical renormalized vertex functions $\Gamma_{\mathrm{c}}^{(n,l)}$. For simplicity, the details are only worked out for the symmetric phase, but the analysis can be extended to the broken symmetry phase without additional difficulties.

Let $\Gamma(M,K;\kappa,\lambda)$, $\kappa\leqslant\kappa_{\mathrm{c}}(\lambda)$, be the generating functional of the bare vertex functions $\Gamma^{(n,l)}$ as defined in sect. 2 of [1] and set $\Delta\kappa=\kappa-\kappa_{\mathrm{c}}$. Our starting point is the (trivial) identity

$$\Gamma(M, K; \kappa, \lambda) = \lim_{L\to\infty} \{\Gamma(M, K + \Delta\kappa\theta_L; \kappa_c, \lambda) - \Gamma(0, \Delta\kappa\theta_L; \kappa_c, \lambda)\}, \quad (4.16)$$

where $\theta_L(x)$, $x \in \mathbb{Z}^4$, denotes the cutoff function

$$\theta_L(x) = \begin{cases} 1 & \text{if } |x_\mu| \leqslant L \text{ for all } \mu = 0, \dots, 3 \\ 0 & \text{otherwise} \end{cases} \quad (4.17)$$

(the limit procedure is necessary, because in infinite volume, Γ is only defined for source fields which are decaying at infinity). By eq. (4.16), the bare vertex functions $\Gamma^{(n,l)}$ at $\kappa < \kappa_c$ can thus be calculated in terms of the critical vertex functions. For example, for $n \geqslant 2$ we have

$$\delta_P\left(\sum_{i=1}^n p_i\right)\Gamma^{(n,0)}(p_1, \dots, p_n; \kappa, \lambda)$$
$$= \lim_{L\to\infty} \sum_{l=0}^{\infty} \frac{1}{l!}(\Delta\kappa)^l \int_{-\pi}^{\pi} \frac{d^4q_1}{(2\pi)^4} \cdots \frac{d^4q_l}{(2\pi)^4}$$
$$\times \delta_P\left(\sum_{i=1}^n p_i + \sum_{j=1}^l q_j\right)\Gamma^{(n,l)}(p_1, \dots, p_n; q_1, \dots, q_l; \kappa_c, \lambda)\tilde{\theta}_L(q_1)\dots\tilde{\theta}_L(q_l), \quad (4.18)$$

where the dependence on κ and λ has been indicated explicitly and the notation is otherwise the same as in eq. (I.2.9).

Eq. (4.18) reveals that the massive vertex functions can be reconstructed from the critical theory essentially by mass perturbation theory with $|\Delta\kappa|$ playing the rôle of the bare mass squared and $\mathcal{O}$ being the mass operator. A subtle point to note is that mass insertions at zero momentum in a Feynman diagram with massless propagators usually result in an infrared divergent integral. In eq. (4.18), the limit $L \to \infty$ can therefore be taken only after the summation over l, which makes the propagators massive and the zero momentum limit well-defined.

We now proceed to rewrite eq. (4.18) in terms of renormalized vertex functions. To this end, define a new set of massive renormalized vertex functions $\Gamma_m^{(n,0)}$ through

$$\delta_P\left(\sum_{i=1}^n p_i\right)\Gamma_m^{(n,0)}(p_1, \dots, p_n; \mu, m_c, g_c)$$
$$= \lim_{L\to\infty} \sum_{l=0}^{\infty} \frac{1}{l!}(-m_c^2)^l \int_{-\pi}^{\pi} \frac{d^4q_1}{(2\pi)^4} \cdots \frac{d^4q_l}{(2\pi)^4}$$
$$\times \delta_P\left(\sum_{i=1}^n p_i + \sum_{j=1}^l q_j\right)\Gamma_c^{(n,l)}(p_1, \dots, p_n; q_1, \dots, q_l; \mu, g_c)\tilde{\theta}_L(q_1)\dots\tilde{\theta}_L(q_l), \quad (4.19)$$

where $m_c \geqslant 0$ denotes an arbitrary mass parameter. From eq. (4.18), one then obtains

$$\Gamma_R^{(n,0)}(p_1,\ldots,p_n;m_R,g_R)=X_\phi^{n/2}\Gamma_m^{(n,0)}(p_1,\ldots,p_n;\mu,m_c,g_c), \qquad (4.20)$$

where X_ϕ, m_R and g_R are functions of μ, m_c and g_c determined by

$$\Gamma_m^{(2,0)}(p,-p)=-X_\phi^{-1}\{m_R^2+p^2+O(p^4)\}, \qquad (p\to 0), \qquad (4.21)$$

$$\Gamma_m^{(4,0)}(0,0,0,0)=-X_\phi^{-2}g_R. \qquad (4.22)$$

These values of m_R and g_R correspond to

$$\kappa=\kappa_c(\lambda)-Z_c^{\mathcal{O}}(\mu,\lambda)m_c^2 \qquad (4.23)$$

and a bare coupling λ as specified by μ and g_c. Furthermore, when deriving eq. (4.20), one also gets the relation

$$X_\phi=Z_R(\kappa,\lambda)/Z_c(\mu,\lambda), \qquad (4.24)$$

which we shall use later on.

Eqs. (4.19)–(4.22) show that the renormalized vertex functions $\Gamma_R^{(n,0)}$ can be reconstructed from the critical vertex functions $\Gamma_c^{(n,l)}$ without recourse to any bare quantities. A crucial observation now is that the newly introduced vertex functions $\Gamma_m^{(n,0)}$ have scaling properties, which are completely analogous to those of $\Gamma_R^{(n,l)}$, provided that we treat μ and m_c as physical masses. In other words, if we introduce a lattice spacing a as in sect. 4.2 of [1], the scaled vertex functions

$$\bar{\Gamma}_m^{(n,0)}=a^{n-4}\Gamma_m^{(n,0)}(a\bar{p}_1,\ldots,a\bar{p}_n;a\bar{\mu},a\bar{m}_c,g_c) \qquad (4.25)$$

are independent of a up to scaling violation terms, which vanish like a power of a for $a\to 0$. This property follows from the definition (4.19) and the scaling behaviour of $\Gamma_c^{(n,l)}$. It implies, in particular, that X_ϕ, m_R/μ and g_R as given by eqs. (4.21), (4.22), depend only on g_c and m_c/μ, if m_c and μ are small compared to the ultra-violet cutoff Λ (which is equal to 1 in lattice units). Another way to say this is that $\Gamma_R^{(n,0)}$ and $\Gamma_m^{(n,0)}$ are related by a *finite* renormalization transformation.

We finally mention that the analysis could of course be extended to the vertex functions $\Gamma_R^{(n,l)}$ with $l\neq 0$ and one would then be able to show that the renormalization constant

$$X_{\mathcal{O}}=Z_R^{\mathcal{O}}(\kappa,\lambda)/Z_c^{\mathcal{O}}(\mu,\lambda), \qquad (4.26)$$

is also "finite", i.e. that it depends only on g_c and m_c/μ in the scaling region.

4.3. TRIVIALITY OF THE CRITICAL THEORY AND THE RELATION BETWEEN THE CONSTANTS C_i AND C_i'

The integration constants $C_i(\lambda)$ and $C_i'(\lambda)$ are determined by the asymptotic behaviour of m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ as one approaches the critical line keeping the bare coupling λ fixed. The framework developed above is ideal to study this limit, since it allows us to compare theories with the same λ but different renormalized masses m_c.

Since we are here only interested in what happens very close to the critical line, we may assume $m_c \ll 1$, and we are also free to choose $\mu \ll 1$. Scaling violations can then be safely neglected and from the discussion in the preceding subsection, we conclude that universal relations of the form

$$g_c = F_1(g_R, m_R/\mu), \tag{4.27}$$

$$Z_c = F_2(g_R, m_R/\mu) Z_R, \tag{4.28}$$

$$Z_c^{\mathcal{O}} = F_3(g_R, m_R/\mu) Z_R^{\mathcal{O}} \tag{4.29}$$

hold, where all quantities involved are defined at some fixed bare coupling λ. The functions F_i are not known in closed form, but they can easily be calculated in low orders of perturbation theory. In particular, for the symmetric phase we have

$$F_1 = g_R - \frac{3g_R^2}{32\pi^2}\left(2 + \ln\tfrac{3}{4} + \ln\frac{m_R^2}{\mu^2}\right) + \mathrm{O}(g_R^3), \tag{4.30}$$

$$F_2 = 1 + \mathrm{O}(g_R^2), \tag{4.31}$$

$$F_3 = 1 + \mathrm{O}(g_R), \tag{4.32}$$

and similar formulae can be derived in the broken symmetry phase.

We now specialize to the symmetric phase, set $\mu = m_R$ and let m_R go to zero. Since λ is fixed, g_R also has to go to zero in this limit according to the scaling law (I.4.30). In eq. (4.27), F_1 may therefore be evaluated in perturbation theory and it follows that

$$g_c = g_R - \frac{3g_R^2}{32\pi^2}\left(2 + \ln\tfrac{3}{4}\right) + \mathrm{O}(g_R^3), \tag{4.33}$$

where g_c is defined at scale $\mu = m_R$. Thus we have

$$\lim_{\mu \to 0} g_c(\mu, \lambda) = 0 \tag{4.34}$$

and the continuum limit of the critical theory is hence trivial (as expected).

We have earlier derived the scaling laws (4.13)–(4.15) assuming that g_c is small for some μ. By the above considerations, this is in fact the case and g_c, Z_c and $Z_c^{\mathcal{O}}$ hence scale as anticipated. Actually, eq. (4.13) also follows directly from (4.33) and the scaling law (I.4.30) for g_R with the constant C_1^c being given by

$$C_1^c(\lambda) = \tfrac{1}{2}\sqrt{3}\,e C_1(\lambda) \tag{4.35}$$

for all λ. We emphasize that this is an exact result even though we only needed to work out the function $F_1(g_R, m_R/\mu)$ to one-loop order*.

The constants C_2^c, C_3^c may be related to C_2, C_3 in a similar way. In fact, setting $\mu = m_R$ and using eqs. (4.28)–(4.32), one readily obtains

$$C_i^c(\lambda) = C_i(\lambda) \qquad \text{for } i = 2,3. \tag{4.36}$$

The import of eqs. (4.35), (4.36) is that they establish a one-to-one correspondence between the scaling properties of the critical theory and those of the massive theory in the symmetric phase. In particular, the results of [1] could be transferred to the critical theory and, for specified λ, any low energy amplitude could then be calculated to some reasonable estimated accuracy.

We now turn to the broken symmetry phase. Since we already know that the critical theory is trivial, we can invert the above argumentation to show that the theory is also trivial in the broken symmetry phase, in particular, the scaling laws (3.6)–(3.9) hold. Furthermore by working out the functions F_i in perturbation theory, we can then relate the integration constants C_i' to the constants C_i^c as in the symmetric phase and finally obtain the long heralded result

$$C_1'(\lambda) = e^{1/6} C_1(\lambda), \tag{4.37}$$

$$C_i'(\lambda) = C_i(\lambda) \qquad \text{for } i = 2,3. \tag{4.38}$$

By passing through the critical theory, we have thus managed to transfer the information contained in the asymptotic scaling behaviour of m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ in the symmetric phase to the other side of the critical line. In what follows, we shall not refer to the critical theory again, in fact, the relations (4.37), (4.38) are all what is needed to extend the results of [1] to the broken symmetry phase (sect. 5).

4.4. CALCULATION OF $C_i(\lambda)$ FOR $0 < \lambda \leqslant \infty$

As we have discussed in ref. [1], the dependence of m_R, g_R, Z_R and $Z_R^{\mathcal{O}}$ on κ and λ can be determined in the whole symmetric phase region by combining data from the high temperature expansion with the renormalization group equations. In

* It has probably not escaped the reader that our calculations are formally very similar to those needed to match the Λ-parameters of different renormalization schemes in non-abelian gauge theories.

TABLE 1
Values of $\ln C_i(\lambda)$ versus λ

$\bar{\lambda}$	$\ln C_1$	$\ln C_2$	$\ln C_3$
0.01	44.3(2)	1.3799(3)	−4.072(8)
0.02	23.3(2)	1.3738(4)	−3.84(1)
0.03	16.2(2)	1.3677(4)	−3.70(1)
0.04	12.6(2)	1.3616(5)	−3.60(2)
0.05	10.4(2)	1.3556(6)	−3.53(2)
0.06	9.0(2)	1.3496(7)	−3.45(2)
0.07	7.9(2)	1.3437(8)	−3.39(2)
0.08	7.1(2)	1.3378(8)	−3.34(3)
0.09	6.4(2)	1.3321(9)	−3.30(3)
0.10	5.9(2)	1.326(1)	−3.26(3)
0.20	3.5(3)	1.277(3)	−2.98(4)
0.30	2.7(2)	1.249(4)	−2.81(5)
0.40	2.3(2)	1.248(5)	−2.71(7)
0.50	2.0(2)	1.275(6)	−2.66(7)
0.60	1.9(2)	1.330(7)	−2.65(8)
0.70	1.8(2)	1.411(8)	−2.67(9)
0.80	1.7(2)	1.520(9)	−2.72(9)
0.90	1.6(2)	1.662(9)	−2.8(1)
1.00	1.5(2)	1.87(1)	−3.0(1)

The parameter $\bar{\lambda}$ is defined in [1] and the errors are obtained by propagating the errors of the initial data at $\kappa = 0.95\kappa_c$, where the integration of the renormalization group equations was started.

particular, the constants C_i can be calculated by first integrating the renormalization group equations (I.4.26)–(I.4.28) from $m_R \simeq 0.5$ down to (say) $m_R = 0.001$ numerically. Then, for still smaller values of m_R, scaling violations in β, γ, δ are negligible and the integration can be carried on analytically to the limit $m_R \to 0$, where the C_i's are extracted from the asymptotic behavior of the solution. The result of this calculation for a selection of values of λ is shown in table 1.

For small λ the constants $C_i(\lambda)$ can also be calculated in perturbation theory. For example, to determine $\ln C_1$ up to terms of $O(\lambda)$, we start with the one-loop formula

$$g_R = \frac{6\lambda}{\kappa_c^2} + \left(\frac{6\lambda}{\kappa_c^2}\right)^2 \frac{3}{32\pi^2}\left[\ln m_R^2 - 3.7920\right] + O(\lambda^3), \tag{4.39}$$

where we have neglected scaling violations (which is permissible for $m_R \to 0$). Inserting this relation into the scaling law (I.4.30) and using the small λ expansion of $\kappa_c(\lambda)$ (appendix C of ref. [1]), then yields

$$\ln C_1 = \frac{\pi^2}{72\lambda} - \frac{17}{27}\ln\lambda + 1.1159 + O(\lambda). \tag{4.40}$$

To the order stated, this expansion is exact. It reproduces the values of $\ln C_1$ listed in table 1 within the quoted errors up to $\bar{\lambda} = 0.2$ and thus provides an excellent check of our calculations. We have also worked out the small λ expansions of C_2 and C_3 and found agreement with the data of table 1 (to the extent expected) in these cases too.

5. Integration of the renormalization group equations in the broken symmetry phase

We now proceed to calculate m_R, g_R, Z_R and Z_R^{φ} as a function of κ, λ in the broken symmetry phase $\kappa > \kappa_c$. The idea is, first to use the scaling laws (3.6)–(3.9) and the known values of the integration constants $C_i'(\lambda)$ to determine these quantities very close to the critical line, where the corrections to the scaling laws are negligible. After that the larger values of κ are reached by integrating the renormalization group equations (I.4.26)–(I.4.28) and (3.10) numerically using the 3-loop formulae for β, γ, δ. Of course, we have to make sure that this approximation is valid in the range of integration and we therefore study this question first.

5.1 THE TREE LEVEL UNITARITY BOUND AND THE APPLICABILITY OF RENORMALIZED PERTURBATION THEORY

As in sect. 7 of [1], the tree level unitarity bound on g_R is obtained from the requirement that the lowest order expression for the S-wave scattering matrix does not violate unitarity in the elastic region. Due to the contribution of one-particle exchange diagrams, the bound

$$g_R \lesssim 47 \tag{5.1}$$

derived in this way is somewhat lower than in the symmetric phase, but there is no qualitative difference.

In general, the perturbation expansion of low energy quantities is rather well convergent when the bound (5.1) is satisfied. In case of the physical particle mass

$$m = m_R\left\{1 - 0.01465\alpha_R - 0.02739\alpha_R^2 + O\left(g_R^3\right)\right\}, \tag{5.2}$$

$$\alpha_R = g_R/16\pi^2, \tag{5.3}$$

the higher order corrections are actually very small so that $m = m_R$ should be an excellent approximation*. For other quantities such as the 3-point coupling

$$\Gamma_R^{(3,0)}(0,0,0) = -m_R\sqrt{3g_R}\left\{1 + \tfrac{3}{2}\alpha_R + \tfrac{3}{4}\alpha_R^2 + O\left(g_R^3\right)\right\}, \tag{5.4}$$

* In eqs. (5.2)–(5.6) we have neglected scaling violations.

and the (amputated) full 4-point function

$$G_{\mathrm{R}}^{(4,0)}(0,0,0,0)=\Gamma_{\mathrm{R}}^{(4,0)}(0,0,0,0)+\frac{3}{m_{\mathrm{R}}^{2}}\left(\Gamma_{\mathrm{R}}^{(3,0)}(0,0,0)\right)^{2}$$

$$=8g_{\mathrm{R}}\left\{1+\tfrac{45}{16}\alpha_{\mathrm{R}}+\tfrac{45}{16}\alpha_{\mathrm{R}}^{2}+\mathrm{O}\left(g_{\mathrm{R}}^{3}\right)\right\}, \tag{5.5}$$

the perturbation expansion is however less well behaved, but the rate of convergence is about what one expects on the basis of (5.1).

To one-loop order, the S-wave scattering length a_0 is given by

$$a_0=\frac{g_{\mathrm{R}}}{8\pi m_{\mathrm{R}}}\left\{1+4.5491\alpha_{\mathrm{R}}+\mathrm{O}\left(g_{\mathrm{R}}^{2}\right)\right\}. \tag{5.6}$$

Since a_0 is positive, the particle interactions at large distances are attractive and it is conceivable that bound states form when g_{R} is sufficiently big. At the threshold value g_{R}^* of g_{R} where this happens, the scattering length a_0 diverges and from eq. (5.6) we thus estimate

$$g_{\mathrm{R}}^* \simeq 35 \tag{5.7}$$

(at this value of g_{R}, the one-loop "correction" is equal to the first term in eq. (5.6)). Of course, this argumentation is extremely crude and eq. (5.7) can only be taken as a rough indication for the minimal coupling strength required for bound state formation.

We finally remark that the expansions of the Callan-Symanzik coefficients β, γ, δ up to 3-loops (appendix A) are rather well convergent in the range (5.1), at least it seems reasonable to apply these formulae for $0 \leqslant g_{\mathrm{R}} \leqslant 31$, which will turn out to be the relevant domain in what follows.

5.2 RENORMALIZATION GROUP TRAJECTORIES AND THE TRIVIALITY BOUND

Following the steps sketched at the beginning of this section, we now integrate the renormalization group equations from the immediate neighborhood of the critical line up to $m_{\mathrm{R}}=0.5$. For the Callan-Symanzik coefficients β, γ, δ, we take the 3-loop formulae with the exact lattice expressions for the tree level and 1-loop coefficients (eqs. (A.1)–(A.6)). The results of this calculation, presented in table 2, show that the coupling g_{R} at $m_{\mathrm{R}}=0.5$ is monotonically increasing with λ and reaches a maximal value of about 31 in the Ising limit. For $m_{\mathrm{R}} \leqslant 0.5$, the coupling g_{R} is hence smaller than $\frac{2}{3}$ of the tree level unitarity bound (5.1), in particular, the use of the 3-loop approximation for β, γ, δ appears to be justified a posteriori. To sum up we have thus achieved an essentially complete solution of the ϕ^4 theory in the broken symmetry phase in a range of κ, which includes the scaling region $m_{\mathrm{R}} \leqslant 0.5$.

TABLE 2
Values of κ, g_R, Z_R, and $Z_R^{\emptyset}$ versus λ at $m_R = 0.5$ as obtained by integrating the renormalization group equations starting from the critical line

$\bar{\lambda}$	κ	g_R	Z_R	$Z_R^{\emptyset}$
0.01	0.1277(1)	1.197(6)	3.907(1)	0.0161(2)
0.02	0.1285(2)	2.32(2)	3.878(1)	0.0164(2)
0.03	0.1292(2)	3.37(4)	3.850(2)	0.0166(3)
0.04	0.1299(2)	4.36(7)	3.821(2)	0.0168(4)
0.05	0.1307(2)	5.3(1)	3.794(3)	0.0170(4)
0.06	0.1314(2)	6.2(1)	3.767(3)	0.0173(5)
0.07	0.1322(2)	7.0(2)	3.740(4)	0.0176(6)
0.08	0.1329(2)	7.9(2)	3.714(4)	0.0178(6)
0.09	0.1337(2)	8.6(3)	3.689(5)	0.0181(7)
0.10	0.1344(2)	9.4(3)	3.664(5)	0.0183(8)
0.20	0.1408(3)	15(1)	3.46(2)	0.021(1)
0.30	0.1445(3)	20(1)	3.34(2)	0.022(2)
0.40	0.1444(4)	23(2)	3.31(3)	0.024(2)
0.50	0.1402(4)	25(2)	3.39(3)	0.024(2)
0.60	0.1324(4)	27(2)	3.57(4)	0.024(3)
0.70	0.1217(4)	28(3)	3.86(5)	0.023(3)
0.80	0.1089(4)	29(3)	4.29(6)	0.022(3)
0.90	0.0942(4)	30(3)	4.94(8)	0.020(3)
1.00	0.0764(3)	31(3)	6.1(1)	0.017(2)

To give an impression of how this solution looks and what the typical errors involved are, we have listed g_R, Z_R, $Z_R^{\emptyset}$ and κ versus m_R for three values of $\bar{\lambda}$ in table 3. The similarity of these data lists with the corresponding ones in the symmetric phase (table 3 of ref. [1]) is striking, the only notable difference being the substantially smaller errors for κ at $\bar{\lambda} = 1$ quoted here. The reason for this is that κ is calculated differently in the two cases. Here we compute $\kappa - \kappa_c$ by integrating eq. (3.10) starting at the critical line and then add κ_c as determined from the high temperature expansion of the susceptibility χ_2 (table 1 of [1]), while in the symmetric phase, we integrated the renormalization group equations starting at $\kappa = 0.95\kappa_c$. Of course, the procedure applied in the broken symmetry phase works as well in the symmetric phase and in this way, the errors quoted for κ in table 3 of [1] could actually be reduced to the level reported here.

In order to have some sort of quantitative check on the applicability of perturbation theory for the renormalization group functions β, γ, δ, we have repeated our calculations using the 2-loop instead of the 3-loop expressions. The result was that no significant change would be implied in table 3 by this substitution except for g_R at $\bar{\lambda} = 1$, although even in this case the data agreed within the combined error margins.

The flow of the renormalization group trajectories in the scaling region is displayed in fig. 2. While the shape of the curves is similar in both phases, the

TABLE 3
Results from the solution of the renormalization group equations in the broken symmetry phase at $\bar{\lambda} = 0.01, 0.10$ and 1.00

$\bar{\lambda}$	m_R	g_R	Z_R	Z_R^ϕ	κ
0.01	0.50	1.197(6)	3.907(1)	0.0161(2)	0.1277(1)
	0.40	1.202(6)	3.929(1)	0.0161(2)	0.1270(1)
	0.30	1.204(6)	3.946(1)	0.0160(2)	0.1264(1)
	0.20	1.199(5)	3.959(1)	0.0160(2)	0.1260(1)
	0.10	1.184(5)	3.966(1)	0.0161(2)	0.1258(1)
	0.09	1.181(5)	3.967(1)	0.0161(2)	0.1258(1)
	0.08	1.178(5)	3.967(1)	0.0161(2)	0.1258(1)
	0.07	1.175(5)	3.968(1)	0.0161(2)	0.1258(1)
	0.06	1.171(5)	3.968(1)	0.0161(2)	0.1258(1)
	0.05	1.167(5)	3.968(1)	0.0162(2)	0.1257(1)
	0.04	1.161(5)	3.968(1)	0.0162(2)	0.1257(1)
	0.03	1.154(5)	3.969(1)	0.0162(2)	0.1257(1)
	0.02	1.144(5)	3.969(1)	0.0163(2)	0.1257(1)
	0.01	1.127(5)	3.969(1)	0.0163(2)	0.1257(1)
0.10	0.50	9.4(3)	3.664(5)	0.0183(8)	0.1344(2)
	0.40	9.1(3)	3.685(5)	0.0184(8)	0.1336(2)
	0.30	8.7(3)	3.702(5)	0.0186(8)	0.1330(2)
	0.20	8.1(2)	3.715(5)	0.0190(8)	0.1326(1)
	0.10	7.4(2)	3.726(5)	0.0195(8)	0.1323(1)
	0.09	7.3(2)	3.727(5)	0.0196(8)	0.1323(1)
	0.08	7.2(2)	3.728(5)	0.0197(8)	0.1323(1)
	0.07	7.1(2)	3.730(5)	0.0198(8)	0.1323(1)
	0.06	6.9(2)	3.731(5)	0.0200(8)	0.1323(1)
	0.05	6.8(2)	3.732(5)	0.0201(8)	0.1323(1)
	0.04	6.6(2)	3.733(5)	0.0203(8)	0.1323(1)
	0.03	6.4(1)	3.734(5)	0.0205(8)	0.1323(1)
	0.02	6.1(1)	3.736(4)	0.0209(8)	0.1322(1)
	0.01	5.7(1)	3.739(4)	0.0214(8)	0.1322(1)
1.00	0.50	31(3)	6.1(1)	0.017(2)	0.0764(3)
	0.40	27(2)	6.12(9)	0.017(2)	0.0759(2)
	0.30	24(2)	6.18(9)	0.018(2)	0.0754(2)
	0.20	20(1)	6.24(8)	0.019(2)	0.0751(1)
	0.10	16.4(9)	6.30(7)	0.020(2)	0.07485(9)
	0.09	16.0(8)	6.31(7)	0.020(2)	0.07483(8)
	0.08	15.5(8)	6.32(7)	0.020(2)	0.07481(8)
	0.07	15.0(7)	6.32(7)	0.020(2)	0.07480(8)
	0.06	14.4(7)	6.33(7)	0.021(2)	0.07479(8)
	0.05	13.8(6)	6.34(7)	0.021(2)	0.07478(8)
	0.04	13.1(6)	6.35(7)	0.021(2)	0.07477(8)
	0.03	12.3(5)	6.35(7)	0.022(2)	0.07476(8)
	0.02	11.4(4)	6.37(7)	0.022(3)	0.07476(8)
	0.01	10.0(3)	6.38(7)	0.023(3)	0.07476(7)

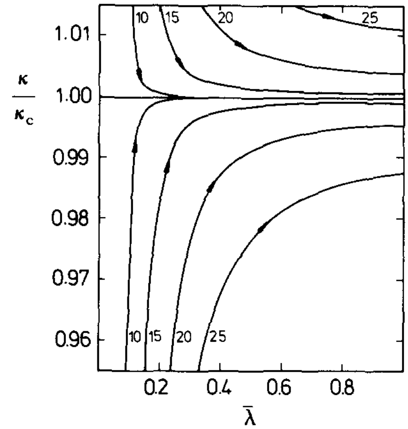

Fig. 2. Quantitative drawing of the renormalization group trajectories (curves of constant coupling g_R) in the plane of bare parameters for $g_R = 10$, 15, 20 and 25. The top and bottom of the diagram correspond approximately to $m_R = 0.5$ and arrows are in the direction of decreasing m_R (increasing ultra-violet cutoff in other words).

interval of κ corresponding to $m_R \lesssim 0.5$ is roughly a factor 3 smaller in the broken symmetry phase than in the symmetric phase. This difference essentially originates in the fact that the coefficient ε in the renormalization group equation (3.10), which describes the evolution of κ as a function of m_R, is differently related to β, γ, δ in the two phases.

As shown in fig. 2, the maximal value of the cutoff Λ in units of m_R at fixed g_R is attained in the Ising limit $\bar{\lambda} = 1$. This observation together with the data listed in table 3 leads to the triviality bound plotted in fig. 3, where instead of the coupling g_R, we have taken m_R/v_R as the independent variable, because in the context of the Higgs model, this quantity has a more direct physical significance (recall $g_R = 3m_R^2/v_R^2$ in our renormalization scheme). A striking feature of fig. 3 is that the upper bound on the cutoff is very rapidly rising when m_R/v_R decreases from 3 to 2. For example, if we assume the standard model value $v_R = 250$ GeV for the purpose of illustration, a mass $m_R = 750$ GeV would imply $\Lambda \leqslant 1.9(2)$ TeV, whereas for $m_R = 500$ GeV the maximal allowed value of Λ would be as high as 19(3) TeV.

6. Conclusions

The most conspicuous result of our analysis of the one-component lattice ϕ^4 theory in the broken symmetry phase is that, concerning the scaling behaviour, there is practically no difference to what happens in the symmetric phase [1]. In particular, as soon as the ultra-violet cutoff Λ is larger than two times the physical particle mass, the renormalized coupling g_R cannot exceed a maximal value of about $\frac{2}{3}$ of the tree level unitarity bound and renormalized perturbation theory should hence give an essentially correct description of the particle interactions at low energies.

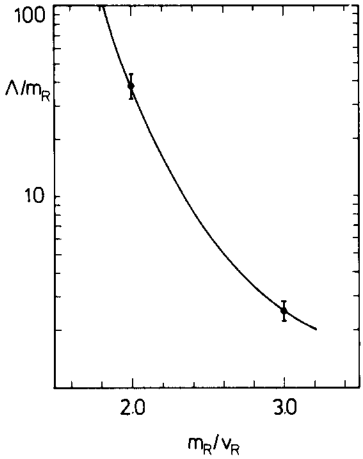

Fig. 3. Maximal value of the ultra-violet cutoff Λ in units of m_R for given m_R/v_R (v_R: renormalized vacuum expectation value of the field ϕ). The size of the estimated errors as quoted in table 3 is indicated at two representative points.

From the triviality bound shown in fig. 3, one obtains an upper bound

$$m_R \leqslant 3.2 v_R \tag{6.1}$$

on the particle mass m_R, if one requires $\Lambda \geqslant 2m_R$. Since we have only considered the one-component model in this paper, we cannot assign any immediate physical significance to this bound, but it is nevertheless interesting to see what it would imply in the context of the standard model. There m_R would be the Higgs particle mass m_H and the vacuum expectation value v_R would be related to the W boson mass m_W through $v_R \simeq 3m_W$. Thus, eq. (6.1) amounts to $m_H \lesssim 9.6 m_W$, which is actually rather close to the bounds obtained recently in Monte Carlo studies of the lattice SU(2) Higgs model [8–10] and those derived earlier on the basis of certain theoretical or phenomenological assumptions (ref. [11] and references quoted there).

In the scaling region $\Lambda \geqslant 2m_R$, the maximal possible value of g_R is only slightly lower than our estimate eq. (5.7) of the minimal coupling g_R^* which is required for bound state formation. We therefore cannot exclude the existence of bound state particles for (say) $\Lambda \lesssim 10 m_R$ and large bare coupling λ, but it is quite clear that for all $\Lambda \geqslant 2m_R$, bound state formation is marginal and the binding energies should be rather small.

Although our analysis did not involve any unjustified approximations, it would nevertheless be very important to check our results by a large scale numerical simulation similar to the one already performed in the symmetric phase [2], because we cannot be absolutely sure that the (weak) qualitative assumptions that we made along the way (for example, that the renormalized perturbation theory is applicable when g_R is small) are in fact true. As discussed in subsect. 5.2, the whole scaling region $\Lambda \geqslant 2m_R$ is squeezed into the interval $\kappa_c < \kappa \lesssim 1.015\kappa_c$ so that in such a simulation κ must be tuned on an even finer scale than is already necessary in the symmetric phase. Also, compared to the symmetric phase, finite size effects are expected to be significantly enhanced. One of the reasons for this is that in finite volume there exists a slow mode, which is associated with fluctuations of the total "magnetization" of the system between the classically degenerate ground states [12, 13]. Another source for finite size effects are one-particle exchange processes "around the world" [14, 15]. In the broken symmetry phase, these make a rather large contribution to the volume dependence of the particle mass, because the leading ("pole") term in the mass shift formula of refs. [14, 15] does not vanish in this case. Finally, we note that for given m_R and g_R, the scattering length a_0 in the broken symmetry phase is at least 4 times as big as in the symmetric phase so that the finite size shift of the lowest two-particle energy level will also be enhanced by about this factor [2, 16] (in this calculation additional difficulties must be expected when bound states exist).

The solution of the ϕ^4 theory in the broken symmetry phase presented in this paper is based on the observation that the scaling properties in the two phases of the model are related in a one-to-one fashion through a set of (computable) linear relations. We expect the same is true for any "trivial" field theory irrespective of whether there are Goldstone particles, gauge fields or other complications. In particular, our methods should apply straightforwardly to the 4-component ϕ^4 theory and thus we hope to come back to this physically more interesting case in the near future [17].

Appendix A

PERTURBATION EXPANSION OF THE CALLAN-SYMANZIK COEFFICIENTS β, γ AND δ IN THE BROKEN SYMMETRY PHASE

The method of calculation is exactly the same as in the symmetric phase (appendix A of ref. [1]) so that here we merely quote the results. The universal coefficients β_ν, γ_ν, and δ_ν as defined through eqs. (I.A.1)–(I.A.3) are given up to three loops in table 4. For the full m_R-dependent coefficients u_ν, v_ν and w_ν (eqs.

TABLE 4
Perturbation expansion coefficients for β, γ, and δ in the broken symmetry phase

ν	$(16\pi^2)^\nu\beta_\nu$	$(16\pi^2)^\nu\gamma_\nu$	$(16\pi^2)^\nu\delta_\nu$
1	3	0	−1
2	−17/3	−7/24	13/12
3	14.715616	−1.1224089	0.13543414

TABLE 5
Coefficients in the Chebyshev expansion (I.B.9) of the integral J_4

n	$c_{4,n}$
0	1.0159974
1	0.0211666
2	0.0034174
3	−0.0030079
4	−0.0012099
5	0.0001196
6	0.0000626
7	−0.0000110
8	−0.0000004

(I.A.4)–(I.A.6)), we have up to one loop

$$u_0 = -\frac{4m_R^2}{16-m_R^2}, \tag{A.1}$$

$$u_1 = -\frac{u_0}{16-m_R^2}\left\{J_1(m_R) + \left(32-\tfrac{3}{2}m_R^2\right)J_2(m_R) + \left(256-48m_R^2+\tfrac{7}{2}m_R^4\right)J_3(m_R)\right.$$
$$\left. +3m_R^2\left(16-m_R^2\right)\left(8+m_R^2\right)J_4(m_R)\right\}, \tag{A.2}$$

$$v_0 = -\frac{m_R^2}{16-m_R^2}, \tag{A.3}$$

$$v_1 = -\frac{v_0}{16-m_R^2}\left\{J_1(m_R) + 8J_2(m_R) - \left(128-2m_R^4\right)J_3(m_R)\right.$$
$$\left. +3m_R^2\left(16-m_R^2\right)\left(8+m_R^2\right)J_4(m_R)\right\}, \tag{A.4}$$

$$w_0 = \frac{2m_R^2}{16-m_R^2}, \tag{A.5}$$

$$w_1 = \frac{w_0}{16-m_R^2}\left\{\left(1-\tfrac{1}{8}m_R^2\right)J_1(m_R) + \left(8-5m_R^2+\tfrac{1}{4}m_R^4\right)J_2(m_R)\right.$$
$$\left. -\left(128-m_R^4+\tfrac{1}{8}m_R^6\right)J_3(m_R) - 3m_R^2\left(16-m_R^2\right)\left(8+m_R^2\right)J_4(m_R)\right\}. \tag{A.6}$$

The integrals $J_p(\mu)$ in these equations are the same as in [1] (eq. (I.A.13) and appendix B). An accurate numerical representation, using Chebyshev polynomials, was given there for J_1, J_2, and J_3 in the range $0 < \mu \leqslant 1$. This representation (eq.

(I.B.9)) is also valid for J_4 to a relative accuracy of better than 10^{-6}, if we choose $N=8$ and insert the values listed in table 5 for the coefficients $c_{4,n}$.

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