QUANTUM FIELD THEORY ON RIEMANN SURFACES AND THE UNITARITY PROBLEM

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By the use of the Klein method instead of the theta-function method of Jacobi we are able to relate a conformal quantum theory on Riemann surfaces to the corresponding flat-space field theory and its Virasoro algebra. Physical positivity holds on a distinguished real subset in the manifold with nontrivial Hausdorff dimension which in the general case $g>1$ cannot be shifted by a hamiltonian. Our picture of obtaining curved two-dimensional quantum field theories by applying a special diffeomorphism to flat ones resembles that of the Hawking–Unruh effect.

The modern covariant formulation [1] of relativistic strings is based on perturbative Feynman integrals. The g-loop order requires the study of special two-dimensional conformal quantum field theories on Riemann surfaces of genus g and their dependence on Teichmüller parameters. Using the method of Jacobi, physicists have learned [#1] how to calculate free-field genus-g correlation functions of arbitrary spin in terms of Jacobi theta functions. The price to be payed for this formal elegance was very high (up to now). There is no simple natural relation of this approach to the flat-space conformal invariant field theory with its circular diffeomorphism group and the Virasoro algebra. From experience with the $g=1$ case and its temperature formalism, one knows that there exists a one-dimensional "real" submanifold on which the free-field correlation functions for spin $s\neq 0$ fields are positive definite in the sense of Wightman, i.e., have positive-definite Källen–Lehmann spectral functions. Using the θ-function representation it becomes however quite cumbersome to derive this unitarity statement, even in the case $g=1$. In the $g>1$ case I have not beeen able to see a generalization of this unitarity structure by just studying the explicit θ-function representation [#2]. There exists however an alternative method which, although not leading (up to now) to such elegant compact formulas, nevertheless has the elegance of a field theoretic method. Let us first revisit the $g=1$ case. Consider first the special case of an $s=1$ field (a current) on a rectangular torus. By averaging over the lattice points $(n,m\beta)$, I define a new field on the u light-cone:

$$J^{\beta}(u)=\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{n,m}^{N}J_{(n,m\beta)}(u)\ . \tag{1}$$

For positive m this is the action of the semigroup of the light-cone hamiltonian on a free field. The sum in (1) ranges over N lattice points with $N\to\infty$. The two-point function

$$\langle J(u)J(u')\rangle=\frac{1}{2\pi\mathrm{i}}\frac{1}{(u-u'-\mathrm{i}\epsilon)^2} \tag{2}$$

implies

$$\begin{aligned}\langle J^{\beta}(u)J^{\beta}(u')\rangle&=\sum_{n,m}\langle J(u)J_{(n,m\beta)}(u')\rangle\\&=\frac{1}{2\pi\mathrm{i}}\sum_{n,m}\frac{1}{(u-u'+n+\mathrm{i}\beta m)^2}\\&=\frac{1}{2\pi\mathrm{i}}\frac{1}{u-u'-\mathrm{i}\epsilon}\\&+\frac{1}{2\pi\mathrm{i}}\sum_{(n,m)\neq(0,0)}\left(\frac{1}{(u-u'+n+\mathrm{i}\beta m)^2}-\frac{1}{(n+\mathrm{i}\beta m)^2}\right)\ .\end{aligned} \tag{3}$$

[#1] See ref. [2] for a recent account.
[#2] I am indebted to R. Schrader for informing me that D. Friedan is working on this problem.

The last expression has been obtained by symmetric summation in m and represents the Weierstrass p-function restricted to the real light-cone variable $z=u$.

The positivity follows either directly from (1) by smearing with an (anti)analytic test function or from the form of the Källen–Lehmann spectral function (the Fourier transform of (3)):

$$\rho^{\beta}(p)=\theta(p)p^{3/2}\delta_{\mathrm{pcr.}}(p) \times \sum_{n\geq 0}[\exp(\beta np)+\exp(-\beta np)] . \qquad (4)$$

Three comments are in order:

(1) Spinors have nontrivial transformation law under global conformal transformations: they live on a two-fold covering of Minkowski space. Appropriate signs in the sum (1) can easily be attached, which account for the different spin structures on the torus.

(2) The correlation functions of J^{τ} for τ complex is not positive definite. However, if one adds the conjugate situations $-\tau^*$, positivity is restored. For string theoretic purpose this is sufficient for getting positivity after integration over the Teichmüller parameter τ.

(3) The resulting torus correlation function has the remaining H-symmetry. The existence of an abelian symmetry implies the positivity on each real line parallel to the light-cone line. This means that the (thermal) reference state, which is obtained via the Gel'fand–Segal reconstruction of the Hilbert space from the set of positive expectation values, is not a lowest weight state of the symmetry generator. To be more precise, there are two operators H. One is obtained by integrating the hamiltonian density. This operator is bounded below, but the thermal reference state is not an eigenstate. The symmetry generator on the other hand cannot be obtained in the field algebra. Its construction requires the additional use of the nontrivial commutant of the field algebra, which turns out to be (anti)isomorphic to the field algebra [3]. These two copies exist also for the total Virasoro algebra.

The generalization of this construction to $g>1$ is conceptually simple but computationally involved. Take a fuchsian group G, i.e. a discrete subgroup of SL(2, ℝ) generating the particular compact Riemann surface [4]. Such a group is generated by a finite number of elements. These elements correspond to the fundamental domain which has a hyperbolic polygon as its boundary. Such a polygon can always be choosen to touch the light-cone. A polygon side which lies on the light-cone (i.e. the real axis) is called a free side and the others are inner sides. Inner sides are pairwise equivalent by the generators of the fuchsian group but free sides remain distinguished. Let us again consider an $s=1$ free field $I(n)$ which has a trivial transformation law under the covering SL(2, ℝ). In analogy to the previous case we define

$$J^{\mathrm{G}}(u)=\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{g_i\in\mathrm{G}}^{N}U(g_i)J(u)U^{+}(g_i) . \qquad (5)$$

Now the U are true unitary operators in the Fock space. The Nth approximant clearly leads to a positive-definite two-point function and this state of affairs cannot change in the limit $N\to\infty$. The resulting two-point function

$$\langle J^{\mathrm{G}}(u)J^{\mathrm{G}}(u')\rangle=\sum_{g_i\in\mathrm{G}}\langle J(u)U(g_i)J(u')\rangle_0 \qquad (6)$$

depends on the particular fuchsian group. The free side of the polygon lying on the original light-cone is the real submanifold on which one has Wightman positivity. It is distinguished and unlike the torus case there exists no hamiltonian which shifts it to other places. The free line represents points which in the hyperbolic metric are at infinity, i.e. such a situation describes a noncompact surface. For a compact Riemann surface a point on the light-cone line can be represented by a sequence of points on the surface, i.e., the distinguished set is not a smooth one-dimensional submanifold but rather a quite complicated topological subset of Hausdorff dimension larger than one. The field theoretic generation of the Teichmüller parameters proceeds as follows. Consider special non-Möbius diffeomorphisms of $\mathrm{Diff}(\mathbb{R})\simeq\mathrm{Diff}(S^1)$ generated by the subgroups belonging to the infinitesimal generators L_0, L_n, L_{-n}. These transformations have the form

$$u\to f_n(u)=(h_n^{-1}\circ A\circ h_n)(u) . \qquad (7)$$

h is a covering transformation which in the compact (S^1) language corresponds to

$$h: z\to z^n ,$$

whereas

$$A=(au+b)/(cu+d) . \qquad (8)$$

The result of all three transformations is a bona fide (non-covering) transformation from Diff(ℝ). In addition to demanding $f(+\infty)=+\infty$ we normalize the f at two finite points, say $f(0)=0$ and $f(1)=1$. For odd n the only normalized maps of the form (7) are the identity. For $n=4m+2$, the Möbius part consists of just the translation, whereas for $n=4m$ one obtains all affine Möbius transformations. A straightforward calculation reveals that for $u_2>0$

$$\frac{1}{c_n} < \frac{f(u_1+u_2)-f(u_1)}{f(u_1)-f(u_1-u_2)} < c_n \,, \tag{9}$$

where c_n depends on n but not on u_i. Diffeomorphisms with this property are called (normalized) "quasisymmetric" transformations. They form a subgroup of Diff(ℝ) and possess a deep relation to the Beltramie form and "quasiconformal" transformations [4]. They form the points of the universal infinite-dimensional Teichmüller space used by Bers [5]. The smallest c_n is a measure from the distance to a Möbius transformation which in the case of our normalization must be the identity. The finite-dimensional Teichmüller spaces of compact Riemann surfaces are subspaces of a quasisymmetric diffeomorphism fullfilling

$$(fGf^{-1})(u) = \text{Möbius transformation} \,, \tag{10}$$

where G represents the previously introduced Fuchs group. If the set of Möbius transformations on the right-hand side form again the group G, i.e., for automorphisms f of G one obtains the so-called modular transformation. Dividing out this discrete group of diffeomorphisms one descends from the Teichmüller space of all "G-covariant" diffeomorphisms (10) to the Riemann moduli space. It is fairly easy to see that the level-n quasisymmetric transformation of form (7) can never have property (10); one needs infinite products

$$f(u) = \prod_n f_n(u) \,. \tag{11}$$

The ansatz (11) may be converted into a Virasoro algebra relation by using the fact that the f_n lie on one parametric subgroup. The only finite transformations with such a relation in the Virasoro algebra are the unitary representations of the fuchsian group G. We hope to return to some explicit algebraic calculation identifying elements of the Virasoro algebra (involving infinite high level elements) with the tangent vectors to the origin of Teichmüller space. Our main purpose in this note was to point out that there exists a thoroughly quantum field theoretical approach to Riemann surfaces and Teichmüller spaces in which the link with the flat-space conformal field theory with its Virasoro generators is essentially used.

In some sense our approach resembles the Hawking–Unruh [5] construction of generating new correlation functions with horizons and a Hawking temperature (→Teichmüller parameters?) by doing just (acceleration) transformations on flat-space correlation functions. Related mathematical topics are the determination of the commutant algebra for the J^G and the connection with possible modular properties of the local algebras of quantum field theory [6].

References

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