GENERALIZED BOGOLIUBOV TRANSFORMATIONS IN LATTICE GAUGE THEORY WITH STAGGERED FERMIONS

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A transformation is investigated that mixes quarks with composites of $N-1$ antiquarks in a gauge-invariant way for QCD with gauge group SU(N). An infinite family of identities among fermionic Green functions is derived in the form of a generating functional.

A popular choice for the lattice discretization of euclidean fermions is the staggered action [1]

$$S=\sum_{x\mu}\Gamma_\mu(x)[\bar\psi(x)U_\mu(x)\psi(x+\mu)$$
$$-\bar\psi(x+\mu)U_\mu^\dagger(x)\psi(x)]+m\sum_x\bar\psi(x)\psi(x)$$
$$=:\sum_x(\bar\psi\not{\Delta}\psi+m\bar\psi\psi)\,, \tag{1}$$

where $\Gamma_\mu(x)$ are the standard phase factors stemming from the Dirac matrices, and $U_\mu(x)$ is an arbitrary SU(N) Wilson type gauge field. From the Grassmann fields ψ, $\bar\psi$ we form the local (anti)baryon and meson composites

$$B(x)=(1/N!)\epsilon_{a_1\ldots a_N}\psi_{a_1}(x)\ldots\psi_{a_N}(x)\,,$$
$$\bar B(x)=(1/N!)\epsilon_{a_1\ldots a_N}\bar\psi_{a_N}(x)\ldots\bar\psi_{a_1}(x)\,, \tag{2}$$
$$M(x)=\bar\psi_a(x)\psi_a(x)\,. \tag{3}$$

In (2) and (3) we exhibit the color index $a_i=1,\ldots,N$, which is the only index carried by ψ, $\bar\psi$ and $\epsilon_{a_1\ldots a_N}$ is the SU(N)-invariant antisymmetric symbol. It is known [2] that for $N=2$ mesons and baryons "are the same". This case is popular, because the formidable numerical problem of incorporating fermions is somewhat ameliorated as compared to the physical value $N=3$. A more precise statement is that for $N=2$ and vanishing mass in (1) there is an additional global SU(2) symmetry under which $(B,\bar B,M)$ transform as a triplet. This becomes manifest if we introduce a field [#1] $\chi_a^\alpha(x)$ with a new index $\alpha=1,2$

$$\chi_a^1(x)=\epsilon_{ab}\bar\psi_b(x)\,,\quad \chi_a^2(x)=\psi_a(x)\,. \tag{4}$$

An easy rearrangement of terms shows that (1) reads in terms of χ_a^α ($N=2$)

$$S=\sum_{x\mu}\Gamma_\mu(x)[\epsilon U_\mu(x)]_{ab}\epsilon^{\alpha\beta}\chi_a^\alpha(x)\chi_b^\beta(x+\mu)$$
$$+\frac{m}{2}\sum_x\epsilon_{ab}\tau_1^{\alpha\beta}\chi_a^\alpha(x)\chi_b^\beta(x)\,, \tag{5}$$

where τ is a Pauli matrix. Clearly, for $m=0$, S is invariant under a global SU(2) acting on α, i.e., mixing ψ and $\bar\psi$. Such a mixing also occurs as the analogue of Bogoliubov transformations [3], if the BCS model partition function is formulated as a Grassmann functional integral over non-relativistic fermion fields [4]. The baryon and meson fields assume the form

$$\boldsymbol{P}(x)=\tfrac12(\epsilon\boldsymbol{\tau})^{\alpha\beta}\epsilon_{ab}\chi_a^\alpha(x)\chi_b^\beta(x)$$
$$=(\bar B(x)+B(x),\,\mathrm{i}(\bar B(x)-B(x)),\,M(x))\,. \tag{6}$$

We see that for $N=2$ a mass term is similar to an external field "magnetizing" $\boldsymbol{P}$ in a fixed direction, and fermion number is the left-over symmetry of rotations around that axis.

[#1] For euclidean fermions ψ and $\bar\psi$ are independent integration variables.

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In this letter we discuss consequences of the possibility to mix ψ and $\bar{\psi}$ in a gauge-invariant way also for $N\geqslant 3$. To that end we consider a transformation

$$\psi_a \to \psi'_a = \psi_a + \alpha\varphi_a , \quad \bar{\psi}_a \to \bar{\psi}'_a = \bar{\psi}_a + \bar{\alpha}\bar{\varphi}_a , \tag{7}$$

with

$$\varphi_a = [1/(N-1)!]\, \epsilon_{a_1 \dots a_{N-1} a} \bar{\psi}_{a_{N-1}} \dots \bar{\psi}_{a_1} ,$$
$$\bar{\varphi}_a = [1/(N-1)!]\, \epsilon_{a_1 \dots a_{N-1} a} \psi_{a_1} \dots \psi_{a_{N-1}} . \tag{8}$$

The parameters α, $\bar{\alpha}$ are (anti)commuting scalars if N is even (odd). Note, that we always mix odd Grassmann numbers, and that (7) is a gauge-covariant equation. For $N=2$ the fermion number phase group together with (7) compose the extra SU(2) symmetry [#2]. For $N>2$, however, the kinetic term varies under (7), and also the jacobian of the transformation has to be worked out as (7) is non-linear. The variations of gauge-invariant composites are as follows

$$M \to M + N(\bar{B}\alpha + \bar{\alpha}B) + [\bar{\alpha}\alpha/(N-1)!](-M)^{N-1} ,$$
$$B \to B + [\alpha/(N-1)!](-M)^{N-1} - \delta_{N,2}\alpha^2 \bar{B} ,$$
$$\bar{B} \to \bar{B} + [\bar{\alpha}/(N-1)!](-M)^{N-1} - \delta_{N,2}\bar{\alpha}^2 B \tag{9}$$

and

$$S \to S + S_\alpha + S_{\bar{\alpha}} + S_{\bar{\alpha}\alpha} , \tag{10}$$

with

$$S_\alpha = \sum_x \bar{\psi}\not{\Delta}\varphi\alpha + mN\sum_x \bar{B}\alpha ,$$
$$S_{\bar{\alpha}} = \sum_x \bar{\alpha}\bar{\varphi}\not{\Delta}\psi + mN\sum_x \bar{\alpha}B ,$$
$$S_{\bar{\alpha}\alpha} = \sum_x \bar{\alpha}\bar{\varphi}\not{\Delta}\varphi\alpha + \frac{m}{(N-1)!}\sum_x \bar{\alpha}\alpha(-\bar{\psi}\psi)^{N-1} . \tag{11}$$

The possibility of non-linear changes of variables in Grassmann integrals has already been mentioned in ref. [5] and presented in detail in ref. [6]. For our purpose it is adequate to consider a generic integral over an n-dimensional Grassmann algebra,

$$I = \int d\eta_1 \dots d\eta_n f(\eta) , \tag{12}$$

[#2] This is strictly true for infinitesimal α, $\bar{\alpha}$; otherwise the field has to be rescaled trivially to define a proper SU(2) mixing.

and a "general coordinate transformation",

$$\eta_\mu \to \eta'_\mu(\eta) . \tag{13}$$

Here η'_μ is assumed to be odd, i.e., even monomials in the expansion of η'_μ have c-number coefficients, and odd ones, if they occur, have anticommuting coefficients. Moreover we want (13) to be invertible as a power series, which is the case if its linear part $\partial\eta'_\mu/\partial\eta_\nu|_{\eta=0}$ is a non-singular c-number matrix. Then it follows from results in ref. [6] that

$$\int d\eta_1 \dots d\eta_n f(\eta) = \int d\eta_1 \dots d\eta_n \, \det(\partial\eta'_\mu/\partial\eta_\nu)^{-1} f(\eta') \tag{14}$$

holds. As in the linear case the only difference as compared to ordinary integrals is the exponent of the jacobian determinant. Note that only even Grassmann elements appear under det( )$^{-1}$ which can be defined purely algebraically. Also, left and right differentiation [5] give the same matrix elements. For transformation (7) the resulting jacobian is given by

$$\exp(S_J) = \prod_x \det\left(\frac{\partial(\psi'(x), \bar{\psi}'(x))}{\partial(\psi(x), \bar{\psi}(x))}\right)^{-1}$$
$$= \prod_x (1+\bar{\alpha}\alpha)^{-2} \quad \text{for } N=2 ,$$
$$= \prod_x 1 - 2[\bar{\alpha}\alpha/(N-2)!](-\bar{\psi}\psi)^{N-2}$$
$$\text{for } N>2 , \tag{15}$$

and thus for the non-trivial cases $N>3$

$$S_J = \sum_x \{-2[\bar{\alpha}\alpha/(N-2)!](-\bar{\psi}\psi)^{N-2}$$
$$+ \delta_{N,4}\tfrac{1}{2}(\bar{\alpha}\alpha)^2(\bar{\psi}\psi)^4\} , \tag{16}$$

where the nilpotency properties $(\bar{\psi}\psi)^{N+1}=0$ and $(\bar{\alpha}\alpha)^2=0$ for N=odd have been used. Note that for the physical case $N=3$ the chiral condensate appears in (16). If we now combine our results it can been shown that

$$\int D\psi\, D\bar{\psi} \exp(S)$$
$$= \int D\psi\, D\bar{\psi} \exp(S + S_\alpha + S_{\bar{\alpha}} + S_{\bar{\alpha}\alpha} + S_J) \tag{17}$$

or

$$\langle \exp(S_\alpha + S_{\bar{\alpha}} + S_{\bar{\alpha}\alpha} + S_J) \rangle = 1 . \tag{18}$$

Differentiation of the RHS of (18) with respect to the general x-dependent α, $\bar{\alpha}$ produces gauge-invariant identities. Since we worked with an arbitrary background gauge field they hold for both dynamical and quenched staggered fermions. One example of an identity following from (18) (order $\bar{\alpha}\alpha$ at one site) is

$$\begin{aligned}
&\langle [(\bar{\psi}\not{\Delta})(x)\varphi(x)+mN\bar{B}(x)] \\
&\quad \times[\bar{\varphi}(x)(\not{\Delta}\psi)(x)+mNB(x)] \\
&\quad -[m/(N-1)!][\bar{\psi}\psi(x)]^{N-1} \\
&\quad -[2/(N-2)!][\bar{\psi}\psi(x)]^{N-2}\rangle \\
&\quad =0\,, \qquad (19)
\end{aligned}$$

or for $m=0$ and $N=3$

$$2\langle\bar{\psi}\psi(x)\rangle=\langle(\bar{\psi}\not{\Delta})(x)\varphi(x)\bar{\varphi}(x)(\not{\Delta}\psi)(x)\rangle. \qquad (20)$$

Such a relation could in principle be used or monitored in numerical simulations. Clearly, (20) is easily checked in terms of Feynman diagrams, but the Bogoliubov transformation systematically produces an infinite family of such gauge-invariant identities.

One of the original motivations to develop generalized Bogoliubov transformations for staggered fermions was related to the dimer simulation of baryons at strong coupling [7]. These sources conjugate to B, $\bar{B}$ had to be introduced to run the algorithm, and then they had to be numerically extrapolated to zero strength. A transformation with α, $\bar{\alpha}$ constant and non-zero ($N=2$ or 4) produces the source terms automatically without changing the physics. A closer inspection of the new terms in (11) and (16) revealed, however, that it is unavoidable to produce new negative amplitudes in the dimer model along with B, $\bar{B}$ sources. Thus the notorious negative weight problem for fermions reappears and renders the Bogoliubov transformed version of staggered fermions useless for Monte Carlo simulation by the dimer method. Nevertheless, we thought that the application of non-linear changes of Grassmann variables is of interest, and that identities contained in (18) may be useful in other contexts.

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