HIERARCHICAL STRUCTURE OF FERMION MASSES AND MIXINGS

Johan BIJNENS [1] and Christof WETTERICH
Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg, Fed. Rep. Germany

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We examine patterns where ratios of the fermion masses and the W-boson mass ($x_i = m_i/m_W$) are proportional to powers of a small parameter λ ($x_i = c_i\lambda^{P_i}$). For a simple estimate of the uncertainty in the coefficients c_i we determine the allowed values of P_i and the corresponding range of λ. Using this information we search for realistic patterns in a large class of anomaly-free $SU(3)\times SU(2)\times U(1)\times U(1)$ models where λ is related to a symmetry breaking scale and the P_i follow from the quantum numbers. No realistic model is found. In contrast, realistic mass patterns can be induced from an anomalous U(1) symmetry.

It has been proposed [1] that small quantities appearing in the fermion mass matrices correspond to different powers of a small parameter λ. Models have been constructed where *all* small mixing angles and small mass ratios $x_i = m_i/m_W$ can be understood in terms of a symmetry, $x_i = c_i\lambda^{P_i}$ [2]. The parameter λ is a ratio of symmetry breaking scales and the various powers of λ follow from the quantum numbers under this symmetry. No small quantities besides λ are needed. In particular all the dimensionless couplings (Yukawa, gauge and scalar) are supposed to be of the same order of magnitude.

First we discuss in what sense λ and P_i determine the various quantities. Then we give an approximate diagonalization of the fermion mass matrices and use this to estimate the uncertainty in c_i. This information together with the experiental values of the fermion masses and mixings then fix the allowed regions of λ and powers P_i. A typical Yukawa coupling of the order of the weak gauge coupling leads to a fermion mass of order m_W. We write the dimensionless mass ratios and the mixing angles as

$$x_i = m_i/m_W = c_i\lambda^{P_i} \tag{1}$$

$$\theta_{ij} = c_{ij}\lambda^{P_{ij}} . \tag{2}$$

In (2) θ_{ij} is the mixing angle between generation i and j. We now want to fix λ and P_i, P_{ij} from the x_i and θ_{ij}. This is course depends on the allowed range of values for the c_i and c_{ij}. These quantities cannot be understood purely in terms of symmetry and their values depend on specific details of a model. For the models considered in ref. [2] these coefficients are given by ratios of dimensionless coupling constants. In the context of higher dimensional unification they correspond to generalized Clebsch–Gordan coefficients [3]. In addition the c_i often have several contributions. The number of contributions typically increases with a higher power P_i. We therefore expect a larger uncertainty for the smaller quantities, in particular for the first-generation masses. We will take the c_i to be equal to one within a multiplicative uncertainty Δ_i, which reflects our lack of knowledge of the details of a model.

$$1/\Delta_i \leqslant c_i \leqslant \Delta_i . \tag{3}$$

So if x_i^+ and x_i^- are the experimental upper and lower bound for x_i the allowed values for λ for a given P_i are those that satisfy

$$x_i^-/\Delta_i \leqslant \lambda^{P_i} \leqslant \Delta_i x_i^+ . \tag{4}$$

In this letter we will take for the masses of the third generation a standard uncertainty $\Delta = 2$. The uncertainty for the other x_i, θ_{ij} is taken as $\sqrt{n_i}\Delta$ and $\sqrt{n_{ij}}\Delta$ with n_i discussed below.

The powers P_i and the coefficients c_i come from a diagonalization of the fermion mass matrices. We will

[1] Present address: Sektion Physik, Universität München, Theresienstr. 37, D-8000 Munich 2, Fed. Rep. Germany.

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perform this diagonalization explicitly. The elements of the up quark mass matrix M_U are given by

$$u_{ij} = c_{ij}^{U} \lambda^{U_{ij}} m_W . \tag{5}$$

Here i labels the species of right-handed quarks u_i^c and j stands for the generation of left-handed quarks u_j. We assume the matrix to be properly ordered so that u_{33} is the largest element, i.e. the mass of the top quark m_t. We are only interested in the power of λ and neglect unnatural cancellations. This allows us to use the observed smallness of the mixings with the third generation to perform a simplified diagonalization of M_U. We first rotate the elements u_{13} and u_{23} to zero. The 33 element of the resulting matrix v_{ij} determines the top quark mass ($m_t = v_{33} \simeq u_{33}$). The other matrix elements induced by this rotation are of order [#1]

$$v_{11} = u_{11} + \frac{u_{31}u_{13}}{m_t} + \frac{u_{21}u_{23}u_{13}}{m_t^2} , \tag{6}$$

$$v_{12} = u_{12} + \frac{u_{32}u_{13}}{m_t} + \frac{u_{22}u_{23}u_{13}}{m_t^2} , \tag{7}$$

$$v_{21} = u_{21} + \frac{u_{31}u_{23}}{m_t} + \frac{u_{11}u_{23}u_{13}}{m_t^2} , \tag{8}$$

$$v_{22} = u_{22} + \frac{u_{32}u_{23}}{m_t} + \frac{u_{12}u_{23}u_{13}}{m_t^2} , \tag{9}$$

$$v_{31} = u_{31} + \frac{u_{21}u_{23}}{m_t} + \frac{u_{11}u_{13}}{m_t} , \tag{10}$$

$$v_{32} = u_{32} + \frac{u_{22}u_{23}}{m_t} + \frac{u_{12}u_{13}}{m_t} . \tag{11}$$

Next we rotate away the elements v_{31} and v_{32}. This defines the contributions from M_U to the mixing angles with the third generation:

$$\theta_{13}^{U} = \frac{u_{31}}{m_t} + \frac{u_{21}u_{23}^*}{m_t^2} + \frac{u_{11}u_{13}^*}{m_t^2} , \tag{12}$$

$$\theta_{23}^{U} = \frac{u_{32}}{m_t} + \frac{u_{22}u_{23}^*}{m_t^2} + \frac{u_{12}u_{13}^*}{m_t^2} . \tag{13}$$

This, of course, again induces elements in the top quark column (u_{13}, u_{23}). They are, however, suppressed by the smallness of the angles θ_{i3} and the small relative size of v_{ij} for $i, j = 1, 2$. We neglect them and consider only the remaining 2×2 matrix for the lower generations. Up to negligible corrections $\sim \theta_{13}^{U}\theta_{23}^{U}$ this matrix is given by v_{ij} ($i, j = 1, 2$). This is easily diagonalized and one obtains

$$m_c = u_{22} + \frac{u_{32}u_{23}}{m_t} + \frac{u_{12}u_{23}u_{13}^*}{m_t^2} , \tag{14}$$

$$\theta_{12}^{U} = \frac{u_{21}}{m_c} + \frac{u_{31}u_{23}}{m_c m_t} + \frac{u_{11}u_{23}u_{13}^*}{m_c m_t^2} , \tag{15}$$

$$m_u = u_{11} + \frac{u_{31}u_{13}}{m_t} + \frac{1}{m_c}\left(u_{12} + \frac{u_{32}u_{13}}{m_t}\right)\left(u_{21} + \frac{u_{31}u_{23}}{m_t}\right) + \frac{u_{21}u_{23}u_{13}^*}{m_t^2} . \tag{16}$$

We have neglected terms which are proportional to other terms up to a factor of order one or smaller.

The diagonalization of M_D is similar. The final mixing angles are a combination from M_U and M_D.

$$\theta_{ij} = \theta_{ij}^{U} + \theta_{ij}^{D} . \tag{17}$$

For the lepton mass matrix nothing is known about mixing angles. We nevertheless adopt the same procedure and take care of the large mixing case by considering the additional contributions to the effective 2×2 matrix in the second step.

From (12)–(16) we can easily compute the powers P_i, P_{ij} in terms of U_{ij}, D_{ij} and L_{ij} like

$$P_b = D_{33} , \tag{18}$$

$$P_s = \min(D_{22}, D_{32} + D_{23} - D_{33} , D_{12} + D_{23} + D_{13} - 2D_{33}) . \tag{19}$$

For the uncertainty factors we choose n_i as the number of undetermined matrix elements on the right-hand side of the corresponding formulae (12)–(16). Here the contributions involving more than one factor of the heaviest mass are denoted with an asterisk and are not counted in the uncertainty since they are important only under relatively rare circumstances. For example, from (8) one obtains $n_s = 3$, $n_c = 4$. (We note that m_t, in contrast with all other mass values should be treated as an unknown matrix element.) The n_i derived from (12)–(16) are given in table 1.

[#1] Remember that we only determine the order of magnitude, not the exact value.

Table 1

Quantity	Experimental value	n_i	$y_i^- - y_i^+$
m_t	$\geqslant 41$ GeV	1	$0.11-\infty$
m_b	4.0 ± 0.1 GeV	1	0.010–0.042
m_τ	1784.2 ± 3.2 MeV	1	0.011–0.043
m_c	0.88 ± 0.03 GeV	4	0.011–0.019
m_s	105 ± 35 MeV	3	0.00011–0.0025
m_μ	105.695 MeV	3	0.00037–0.0044
m_u	3.1 ± 0.9 MeV	12	1.6×10^{-6}–0.00014
m_d	5.4 ± 1.6 MeV	9	3.4×10^{-6}–0.00022
m_e	0.511003 MeV	9	1.0×10^{-6}–3.7×10^{-5}
θ_{23}	0.039–0.050	3	0.011–0.17
θ_{13}	0.0–0.008	3	0.0–0.027
θ_{12}	0.219–0.225	7	0.039–1.19

This simple counting rule for the uncertainty can be motivated by the following reasoning: For two matrix elements with uncertainty factors Δ_1, Δ_2, the uncertainty of the product (or ratio) is approximately $\Delta_{12}=\sqrt{\Delta_1^2+\Delta_2^2}$ if the two Δ_i are treated as statistically independent errors. The error of a sum or difference cannot be so easily estimated but a square root addition $\Delta_{1+2}=\sqrt{\Delta_1^2+\Delta_2^2}$ reflects at least some qualitative features. Our rule for the error then follows if all matrix elements have the same uncertainty factor Δ and all terms in (12)–(16) contribute equally. One may argue that often not all contributions to a given quantity are important and therefore the uncertainty for the lower generations is smaller. On the other hand the uncertainty of a given matrix element also tends to increase with the power of λ since usually more ratios of dimensionless couplings are involved (see refs. [2,4] for examples.) No more accurate estimate of the uncertainty involved seems possible without using more detailed information about specific models. Our simple estimate should be regarded as an educated guess which qualitatively reproduces the increase of uncertainty for the lower generations.

We now turn to the determination of the allowed regions in λ and the corresponding P_i. We assume first that the rough equality of Yukawa couplings holds at some large scale $M=10^{17}$ GeV. The generation symmetry is spontaneously broken somewhat below this scale. We have to correct for the different scale dependence of lepton, quark and the W-boson masses according to the different renormalization group equations of the corresponding dimensionless couplings. The relevant multiplicative factors in the one-loop approximation for a small top mass ($m_t \leqslant 100$ GeV) for the rescaling from 100 GeV to 10^{17} GeV are 0.76 for the leptons, 0.32 for m_t, m_c, m_u, 0.33 for m_b, m_s, m_d and 0.79 for m_W [#2]. A standard uncertainty $\Delta=2$ allows for factors of four in (corrected) masses to be explained by differences in Clebsch-Gordan coefficients. The regions for the different quantities are given approximately by

$$y_i^- = 0.43\, x_i^- / \sqrt{n_i}\Delta \leqslant \lambda^{P_i} \leqslant y_i^+ = 0.43\, x_i^+ \sqrt{n_i}\Delta \tag{20}$$

for the quarks, and

$$y_i^- = x_i^- / \sqrt{n_i}\Delta \leqslant \lambda^{P_i} \leqslant y_i^+ = x_i^+ \sqrt{n_i}\Delta \tag{21}$$

for the leptons. The values $y_i^\pm$ are shown in table 1. Quark masses are taken from ref. [5] except for the recent UA1 lower bound on the top quark mass [6]. The running quark masses at $\mu=100$ GeV are quoted (neglecting electroweak effects). Values for the mixing angles are taken from ref. [7] and the lepton masses from the particle data book [8]. We use a value of 81.5 GeV for m_W.

The allowed values for λ for the different quantities in terms of the P_i are plotted in fig. 1. The allowed regions of λ can be divided according to P_b equal to 1, 2, 3 and P_c 1 or 2. There is no solution

[#2] For a 160 GeV top mass a further multiplicative factor of 1.3 for m_t, 1.1 for m_b, 1.15 for m_c, m_u and 1.2 for m_s, m_d and the lepton masses is needed.

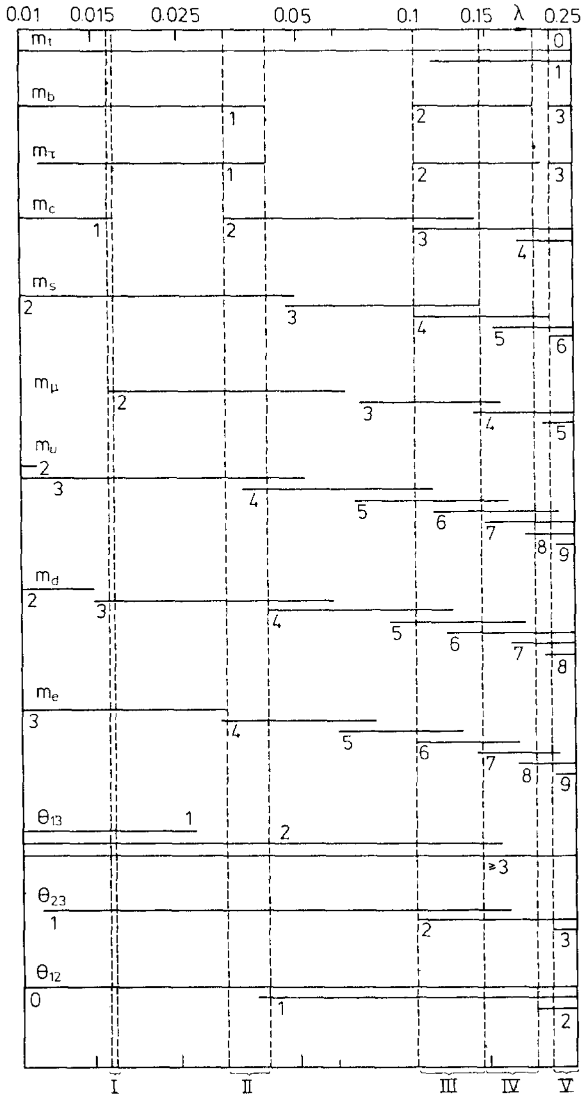

Fig. 1. The allowed regions for λ in terms of the power P_i for all masses and mixing angles for the unification scenario.

for $\lambda \leqslant 0.019$ and we do not consider $\lambda \geqslant 0.25$ because the distinction between differences in c_i and different powers of λ disappears. We have subdivided the region for $P_b = 2$ (III and IV in fig. 1). The allowed values of P_i for the other quantities are given in table 2. The SU(5) example discussed in ref. [2] corresponds to case II.

The above regions are those relevant for generation symmetries broken at a large scale. For comparison we have done a similar analysis without renormalization group corrections for the fermion masses. This scenario is more relevant for composite models. Yukawa couplings here are a consequence of strong interactions between bound states. We took this into account by replacing m_W in (1) by the vacuum expectation value $v = 175$ GeV. The resulting values for λ and P_i can be found in table 3.

In models with a generation symmetry broken somewhat below the unification scale the powers P_i can be computed in terms of the generation quantum numbers [2]. We can use the results in table 2 to decide if a given set quantum numbers leads to a realistic fermion mass pattern. We have investigated a three-parameter (m, p, r) set of anomaly-free U(1)-generation symmetries. These models can all be obtained from compactification of a six-dimensional SO(12) model [9]. The quark and lepton charges are obtained from a linear combination of the $\mathrm{U(1)_I}$ subgroup of a generation group $\mathrm{SU(2)_I}$ and another abelian symmetry $\mathrm{U(1)_q}$:

$$Q = Q_\mathrm{I} + rQ_\mathrm{q} \,. \tag{22}$$

The quantum numbers of the fermions under $\mathrm{SU(2)_I \times U(1)_q}$ are

$$\begin{aligned}
&\mathrm{q}: \quad [\tfrac{1}{2}(3+p)]_{1/2} + [\tfrac{1}{2}(3-p)]_{-1/2}\,, \\
&\mathrm{u^c}: \quad [\tfrac{1}{2}(3-p+2m)]_{1/2} \\
&\qquad + [\tfrac{1}{2}(3+p-2m)]_{-1/2}\,, \\
&\mathrm{d^c}: \quad [\tfrac{1}{2}(3-p-2m)]_{1/2} \\
&\qquad + [\tfrac{1}{2}(3+p+2m)]_{-1/2}\,, \\
&\mathrm{L}: \quad [\tfrac{1}{2}(3-3p)]_{1/2} + [\tfrac{1}{2}(3+3p)]_{-1/2}\,, \\
&\mathrm{e^c}: \quad [\tfrac{1}{2}(3+3p-2m)]_{1/2} \\
&\qquad + [\tfrac{1}{2}(3-3p+2m)]_{-1/2}\,.
\end{aligned} \tag{23}$$

The standard notation is used for the $\mathrm{SU(3)\times SU(2)\times U(1)}_Y$ representation. The number in brackets is the $\mathrm{SU(2)_I}$ representation and the subscript the $\mathrm{U(1)_q}$ quantum number. A negative number in brackets means a mirror particle in the conjugate representation under $\mathrm{SU(3)\times SU(2)\times U(1)}_Y\mathrm{\times U(1)_q}$ whose $\mathrm{SU(2)_I}$ representation is given by the absolute value of the number in brackets. The mirror particles acquire a mass from spontaneous breaking of the U(1) generation symmetry. We eliminate the supermassive quark-mirror pairs, taking

Table 2

Scenario	λ	P_t	P_b	P_τ	P_c	P_s	P_μ	P_u	P_d	P_e	P_{23}	P_{13}	P_{12}
I	0.019	0	1	1	1	2	2	3	3	3	1	$\geqslant 1$	0
II	0.033–0.042	0	1	1	2	2	2	3,4	3,4	3,4	1	$\geqslant 2$	0,1
III	0.10–0.14	0,1	2	2	2,3	3,4	3,4	4–6	4–6	5–7	1,2	$\geqslant 2$	0,1
IV	0.14–0.20	0,1	2	2	3,4	4,5	3,4	5–8	5–7	6–8	1,2	$\geqslant 2$	0–2
V	0.22–0.25	0,1	3	3	3,4	4–6	4,5	6–9	6–8	7–9	2,3	$\geqslant 3$	0–2

into account the mixing with light fermions according to the algorithm for mass matrix diagonalization discussed in detail in section 3 of ref. [4]. This leaves us then with three generations of light fermions which are linear combinations of those in (23). We then allow for an arbitrary charge of the "leading" weak Higgs doublet [2] under the extra U(1) and search for a realistic set of resulting P_i. These are given by the difference of the fermion bilinear quantum numbers and the Higgs ones [2]. We have performed a computerized scan for $p=1$, 3, 5, $m=-5$, $-4,\ldots$, 5 and $r=-11/2$, $-9/2,\ldots$, 11/2. (This leads to integer differences of the U(1) charge between fermion bilinears.) We found no realistic mass patterns corresponding to cases I–V of table 2.

This demonstrates how difficult it is to reproduce realistic masses from higher dimensional field or string theories. (These theories generically fulfil our assumption of dimensionless couplings all of the same order of magnitude so that the structure of mass matrices should be explained by symmetries.) A realistic fermion mass pattern is therefore a very restrictive phenomenological criterion for an acceptable ground states in such theories.

For arbitrary generation symmetries it is in general possible to find quantum numbers to reproduce all the different scenarios discussed here. A rather complete list for scenario II can be found in ref. [2]. We list here possible sets of quantum numbers for the different fermions under an extra U(1) that lead to each of our scenarios:

scenario I:

$$\mathrm{q}\,(1,1,0),\ \mathrm{u^c}\,(2,0,0),\ \mathrm{d^c}\,(2,1,1),$$
$$\mathrm{L}\,(1,1,0),\ \mathrm{e^c}\,(2,1,1)\,,$$

scenario II:

$$\mathrm{q}\,(2,1,0),\ \mathrm{u^c}\,(2,1,0),\ \mathrm{d^c}\,(2,1,1)\,,$$
$$\mathrm{L}\,(2,1,0),\ \mathrm{e^c}\,(2,1,1)\,,$$

scenario III, IV:

$$\mathrm{q}\,(3,2,0),\ \mathrm{u^c}\,(3,1,0),\ \mathrm{d^c}\,(3,2,2)\,,$$
$$\mathrm{L}\,(3,2,0),\ \mathrm{e^c}\,(4,2,2)\,,$$

scenario V:

$$\mathrm{q}\,(4,3,1),\ \mathrm{u^c}\,(3,1,0),\ \mathrm{d^c}\,(3,2,2)\,,$$
$$\mathrm{L}\,(4,2,1),\ \mathrm{e^c}\,(4,2,2)\,.$$

In each of these cases the Higgs doublet has zero charge under the extra U(1). Very similar solutions exist for the composite case.

As an example we assume all $c_i=1$ for the scenarios III, IV mentioned above. The following relations and mass values for $\lambda=1/6$ are obtained:

$$\theta_{12}=\lambda=1/6\,, \tag{24}$$

$$\theta_{23}=\theta_{12}^2=0.028\,, \tag{25}$$

$$\theta_{13}=\theta_{12}^3=0.005\,, \tag{26}$$

$$m_\tau=\theta_{23}m_\mathrm{W}=2.3\ \mathrm{GeV}\,, \tag{27}$$

Table 3

Scenario	λ	P_t	P_b	P_τ	P_c	P_s	P_μ	P_u	P_d	P_e	P_{23}	P_{13}	P_{12}
I	0.015–0.020	0	1	1	1	2	2	2,3	2,3	3	1	$\geqslant 1$	0
II	0.12–0.15	0,1	2	2	2,3	3,4	4	5,6	4,5,6	6,7	1,2	$\geqslant 3$	0,1
III	0.17–0.22	0,1	2	3	3,4	4,5	4,5	5,6,7	5,6,7	7,8,9	2	$\geqslant 3$	0,1

$$m_b = 2.3\, m_\tau = 5.3\ \text{GeV}\,, \tag{28}$$

$$m_c = \theta_{12} m_b = 880\ \text{MeV}\,, \tag{29}$$

$$m_s = m_c^2/m_b = 145\ \text{MeV}\,, \tag{30}$$

$$m_\mu = \theta_{23} m_\tau = 63\ \text{MeV}\,, \tag{31}$$

$$m_{u,d} = \theta_{12}^2 m_s = 4\ \text{MeV}\,, \tag{32}$$

$$m_e = \theta_{12}^3 m_\mu = 0.3\ \text{MeV}\,. \tag{33}$$

Comparison with the mass values in table 1 shows surprisingly good agreement demonstrating that our approach can also work much smaller uncertainty factors. In this particular model the top quark mass is large. Taking for u^c the charges (3, 1, 1) instead of (3, 1, 0) would lead to $m_t/m_c = m_b/m_s$, $m_t = 32$ GeV.

References

[1] S. Dimopoulos, Phys. Lett. B 129 (1983) 417;
S. Dimopoulos and H. Georgi, Phys. Lett. B 140 (1984) 67;
L. Wolfenstein, Phys. Rev. Lett. 51 (1984) 1945;
J. Bagger and S. Dimopoulos, Nucl. Phys. B 244 (1984) 247;
J. Bagger, S. Dimopoulos, H. Georgi and S. Raby, 5th Workshop on Grand unification (Providence, RI.), in: Proc. Providence grand unification (1984) p. 95;
J. Bijnens and C. Wetterich, Phys. Lett. B 176 (1986) 431.
[2] J. Bijnens and C. Wetterich, Nucl. Phys. B 283 (1987) 237.
[3] C. Wetterich, Nucl. Phys. B 261 (1985) 461;
A. Strominger and E. Witten, Commun. Math. Phys. 101 (1985) 341.
[4] J. Bijnens and C. Wetterich, Nucl. Phys. B 292 (1987) 443.
[5] J. Gasser and H. Leutwyler, Phys. Rep. 87 (1982) 77.
[6] S. Geer, talk EPS high energy physics Conf. (Uppsala, Sweden, June 1987).
[7] K. Kleinknecht, in: Proc. Intern. Symp. on Production and decay of heavy hadrons (Heidelberg, May 1986), eds. K.R. Scubert and R. Waldi.
[8] Particle Data Group, Review of particle properties, Phys. Lett. B 170 (1986) 1.
[9] C. Wetterich, Nucl. Phys. B 260 (1985) 402; B 279 (1987) 711.