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The Path Integral on the Pseudosphere

C. GROSCHE* AND F. STEINER

Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 2000 Hamburg 50, Federal Republic of Germany

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A rigorous path integral treatment for the d-dimensional pseudosphere Λ^{d-1}, a Riemannian manifold of constant negative curvature, is presented. The path integral formulation is based on a canonical approach using Weyl-ordering and the Hamiltonian path integral defined on midpoints. The time-dependent and energy-dependent Feynman kernels obtain different expressions in the even- and odd-dimensional cases, respectively. The special case of the three-dimensional pseudosphere, which is analytically equivalent to the Poincaré upper half plane, the Poincaré disc, and the hyperbolic strip, is discussed in detail including the energy spectrum and the normalised wave-functions.

I. INTRODUCTION

Ever since Feynman's fundamental paper [13] there were attempts to calculate path integrals explicitly. Unfortunately, there are essentially only two examples which allow a direct solution: the harmonic oscillator (including, of course, the free particle motion) and the rigid rotator. All other quantum mechanical systems require more sophisticated methods which have been invented only recently. The key to all known solutions is to find a symmetry, often "hidden," which allows a coordinate transformation, which may be non-linear or must be accompanied by a time transformation, to bring the path integral into a manageable form, such that one of the fundamental solutions can be applied (for recent reviews see [18, 26]).

With this paper we continue our previous work [18, 19], where we have formulated a canonical approach to calculate path integrals on curved manifolds. Let us consider the generic case (see [18] and references therein for further details), where the *classical Lagrangian* is given by

$$\mathcal{L}_{\text{Cl}}(q, \dot{q}) = \frac{m}{2} g_{ab} \dot{q}^a \dot{q}^b - V(q) \tag{1}$$

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with metric g_{ab} and line element $ds^2 = g_{ab}\, dq^a\, dq^b$. The *quantum Hamiltonian* reads $(\hbar = 1)$[1]

$$H = -\frac{1}{2m} \Delta_{\text{LB}} + V(q), \tag{2}$$

where Δ_{LB} is the Laplace–Beltrami operator

$$\Delta_{\text{LB}} = \frac{1}{\sqrt{g}} \partial_a g^{ab} \sqrt{g}\, \partial_b \tag{3}$$

(g is the determinant of the metric tensor). In order to express H by position and momentum operators, one constructs the momenta

$$p_a = \frac{1}{i}\left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2}\right), \qquad \Gamma_a = \frac{\partial \ln \sqrt{g}}{\partial q^a}, \tag{4}$$

which are hermitian with respect to the scalar product

$$(f_1, f_2) = \int f_1^* f_2 \sqrt{g}\, dq. \tag{5}$$

In terms of the momentum operators (4) the *Weyl-ordered form of the Hamiltonian* (2) reads

$$H = \frac{1}{8m} (g^{ab} p_a p_b + 2 p_a g^{ab} p_b + p_a p_b g^{ab}) + V(q) + \Delta V(q) \tag{6}$$

with the well-defined *quantum correction* (of order $\hbar^2$)

$$\Delta V = \frac{1}{8m} (g^{ab} \Gamma^d_{ac} \Gamma^c_{bd} - R) = \frac{1}{8m} [g^{ab} \Gamma_a \Gamma_b + 2(g^{ab} \Gamma_b)_{,a} + g^{ab}{}_{,ab}] \tag{7}$$

(R is the scalar curvature; g^{ab} is the inverse of g_{ab}; Γ^a_{bc} are the Christoffel symbols).

Using the Trotter formula $e^{-itH} := e^{-it(A+B)} = s - \lim_{N\to\infty} (e^{-itA/N} e^{-itB/N})^N$ and the short-time approximation to the matrix element $\langle q''| e^{-i\varepsilon H} |q'\rangle$, one obtains the *Hamiltonian path integral* ($\bar{q}^{(j)} := \frac{1}{2}(q^{(j)} + q^{(j-1)})$, $\varepsilon = T/N$, $T = t'' - t'$, $d =$ dimension of the Riemannian manifold),

$$K(q'', q'; T) = [g(q')\, g(q'')]^{-1/4} \lim_{N\to\infty} \int dq^{(1)} \cdots \int dq^{(N-1)} \int \frac{dp^{(1)}}{(2\pi)^d} \cdots \int \frac{dp^{(N)}}{(2\pi)^d}$$
$$\times \exp\left\{ i \sum_{j=1}^{N} \left[p_a^{(j)} (q^{(j)} - q^{(j-1)})^a - \frac{\varepsilon}{2m} g^{ab}(\bar{q}^{(j)})\, p_a^{(j)} p_b^{(j)} - \varepsilon V(\bar{q}^{(j)}) - \varepsilon\, \Delta V(\bar{q}^{(j)}) \right] \right\}, \tag{8}$$

[1] We only consider systems with such a simple structure; see [30] for a discussion of more complicated systems.

and the *Lagrangian path integral* (the momentum integrations can be carried out),

$$K(q'', q'; T)$$
$$= [g(q')\, g(q'')]^{-1/4} \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{Nd/2} \left(\prod_{j=1}^{N-1} \int dq^{(j)}\right) \prod_{j=1}^{N} \sqrt{g(\bar{q}^{(j)})}$$
$$\times \exp\left\{ i \left[\frac{m}{2\varepsilon} g_{ab}(\bar{q}^{(j)})(q^{(j)} - q^{(j-1)})^a (q^{(j)} - q^{(j-1)})^b - \varepsilon V(\bar{q}^{(j)}) - \varepsilon \Delta V(\bar{q}^{(j)}) \right] \right\}. \tag{9}$$

Note that it is crucial that all coordinate-dependent expressions be taken at the midpoints $\bar{q}^{(j)}$. This prescription follows in an unambiguous way from the Weyl-ordering rule (see, e.g., [27, p. 479, 30]). For the correct form of the normalisation $C = [g(q')\, g(q'')]^{-1/4}$ see, e.g., [32].

In [18] we have calculated the path integral for the d-dimensional rotator, i.e., for the quantum mechanical motion on the sphere S^{d-1}. In addition, we have discussed some path integral calculations, which have become important in recent years, i.e., the Coulomb problem (see [7, 23–25, 38]), the Morse potential (see [6]), the Langer transformation in a semiclassical treatment in radial path integrals (see [15]), and general space-time transformations in radial path integrals (see [36–38]). A further application of the Weyl-ordering rule has been presented in [19], where we have explicitly calculated the path integrals for the Poincaré upper half plane and Liouville quantum mechanics, respectively.

In this paper we present the path integral formulation for the pseudosphere Λ^{d-1}. Our work was motivated by the observation that the quantum motion on the pseudosphere Λ^{d-1} is formally similar to the quantum motion on the sphere S^{d-1}, but, of course, very different in its character. To our knowledge, no consistent and complete path integral treatment for the pseudosphere exists up to now.

Recently, there have been two path integral treatments of the pseudosphere. The first is a semiclassical calculation for Λ^2 and Λ^3 due to Gutzwiller [21]. He noticed in the case of Λ^3 a "mysterious phase factor" $\phi = 1/2mR^2$ in the Feynman kernel $K(T)$ which is due to the zero-momentum energy-shift: $E_0 = 1/2mR^2$. This shift did not arise in the semiclassical calculation, but it appears very naturally in deriving $K(T)$ directly from the Schrödinger equation.

A second work on this subject is due to Böhm and Junker [4], who discuss path integrals over compact and non-compact rotation groups. However, these authors missed the essential point leading to the quantum correction (7), and so they got an incorrect energy spectrum for Λ^{d-1}: $\tilde{E}_p^{(d)} = (1/2mR^2)(p^2 + 1/4)$ ($p > 0$, $d = 2, 3, ...$).

Our paper is organized as follows. In Section II we discuss and calculate the path integral for the d-dimensional pseudosphere. We show that the correct energy spectrum reads

$$E_p^{(d)} = \frac{1}{2mR^2}\left[p^2 + \left(\frac{d-2}{2}\right)^2\right] \qquad (p > 0,\ d = 2, 3, 4, ...). \tag{10}$$

In Sections III and IV we discuss in some detail the even- and odd-dimensional cases. On the one hand, it is possible in even dimensions to express the Feynman kernel $K(T)$ in closed form, yielding simple expressions for $d=2, 4$ and finite sums for $d=6, 8, ...$. On the other hand, one can express in all dimensions the Green's function $G(E)$ by associated Legendre functions of the second kind.

In Section V we discuss the pseudosphere Λ^2. Λ^2 is of special interest, because it is analytically equivalent to three further Riemannian spaces: (1) the Poincaré upper half plane U, (2) the Poincaré disc D, and (3) the hyperbolic strip S. These spaces play an important role in the Polyakov approach to string theory (see [10, 11, 17, 31, 35]) and in the theory of quantum chaos (see [1, 21, 39, 40]). In string perturbation theory one considers open or closed Riemannian surfaces of genus g. The order of the perturbation expansion corresponds to g. For a closed Riemannian surface one has, e.g., for $g=1$ the torus and for $g=2$ the double doughnut. By the uniformisation theorem of Klein, Fricke, and Koebe (see, e.g., [3]) these surfaces are conformally equivalent to compact domains (polygons) with $4g$ edges and vertices in these Riemannian spaces (e.g., for $g=2$ an octagon in D, say). Furthermore, these compact domains are fundamental domains of discrete subgroups of $PSL(2, \mathbf{R})$. The action of the group elements are for, e.g., $z \in D$,

$$z \mapsto \frac{az+b}{a^*+b^*z} \qquad (|a|^2-|b|^2=1) \tag{11}$$

which are isometries in D. Under the action of the generators of the group the polygons tessalate D, say. These features have been extensively studied by Poincaré [34] and Fricke and Klein [9]. A more recent discussion is, e.g., due to Fenn [8]. All these spaces have constant negative curvature. This hyperbolic structure is responsible for the fact that classical and quantum motion in the polygons is chaotic.

In our path integral treatment we shall show that the Feynman kernels $K(T)$ on Λ^2 and U can be transformed into each other. Further, having the path integral for Λ^2 it is quite simple to express it in terms of the variables on the Poincaré disc D. This enables us to write down the path integral solution for the disc. We shall briefly mention the path integral formulation for the strip S, but a detailed treatment for S will be given in a forthcoming paper.

Section VI summarizes our results.

The appendices contain further details and some important but tedious calculations. They concern Legendre functions (Appendix A) and the proof of an important path integral equivalence (Appendix B), and the Appendices C and D contain detailed proofs for deriving the Schrödinger equation from the short-time kernels corresponding to the different path integral representations for Λ^{d-1}.

II. The Path Integral on the d-Dimensional Pseudosphere

We are considering the *Schrödinger equation*

$$i\frac{\partial}{\partial t}\psi=-\frac{1}{2mR^2}K^2_{(d)}\psi \tag{1}$$

in d-dimensional pseudospherical polar coordinates (see [41]),

$$\begin{aligned}
x_1&=R\cosh\tau\\
x_2&=R\sinh\tau\cos\theta_{d-2}\\
x_3&=R\sinh\tau\sin\theta_{d-2}\cos\theta_{d-3}\\
&\cdots\\
x_{d-1}&=R\sinh\tau\sin\theta_{d-2}\cdots\sin\theta_2\cos\theta_1\\
x_d&=R\sinh\tau\sin\theta_{d-2}\cdots\sin\theta_2\sin\theta_1,
\end{aligned} \tag{2}$$

where $0\leqslant\tau<\infty$, $0\leqslant\theta_\nu\leqslant\pi$ $(\nu=2,...,d-2)$, $0\leqslant\theta_1\leqslant 2\pi$. The metric in x-space reads as $(G_{ab})=\mathrm{diag}(-1,1,...,1)$ $(a,b=1,...,d)$ such that $\mathbf{x}^2=-R^2=-x_1^2+\sum_{\nu=2}^d x_\nu^2$ with R fixed ($\Lambda^{(d-1)}$ has constant negative Gaussian curvature, $K=-(d-1)(d-2)/2R^2$; we will often also use $\theta_{d-1}=\tau$ and $\theta_1=\phi$). The metric in pseudospherical polar coordinates reads $(g_{ab})=R^2\,\mathrm{diag}(1,\sinh^2\tau,\sinh^2\tau\sin^2\theta_{d-2},...,\sinh^2\tau\cdots\sin^2\theta_2)(a,b=1,...,d-1)$. $K^2_{(d)}$ is the Legendre operator in the space Λ^{d-1}:

$$\begin{aligned}
K^2_{(d)}=&\left[\frac{\partial^2}{\partial\tau^2}+(d-2)\coth\tau\frac{\partial}{\partial\tau}\right]+\frac{1}{\sinh^2\tau}\left[\frac{\partial^2}{\partial\theta^2_{d-2}}+(d-3)\cot\theta_{d-2}\frac{\partial}{\partial\theta_{d-2}}\right]+\cdots\\
&+\frac{1}{\sinh^2\tau\cdots\sin^2\theta_3}\left[\frac{\partial^2}{\partial\theta_2^2}+\cot\theta_2\frac{\partial}{\partial\theta_2}\right]+\frac{1}{\sinh^2\tau\cdots\sin^2\theta_2}\frac{\partial^2}{\partial\phi^2}.
\end{aligned} \tag{3}$$

The *Hamiltonian* reads

$$H=-\frac{1}{2mR^2}K^2_{(d)}. \tag{4}$$

The solutions of the eigenvalue problem (see [2])

$$H\psi=E\psi \tag{5}$$

are the zonal spherical harmonics $H^{(d)}_{p,l,\mu}(u)$ with the spectrum (I.10) (u is a unit vector on Λ^{d-1}: $u=x/R$). The $H^{(d)}_{p,l,\mu}(u)$ can be written as

$$H^{(d)}_{p,l,\mu}(u)=Z_{p,l}(\tau)\cdot S^{(d-1)}_{l,\mu}(\Omega), \tag{6}$$

where the $S_{l,\mu}^{(d-1)}(\Omega)$ are the usual orthonormal spherical harmonics for the S^{d-2} sphere and Ω denotes a $(d-1)$-dimensional unit vector on S^{d-2}. The $Z_{p,l}(\tau)$ read

$$Z_{p,l}(\tau)=\frac{\Gamma(ip+l+(d-2)/2)}{\Gamma(ip)}(\sinh\tau)^{(3-d)/2}\,\mathscr{P}_{ip-1/2}^{(3-d)/2-l}(\cosh\tau) \tag{7}$$

$[p>0,\ l\in\mathbf{N}_0,\ \mu=1,2,\ldots,M,\ M=(2l+d-3)(l+d-4)!/(l!(d-3)!),\ d=4,5,\ldots;$ for $d=2,3$ see (48) and (49)]. The $Z_{p,l}$ are orthonormal

$$\int_0^\infty Z_{p,l}(\tau)\,Z_{p',l}^*(\tau)\sinh^{d-2}\tau\,d\tau=\delta(p-p') \tag{8}$$

and form a complete set

$$\int_0^\infty Z_{p,l}(\tau)\,Z_{p,l}^*(\tau')\,dp=(\sinh\tau\sinh\tau')^{(2-d)/2}\delta(\tau-\tau') \tag{9}$$

(for details see Appendix A). Therefore the $H_{p,l,\mu}^{(d)}$ are orthonormal and form a complete set on Λ^{d-1}.

In order to construct the path integral on Λ^{d-1} we start with the momentum operators which are given by (see (I.4))

$$\begin{aligned}p_\tau&=\frac{1}{i}\left(\frac{\partial}{\partial\tau}+\frac{d-2}{2}\coth\tau\right)\\ p_{\theta_\nu}&=\frac{1}{i}\left(\frac{\partial}{\partial\theta_\nu}+\frac{\nu-1}{2}\cot\theta_\nu\right)\\ p_\phi&=\frac{1}{i}\frac{\partial}{\partial\phi},\end{aligned} \tag{10}$$

and are hermitian with respect to the scalar product

$$(f,g)=\int_0^\infty\sinh^{d-2}\tau\,d\tau\prod_{\nu=2}^{d-2}\int_0^\pi\sin^{\nu-1}\theta_\nu\,d\theta_\nu\int_0^{2\pi}d\phi\,f^*g. \tag{11}$$

Rewriting the Hamiltonian (4) with the help of (I.6) and (I.7) yields (in H no ordering ambiguity arises, because of the special form of g_{ab} for the pseudosphere)

$$H(\{\theta\},\{p_\theta\})=\frac{1}{2mR^2}\left[p_\tau^2+\frac{1}{\sinh^2\tau}p_{\theta_{d-2}}^2+\cdots+\frac{1}{\sinh^2\tau\cdots\sin^2\theta_2}p_\phi^2\right]+\Delta V(\{\theta\}) \tag{12}$$

with

$$\Delta V(\{\theta\})=\frac{1}{8mR^2}\left[(d-2)^2-\frac{1}{\sinh^2\tau}-\cdots-\frac{1}{\sinh^2\tau\cdots\sin^2\theta_2}\right] \tag{13}$$

($\{\cdot\}$ denotes a collection of variables). We thus infer that the *Hamiltonian path integral on the pseudosphere* reads (see (I.8))

$$K^{(d)}(\{\theta''\}, \{\theta'\}; T)$$
$$= C \lim_{N\to\infty} \int \{d\theta^{(1)}\} \cdots \int \{d\theta^{(N-1)}\} \int \frac{\{dp_\theta^{(1)}\}}{(2\pi)^{d-1}} \cdots \int \frac{\{dp_\theta^{(N)}\}}{(2\pi)^{d-1}}$$
$$\times \exp\left\{i \sum_{j=1}^{N} \left[\sum_{\nu=1}^{d-1} p_{\theta_\nu}^{(j)}(\theta_\nu^{(j)} - \theta_\nu^{(j-1)}) - \varepsilon \mathscr{H}(\{\theta^{(j)}, \theta^{(j-1)}\}, \{p_\theta^{(j)}\}) \right]\right\}. \quad (14)$$

C is the normalisation (see (I.8)),

$$C = (\sinh \tau' \sinh \tau'')^{(2-d)/2} \left[\prod_{\nu=2}^{d-2} \sin^{\nu-1}\theta_\nu' \sin^{\nu-1}\theta_\nu'' \right]^{-1/2}, \quad (15)$$

and $\mathscr{H}$ denotes the effective classical Hamiltonian on the lattice,

$$\mathscr{H}(\{\theta^{(j)}, \theta^{(j-1)}\}, \{p_\theta^{(j)}\})$$
$$= \frac{1}{2mR^2} \left\{ [p_\tau^{(j)}]^2 + \frac{1}{\sinh \tau^{(j)} \sinh \tau^{(j-1)}} [p_{\theta_{d-2}}^{(j)}]^2 + \cdots \right.$$
$$\left. + \frac{1}{\sinh \tau^{(j)} \sinh \tau^{(j-1)} \cdots \sin \theta_2^{(j)} \sin \theta_2^{(j-1)}} [p_\phi^{(j)}]^2 \right\} + \Delta V(\{\theta^{(j)}, \theta^{(j-1)}\}) \quad (16)$$

with ΔV given by

$$\Delta V(\{\theta^{(j)}, \theta^{(j-1)}\}) = \frac{1}{8mR^2} \left[(d-2)^2 - \frac{1}{\sinh \tau^{(j)} \sinh \tau^{(j-1)}} - \cdots \right.$$
$$\left. - \frac{1}{\sinh \tau^{(j)} \sinh \tau^{(j-1)} \cdots \sin \theta_2^{(j)} \sin \theta_2^{(j-1)}} \right]. \quad (17)$$

Here some remarks are in order. As mentioned already in the Introduction, the consistent lattice definition of the path integral requires one to take all coordinates $\{\theta\}$ at the midpoints $\bar\theta_a^{(j)} = \frac{1}{2}(\theta_a^{(j)} + \theta_a^{(j-1)})$. However, in our case it is legitimate to make the replacement $\sin^2\bar\theta_a^{(j)} \to \sin \theta_a^{(j)} \sin \theta_a^{(j-1)}$ etc. ("product form"). This follows from the fact that the relevant terms of $O(\varepsilon)$ arising from the above replacement are exactly cancelling each other. A general discussion of path integrals based on the "product form" definition will be given elsewhere.

The momentum integrations in (14) are of Gaussian form and we get the following *Lagrangian path integral on the pseudosphere*,

$$K^{(d)}(\{\theta''\}, \{\theta'\}; T) = \int \{D\theta\}(t) \exp\left\{ i \int_{t'}^{t''} [\mathscr{L}_{\mathrm{Cl}}(\{\theta, \dot\theta\}) - \Delta V(\{\theta\})]\, dt \right\}, \quad (18)$$

where the classical Lagrangian and the integration measure are given by (with the "product form" to be used on the lattice)

$$\mathscr{L}_{\mathrm{Cl}}(\{\theta, \dot{\theta}\}) = \frac{m}{2} R^2[\dot{\tau}^2 + \sinh^2\tau\dot{\theta}_{d-2}^2 + \cdots + (\sinh^2\tau \cdots \sin^2\theta_2)\,\dot{\phi}^2], \tag{19}$$

$$\{D\theta\}(t) \to \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{N(d-1)/2} \prod_{j=1}^{N-1} \sinh^{d-2}\tau^{(j)}\, d\tau^{(j)}\, d\Omega^{(j)}. \tag{20}$$

Here $d\Omega^{(j)}$ denotes the $(d-2)$-dimensional surface element on the unit sphere S^{d-1}:

$$d\Omega^{(j)} = \prod_{k=1}^{d-2} (\sin\theta_k^{(j)})^{k-1}\, d\theta_k^{(j)}. \tag{21}$$

It is worthwhile to note that the normalisation C together with the determinant expressions (see (I.9)) has been exactly cancelled, and that the path integral (18) has the standard canonical measure (20), which can be directly derived by a transformation from Minkowskian to pseudospherical polar coordinates.

In Appendix C we show that from the short-time kernel of Eq. (18) the Schrödinger equation (1) can be deduced, so that the path integral (18) is indeed the correct path integral on Λ^{d-1}. Some details concerning the equivalence of our lattice formulation to the midpoint procedure can be found in Appendix B. We emphasize that this equivalence is a special feature of the pseudosphere (and, of course, the sphere, too; see [18]).

The path integral (18) with the Lagrangian given by (19) is too complicated for explicit calculations. We therefore try to replace (19) by the following expression (this replacement is motivated by the fact that an analogous trick has been successfully employed in the case of the sphere S^{d-1} [18]),

$$\mathscr{L}_{\mathrm{Cl}}(\{\theta, \dot{\theta}\}) \to \tilde{\mathscr{L}}_{\mathrm{Cl}}(\{\theta, \dot{\theta}\}) := \frac{m}{2} R^2\dot{u}^2 - V_{\mathrm{c}}(\{\theta\}), \tag{22}$$

where V_{c} must be determined and u denotes the d-dimensional unit vector on the Λ^{d-1} sphere. With $u^2 = -1$ (Λ^{d-1} is a space of constant negative curvature!),

$$(u^{(1)} - u^{(2)})^2 = -2(1 - \cosh l^{(1,2)}). \tag{23}$$

We note that $l^{(1,2)}$ is nothing but the hyperbolic distance between the points $\{\theta^{(1)}\}$ and $\{\theta^{(2)}\}$ measured in units of R. Using the addition theorem

$$\begin{aligned}\cosh l^{(1,2)} &= \cosh\tau^{(1)}\cosh\tau^{(2)} - \sinh\tau^{(1)}\sinh\tau^{(2)} \\ &\quad\times\left(\cos\theta_{d-2}^{(1)}\cos\theta_{d-2}^{(2)} + \sum_{m=1}^{d-3}\cos\theta_m^{(1)}\cos\theta_m^{(2)} \prod_{n=m+1}^{d-2}\sin\theta_n^{(1)}\sin\theta_n^{(2)}\right. \\ &\quad\left. + \prod_{n=1}^{d-2}\sin\theta_n^{(1)}\sin\theta_n^{(2)}\right)\end{aligned} \tag{24}$$

we can show (see Appendix B) that the following identity holds:[2]

$$\exp\{i\varepsilon\mathscr{L}^N_{\rm Cl}(\{\theta^{(j)}\},\{\theta^{(j-1)}\})] \doteq \exp\left\{-\frac{im}{\varepsilon}R^2(1-\cosh l^{(j,j-1)})-i\varepsilon V_{\rm c}(\{\theta^{(j)}\})\right\}. \tag{25}$$

Here

$$\begin{aligned}\mathscr{L}^N_{\rm Cl}(\{\theta^{(j)}\},\{\theta^{(j-1)}\}) &= \frac{mR^2}{2\varepsilon^2}\Big[(\tau^{(j)}-\tau^{(j-1)})^2+\sinh\tau^{(j)}\sinh\tau^{(j-1)}(\theta^{(j)}_{d-2}-\theta^{(j-1)}_{d-2})^2+\cdots\\ &\quad+(\sinh\tau^{(j)}\cdots\sin\theta^{(j-1)}_2)(\phi^{(j)}-\phi^{(j-1)})^2\Big]\end{aligned} \tag{26}$$

denotes the "classical Lagrangian" (19) on the lattice and

$$V_{\rm c}(\{\theta\})=\frac{1}{8mR^2}\left[-1+\frac{1}{\sinh^2\tau}+\cdots+\frac{1}{\sinh^2\tau\cdots\sin^2\theta_2}\right]. \tag{27}$$

From (13) and (27) we obtain the important relation

$$V_{\rm c}+\Delta V=\frac{(d-1)(d-3)}{8mR^2}. \tag{28}$$

With (22) and (28) the path integral (18) can be rewritten as

$$K^{(d)}(\{\theta''\},\{\theta'\};T)=\int Du(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}R^2\dot{u}^2-\frac{(d-1)(d-3)}{8mR^2}\right]dt\right\}. \tag{29}$$

Equation (29) is our final expression for the *path integral on the pseudosphere* Λ^{d-1}. Its *lattice definition* is given by

$$\begin{aligned}K^{(d)}(\{\theta''\},\{\theta'\};T) &= e^{-iT(d-1)(d-3)/8mR^2}\lim_{N\to\infty}\left(\frac{mR^2}{2\pi i\varepsilon}\right)^{N(d-1)/2}\int\prod_{j=1}^{N-1}du^{(j)}\\ &\quad\times\exp\left\{-\frac{imR^2}{\varepsilon}\sum_{j=1}^{N}[1-\cosh l^{(j,j-1)}]\right\}\end{aligned} \tag{30}$$

($du^{(j)}=\sinh^{d-2}\tau^{(j)}\,d\tau^{(j)}\,d\Omega^{(j)}$). The path integral (29) is, of course, equivalent to the path integral (18), but (29) is much simpler. In Appendix D it is shown that from its short-time kernel the Schrödinger equation (1) can be derived.

[2] We use the symbol $\doteq$ (following DeWitt [5]) to denote "equivalence as far as use in the path integral is concerned."

In order to evaluate the path integral (30) we need an expansion for $e^{-z\cosh l}$. We have ($\mathrm{Re}(z)>0$, $\mathrm{Re}(d)>1$)

$$e^{-z\cosh l}=\sqrt{\frac{2}{\pi z}}\,(z\sinh l)^{(3-d)/2}\int_0^\infty\left|\frac{\Gamma(ip+(d-2)/2)}{\Gamma(ip)}\right|^2\mathscr{P}_{ip-1/2}^{(3-d)/2}(\cosh l)\,K_{ip}(z)\,dp, \tag{31}$$

where $\mathscr{P}_\nu^\mu$ denotes the associated Legendre function of the first kind (see Appendix A) and K_ν a modified Bessel function. Equation (31) can be derived from the integral representation ([20, p. 804], $\mathrm{Re}(z)>0$, $\mathrm{Re}(\mu)<1$)

$$\int_1^\infty e^{-yz}(y^2-1)^{-\mu/2}\,\mathscr{P}_{\nu-1/2}^\mu(y)\,dy=\sqrt{\frac{2}{\pi z}}\,z^\mu K_\nu(z) \tag{32}$$

and the completeness relation (see also Eqs. (9) and (A.11))

$$\int_0^\infty\left|\frac{\Gamma(ip-\mu+1/2)}{\Gamma(ip)}\right|^2\mathscr{P}_{ip-1/2}^\mu(x)\,\mathscr{P}_{ip-1/2}^\mu(y)\,dp=\delta(x-y). \tag{33}$$

Next we must expand (31) into the spherical harmonics on Λ^{d-1}. This is done with the help of the relation [2]

$$(\sinh l^{(1,2)})^{(3-d)/2}\left|\frac{\Gamma(ip+(d-2)/2)}{\Gamma(ip)}\right|^2\mathscr{P}_{ip-1/2}^{(3-d)/2}(\cosh l^{(1,2)})$$

$$=(2\pi)^{(d-1)/2}\sum_{l,\mu}H_{p,l,\mu}^{(d)*}(u^{(1)})\,H_{p,l,\mu}^{(d)}(u^{(2)}). \tag{34}$$

Inserting (34) into (31) yields ($\mathrm{Re}(z)>0$, $d=2,3,...$)

$$e^{-z\cosh l^{(1,2)}}=2\left(\frac{2\pi}{z}\right)^{(d-2)/2}\int_0^\infty dp\,K_{ip}(z)\sum_{l,\mu}H_{p,l,\mu}^{(d)}(u^{(2)})\,H_{p,l,\mu}^{(d)*}(u^{(1)}). \tag{35}$$

For the functions $H_{p,l,\mu}^{(d)}$ we have the orthogonality relation (see Eq. (8))

$$\int_{\Lambda^{d-1}}du\,H_{p',l',\mu'}^{(d)}(u)\,H_{p,l,\mu}^{(d)*}(u)=\delta(p-p')\,\delta_{l,l'}\,\delta_{\mu,\mu'}. \tag{36}$$

Therefore we get for the jth term ($j=1,...,N$) in the path integral (30) (for a direct use of Eq. (35) we first must perform a Feynman–Wick rotation ($\varepsilon\to-i\varepsilon$))

$$\left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2}e^{-i(mR^2/\varepsilon)(1-\cosh l^{(j,j-1)})}$$

$$=\left(\frac{2mR^2}{\pi i\varepsilon}\right)^{1/2}e^{-i(mR^2/\varepsilon)}\int_0^\infty dp^{(j)}K_{ip^{(j)}}\left(\frac{mR^2}{i\varepsilon}\right)\sum_{l^{(j)},\mu^{(j)}}H_{p^{(j)},l^{(j)},\mu^{(j)}}^{(d)*}(u^{(j-1)})\,H_{p^{(j)},l^{(j)},\mu^{(j)}}^{(d)}(u^{(j)}). \tag{37}$$

Using (37) and the orthogonality relation (36) the integrations in (30) can be easily carried out with the result

$$K^{(d)}(\{\theta''\},\{\theta'\};T)$$
$$=\exp\left(-\frac{iT}{8mR^2}(d-1)(d-3)\right)\int_0^\infty dp\,\mu_p(T)\sum_{l,\mu}H^{(d)*}_{p,l,\mu}(u')\,H^{(d)}_{p,l,\mu}(u'') \quad (38)$$

with

$$\mu_p(T):=\lim_{N\to\infty}\left[\left(\frac{2mR^2}{\pi i\varepsilon}\right)^{1/2}\exp\left(\frac{mR^2}{i\varepsilon}\right)K_{ip}\left(\frac{mR^2}{i\varepsilon}\right)\right]^N. \quad (39)$$

To perform the limit we use the asymptotic expansion of the K_ν-Bessel function [20, p. 963],

$$K_\nu(z)\simeq\sqrt{\frac{\pi}{2z}}\exp\left(-z+\frac{\nu^2-1/4}{2z}\right)\qquad\left(|z|\to\infty,\ |\arg(z)|<\frac{3\pi}{2}\right), \quad (40)$$

and get

$$\mu_p(T)=\exp\left(-iT\frac{p^2+1/4}{2mR^2}\right). \quad (41)$$

We thus infer that the *Feynman kernel for the d-dimensional psuedosphere* Λ^{d-1} reads [we set $K^{(d)}(r;T)\equiv K^{(d)}(\{\theta''\},\{\theta'\};T)$, because the Feynman kernel at fixed time T is only a function of the hyperbolic distance $l^{(\prime,\prime\prime)}\equiv r$]

$$K^{(d)}(r;T)=\int_0^\infty dp\sum_{l,\mu}\exp\left(-\left(\frac{iT}{2mR^2}\right)\left[p^2+\frac{(d-2)^2}{4}\right]\right)H^{(d)*}_{p,l,\mu}(u')\,H^{(d)}_{p,l,\mu}(u'') \quad (42)$$

$$=\frac{1}{2\pi}(2\pi\sinh r)^{(3-d)/2}\int_0^\infty dp\left|\frac{\Gamma(ip+(d-2)/2)}{\Gamma(ip)}\right|^2\mathscr{P}^{(3-d)/2}_{ip-1/2}(\cosh r)$$
$$\times\exp\left(-\left(\frac{iT}{2mR^2}\right)\left[p^2+\frac{(d-2)^2}{4}\right]\right). \quad (43)$$

We immediately read off the *normalised wavefunctions*

$$H^{(d)}_{p,l,\mu}(u)=S^{(d-1)}_{l,\mu}(\Omega)\frac{\Gamma(ip+l+(d-2)/2)}{\Gamma(ip)}(\sinh\tau)^{(3-d)/2}\mathscr{P}^{(3-d)/2}_{ip-1/2}(\cosh\tau) \quad (44)$$

and the *energy spectrum*

$$E^{(d)}_p=\frac{1}{2mR^2}\left[p^2+\frac{(d-2)^2}{4}\right]\qquad(p>0) \quad (45)$$

with largest lower bound

$$E_0^{(d)} = \frac{(d-2)^2}{8mR^2}. \tag{46}$$

These results coincide, of course, with the one obtained from the operator approach; see, e.g., [2].

It is a very interesting feature that $E_0^{(d)}$ increases with increasing dimension. Gutzwiller [21] noted this for $E_0^{(4)} = 1/2mR^2$ in a semiclassical path integral calculation. In Ref. [4] this increasing lower bound does not appear. There the largest lower bound is constant for all d reading $\tilde{E}_0^{(d)} = 1/8mR^2$. However, in our calculation this energy shift arises very naturally. Note that it is indispensible in the derivation of the Schrödinger equation from the short-time kernel of the path integral (30).

By a Fourier transformation we obtain the *energy-dependent Feynmann kernel* $G(E)$ (Green's function):

$$G^{(d)}(r; E) = \int_0^\infty dp \frac{1}{(1/2mR^2)[p^2 + (d-2)^2/4] - E} \sum_{l,\mu} H_{p,l,\mu}^{(d)}(u'')\, H_{p,l,\mu}^{(d)*}(u'). \tag{47}$$

$G(E)$ has a cut in the complex E-plane with branch point at the value (46)—in agreement with the continuous spectrum (45).

We close this section by explicitly stating the normalised wavefunctions and the energy spectrum for dimensions $d = 2, 3, 4$ (for $d = 2$ see, e.g., [20, p. 1008]: $\mathscr{P}^{1/2}_{\nu-1/2}(\cosh\tau) = \sqrt{2/\pi}\,\sinh\tau\cosh\nu\tau$; for $d = 4$ see Section III):

$$H_p^{(2)}(\tau) = \sqrt{\frac{1}{2\pi}}\, e^{ip\tau}, \qquad E_p^{(2)} = \frac{p^2}{2mR^2} \qquad (p \in \mathbf{R}) \tag{48}$$

$$\left.\begin{aligned} H_{p,l}^{(3)}(\tau, \phi) &= \sqrt{\frac{p\sinh\pi p}{2\pi^2}}\, \Gamma\left(ip + l + \frac{1}{2}\right) \mathscr{P}^{-l}_{ip-1/2}(\cosh\tau)\, e^{il\phi}, \\ E_p^{(3)} &= \frac{1}{2mR^2}\left(p^2 + \frac{1}{4}\right) \qquad (p > 0, l \in \mathbf{Z}) \end{aligned}\right. \tag{49}$$

$$\left.\begin{aligned} H_{p,l,\mu}^{(4)}(\tau, \theta, \phi) &= \sqrt{\frac{(2l+1)(l+\mu)!}{4\pi(l-\mu)!}\, \frac{\Gamma(ip-l)}{\Gamma(ip+l)}}\, \sinh^l\tau \left[\frac{d^l\cos p\tau}{d(\cosh\tau)^l}\right] P_l^\mu(\cos\theta)\, e^{i\mu\phi}, \\ E_p^{(4)} &= \frac{1}{2mR^2}(p^2+1), \qquad (p > 0, l \in \mathbf{N}_0, \mu = -l, ..., 0, ..., l). \end{aligned}\right. \tag{50}$$

III. The Feynman Kernel in Even Dimensions

In even dimensions it is possible to express Eq. (II.43) in closed form, yielding simple expressions for $d=2, 4$ and finite sums for $d=6, 8, \ldots$. We start with

$$K^{(d)}(r; T)=\frac{1}{2\pi}(2\pi \sinh r)^{(3-d)/2}$$
$$\times\int_0^\infty dp \left|\frac{\Gamma(ip+(d-2)/2}{\Gamma(ip)}\right|^2 \mathscr{P}^{(3-d)/2}_{ip-1/2}(\cosh r)$$
$$\times\exp\left(-\left(\frac{iT}{2mR^2}\right)\left[p^2+\frac{(d-2)^2}{4}\right]\right). \qquad (1)$$

We first rewrite the Legendre functions in terms of Gegenbauer functions. With (see [28, p. 200])

$$(\sinh r)^{-\alpha}\mathscr{P}^{-\alpha}_{\nu-1/2}(\cosh r)=\frac{\Gamma(2\alpha+1)\,\Gamma(\nu-\alpha+1/2)}{2^\alpha\Gamma(\alpha+1)\,\Gamma(\nu+\alpha+1/2)}\mathscr{C}^{\alpha+1/2}_{\nu-\alpha-1/2}(\cosh r) \qquad (2)$$

and using properties of the Γ-function we get for d even

$$K^{(d)}(r; T)=\frac{i(-1)^{(d-2)/2}\Gamma((d-2)/2)}{2\pi^{d/2}}e^{-(iT/8mR^2)(d-2)^2}$$
$$\times\int_0^\infty dp\, p\mathscr{C}^{(d-2)/2}_{ip-(d-2)/2}(\cosh r)\, e^{-(iT/2mR^2)p^2}. \qquad (3)$$

We can now reduce the d-dimensional problem to the case $d=2$. This is done with the help of the following property of the Gegenbauer functions:

$$\mathscr{C}^k_{ip-k}(\cosh r)=\frac{2^{1-k}}{ip\Gamma(k)}\left[\frac{d^k \cos pr}{d(\cosh r)^k}\right]. \qquad (4)$$

This relation can be deduced from [20, p. 1030]

$$\mathscr{C}^{\lambda+k}_{\nu-k}(z)=\frac{\Gamma(\lambda)}{2^k\Gamma(\lambda+k)}\frac{d^k\mathscr{C}^\lambda_\nu(z)}{dz^k} \qquad (5)$$

and $\lim_{\lambda\to 0}\Gamma(\lambda)\,\mathscr{C}^\lambda_\nu(\cosh r)=(2/\nu)\cosh \nu r$. Inserting (4) into (3) we obtain

$$K^{(d)}(r; T)=\frac{1}{2\pi}e^{-(iT/8mR^2)(d-2)^2}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-2)/2}\int_{-\infty}^{\infty}e^{ipr}e^{-(iT/2mR^2)p^2}\,dp$$
$$=\sqrt{\frac{mR^2}{2\pi iT}}e^{-(iT/8mR^2)(d-2)^2}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-2)/2}e^{(imR^2/2T)r^2}, \qquad (6)$$

which yields the recursion formula

$$K^{(d+2)}(r;T) = -\frac{1}{2\pi}\exp\left(-\frac{iT}{2mR^2}(d-1)\right)\frac{d}{d\cosh r}K^{(d)}(r;T)$$

$$= \left(\frac{mR^2}{2\pi iT}\right)\left(\frac{r}{\sinh r}\right)\exp\left(-\frac{iT}{2mR^2}(d-1)\right)\frac{d}{dz}K^{(d)}(r;T) \qquad (7)$$

($z = i(mR^2/2T)\,r^2$). For the first three cases we explicitly obtain

$$K^{(2)}(r;T) = \left(\frac{mR^2}{2\pi iT}\right)^{1/2} e^{(imR^2/2T)r^2} \qquad (8)$$

$$K^{(4)}(r;T) = \left(\frac{mR^2}{2\pi iT}\right)^{3/2}\frac{r}{\sinh r}\exp\left(\left(\frac{imR^2}{2T}\right)r^2 - \frac{iT}{2mR^2}\right) \qquad (9)$$

$$K^{(6)}(r;T) = \left(\frac{mR^2}{2\pi iT}\right)^{5/2}\frac{r^2}{\sinh^2 r}\exp\left(\left(\frac{imR^2}{2T}\right)r^2 - \frac{2iT}{mR^2}\right)\left\{1 - \frac{iT}{mR^2r^2}\left[1 - r\coth r\right]\right\}. \qquad (10)$$

These are the Feynman kernels of the "hyperbolic circle," the Λ^3-pseudosphere (see Gutzwiller [21]) and the Λ^5-pseudosphere, respectively. It is remarkable that the kernel for $d = 2$ is identical to the free particle kernel in **R**, if the euclidean distance is replaced by the hyperbolic distance $R \cdot r$. This is quite different from the euclidean circle where the Feynman kernel can be expressed in terms of a Jacobi θ_3-function, which is an infinite sum over free particle kernels.

We can also calculate the Fourier transform of $K^{(d)}(T)$, the Green's function $G^{(d)}(E)$. We get

$$G^{(d)}(r;E) := i\int_0^\infty K^{(d)}(r;T)\,e^{iTE}\,dT = \sqrt{\frac{imR^2}{2\pi}}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-2)/2}$$

$$\times\int_0^\infty T^{-1/2}\exp\left\{-\frac{mR^2r^2}{2i}\frac{1}{T} + i\left[E - \frac{(d-2)^2}{8mR^2}\right]T\right\}dT$$

$$= mR^2\left[2mR^2E - (d-2)^2/4\right]^{-1/4}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-2)/2}$$

$$\times\sqrt{\frac{2i}{\pi}}\,r\,K_{1/2}\left(ir\sqrt{2mR^2E - (d-2)^2/4}\right)$$

$$= \frac{mR^2}{\sqrt{2mR^2E - (d-2)^2/4}}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-2)/2}$$

$$\times\exp\left(-ir\sqrt{2mR^2E - (d-2)^2/4}\right)$$

$$= \frac{mR^2}{i\pi}\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{(d-3)/2}\mathcal{Q}^{(d-3)/2}_{i\sqrt{2mER^2-(d-2)^2/4}-1/2}(\cosh r). \qquad (11)$$

The last equation is proved in Appendix A. Further we have used the integral [20, p. 340]

$$\int_0^\infty x^{\nu-1} e^{-\beta/x-\gamma x}\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}) \tag{12}$$

and $K_{\pm 1/2}(z) = \sqrt{\pi/2z}\, e^{-z}$. The first three cases read

$$G^{(2)}(r;E) = \sqrt{\frac{mR^2}{2E}}\, e^{-ir\sqrt{2mR^2E}} \tag{13}$$

$$G^{(4)}(r;E) = \frac{imR^2}{2\pi \sinh r}\, e^{-ir\sqrt{2mR^2E-1}} \tag{14}$$

$$G^{(6)}(r;E) = -\frac{mR^2}{4\pi^2}\cdot\frac{\sqrt{2mR^2E-4} - i\coth r}{\sinh^2 r}\, e^{-ir\sqrt{2mR^2E-4}}. \tag{15}$$

IV. The Feynman Kernel in Odd Dimensions

Unfortunately there is no explicit expression for the Legendre functions $\mathscr{P}_\nu^\mu$ in terms of elementary functions when μ is an integer. However, we can express the Green's function $G(E)$ in a simple way in terms of Legendre functions of the second kind. Using the property (A.5) of the Legendre functions we get for d odd

$$\left|\frac{\Gamma(ip+(d-2)/2)}{\Gamma(ip)}\right|^2 \mathscr{P}_{ip-1/2}^{(3-d)/2}(\cosh r) = (-1)^{(d-3/2)} p \tanh \pi p \mathscr{P}_{ip-1/2}^{(d-3)/2}(\cosh r). \tag{1}$$

Inserting into the Feynman kernel yields with $\mathscr{P}_\nu^m(z) = (z^2-1)^{m/2}\, d^m \mathscr{P}_\nu(z)/dz^m$

$$\begin{aligned}
K^{(d)}(r;T) &= \frac{1}{2\pi}(2\pi\sinh r)^{(3-d)/2}\int_0^\infty dp\left|\frac{\Gamma(ip+(d-2)/2)}{\Gamma(ip)}\right|^2 \mathscr{P}_{ip-1/2}^{(3-d)/2}(\cosh r) \\
&\quad\times\exp\left(-\frac{iT}{2mR^2}\left[p^2+\frac{(d-2)^2}{4}\right]\right) \\
&= \frac{1}{2\pi}\left(\frac{-1}{2\pi\sinh r}\right)^{(d-3)/2}\int_0^\infty p\tanh \pi p \mathscr{P}_{ip-1/2}^{(d-3)/2}(\cosh r) \\
&\quad\times\exp\left(-\frac{iT}{2mR^2}\left[p^2+\frac{(d-2)^2}{4}\right]\right) dp \\
&= \frac{1}{2\pi}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-3)/2}\int_0^\infty p\tanh \pi p \mathscr{P}_{ip-1/2}(\cosh r) \\
&\quad\times\exp\left(-\frac{iT}{2mR^2}\left[p^2+\frac{(d-2)^2}{4}\right]\right) dp
\end{aligned} \tag{2}$$

We now perform a Fourier transformation,

$$
\begin{aligned}
&G^{(d)}(r; E)\\
&= i\int_0^\infty e^{iTE}K^{(d)}(r; T)\,dT\\
&= \frac{i}{2\pi}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-3)/2}\int_0^\infty dp\, p\tanh \pi p \mathscr{P}_{ip-1/2}(\cosh r)\\
&\quad\times\int_0^\infty dT\exp\left(-iT\left\{\frac{1}{2mR^2}\left[p^2+\frac{(d-2)^2}{4}\right]-E\right\}\right)\\
&= \frac{mR^2}{\pi}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-3)/2}\int_0^\infty dp\,\frac{p\tanh \pi p}{p^2-[2mR^2E-(d-2)^2/4]}\mathscr{P}_{ip-1/2}(\cosh r) \qquad (3)\\
&= \frac{mR^2}{\pi}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-3)/2}\mathscr{Q}_{i\sqrt{2mER^2-(d-2)^2/4}-1/2}(\cosh r), \qquad (4)
\end{aligned}
$$

where we have used the integral [20, p. 819]

$$\int_0^\infty \frac{x\tanh \pi x}{a^2+x^2}\mathscr{P}_{ix-1/2}(\cosh y)\,dx=\mathscr{Q}_{a-1/2}(\cosh y). \qquad (5)$$

Differentiating $\mathscr{Q}_\nu$ $(d-3)/2$ times and using $\mathscr{Q}_\nu^m(z)=(z^2-1)^{m/2}(d^m\mathscr{Q}_\nu(z)/dz^m)$ we get for the *Green's function on the d-dimensional pseudosphere*

$$G^{(d)}(r; E)=\frac{mR^2}{\pi}\left(\frac{-1}{2\pi\sinh r}\right)^{(d-3)/2}\mathscr{Q}^{(d-3)/2}_{i\sqrt{2mER^2-(d-2)^2/4}-1/2}(\cosh r). \qquad (6)$$

A comparison of Eq. (6) with Eq. (III.11) shows that the representation (6) is generally true—up to a phase factor—irrespective whether the dimension is even or odd.

V. The Pseudosphere Λ^2

The pseudosphere Λ^2 has some special properties which make it the most interesting one among all the others.[3]) Λ^2 is analytically equivalent to three further Riemannian spaces:

(1) The Poincaré disc $D:=\{z=x_1+ix_2=re^{i\phi}\,|\,r<1, \phi\in[0, 2\pi]\}$,

(2) the Poincaré upper half plane $U:=\{\zeta=x+iy\,|\,x\in\mathbf{R}, y>0\}$, and

(3) the hyperbolic strip $S:=\{\eta=X+iY\,|\,X\in\mathbf{R}, Y\in(-\pi/2, \pi/2)\}$.

[3] Throughout this section we put $R=1$; i.e., we consider the pseudosphere Λ^2 with Gaussian curvature $K=-1$.

The study of compact domains in these spaces is of great interest in string theories and quantum chaos (as already mentioned in the Introduction).

1. *The Poincaré Disc*

Let us consider the stereographic projection of Λ^2 onto the (x_1, x_2)-plane with projection center $y=(0, 0, -1)$ ($y_0 = \cosh\tau$, $y_1 = \sinh\tau \sinh\phi$, $y_2 = \sinh\tau \cos\phi$):

$$z = x_1 + ix_2 = re^{i\phi} = \frac{y_1 + iy_2}{1 + y_0} = \tanh\frac{\tau}{2}(\sin\phi + i\cos\phi). \tag{1}$$

The boundary $r=1$ of the disc D corresponds to the points at infinity of the hyperboloid (i.e., the pseudosphere Λ^2). The pseudosphere itself is represented by the interior of the disc. The *classical Lagrangian and Hamiltonian* are, respectively,

$$\mathscr{L}_{\text{Cl}} = 2m\frac{\dot r^2 + r^2\dot\phi^2}{(1-r^2)^2}, \qquad \mathscr{H}_{\text{Cl}} = \frac{(1-r^2)^2}{8m}\left(p_r^2 + \frac{1}{r^2}p_\phi^2\right), \tag{2}$$

and the *quantum Hamiltonian* reads (see Eqs. (II.3 and II.4))

$$H = -\frac{(1-r^2)^2}{8m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right). \tag{3}$$

The geodesic distance d between two points z and z' is given by

$$\cosh d(z, z') = 1 + \frac{2|z-z'|^2}{(1-|z|^2)(1-|z'|^2)}. \tag{4}$$

The metric reads $g_{ab} = [2/(1-r^2)]^2 \operatorname{diag}(1, r^2)$.

Let us construct the path integral on D. Following our prescription outlined in the Introduction, we get

$$\begin{aligned}
\Gamma_r &= \partial_r(\ln\sqrt{g}) = \frac{1}{r} + \frac{4r}{1-r^2}\\
\Gamma_\phi &= \partial_\phi(\ln\sqrt{g}) = 0\\
p_r &= \frac{1}{i}\left(\frac{\partial}{\partial r} + \frac{1}{2r} + \frac{2r}{1-r^2}\right)\\
p_\phi &= \frac{1}{i}\frac{\partial}{\partial\phi}
\end{aligned} \tag{5}$$

and the *quantum correction* ΔV reads

$$\Delta V(r) = \frac{1}{8m}\left[1 + r^2 - \frac{(1-r^2)^2}{4r^2}\right] = \frac{3}{16m} - \frac{1}{32mr^2} + \frac{3}{32m}r^2. \tag{6}$$

Therefore we get for the *path integral on D* (see (I.9))

$$K^D(r'', r', \phi'', \phi'; T)$$

$$\equiv \left[\frac{16r'r''}{(1-r'^2)^2(1-r''^2)^2}\right]^{-1/2} \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^N \prod_{j=1}^{N-1} \int_0^1 dr_{(j)} \int_0^{2\pi} d\phi_{(j)} \prod_{j=1}^{N} \frac{4\bar{r}_{(j)}}{(1-\bar{r}_{(j)}^2)^2}$$

$$\times \exp\left[\frac{2im}{\varepsilon} \frac{(r_{(j)}-r_{(j-1)})^2 + \bar{r}_{(j)}^2(\phi_{(j)}-\phi_{(j-1)})^2}{(1-\bar{r}_{(j)}^2)^2} - i\varepsilon\, \Delta V(\bar{r}_{(j)})\right]. \tag{7}$$

This path integral looks rather complicated, but nevertheless it can be explicitly computed as will be shown in a forthcoming publication. Here we go back to the solution derived in Section II. From Eqs. (II.42) and (II.49) we obtain

$$K^{(3)}(\tau'', \tau', \phi'', \phi', T)$$

$$= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^\infty dp \exp\left[-iT\frac{p^2+1/4}{2m}\right]$$

$$\times \frac{p \sinh \pi p}{\pi} \left|\Gamma(\tfrac{1}{2}+ip+l)\right|^2 e^{il(\phi''-\phi')} \mathscr{P}^{-l}_{-1/2+ip}(\cosh \tau')\, \mathscr{P}^{-l}_{-1/2+ip}(\cosh \tau''). \tag{8}$$

Using for the Legendre functions the representation [20, p. 1010, $|(z-1)/(z+1)| < 1$)

$$\mathscr{P}^\mu_\nu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z-1}{z+1}\right)^{-\mu/2} \left(\frac{z+1}{2}\right)^\nu {}_2F_1\left(-\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1}\right) \tag{9}$$

and introducing, following Helgason [22],

$$\Phi_{p,l}(r) = (1-r^2)^{1/2+ip} r^{|l|} \frac{\Gamma(|l|+1/2+ip)}{|l|!\,\Gamma(1/2+ip)}\, {}_2F_1(\tfrac{1}{2}+ip, |l|+\tfrac{1}{2}+ip; |l|+1; r^2), \tag{10}$$

we can express Eq. (7) with the help of (1) and (10) in terms of the variables of the Poincaré disc D:

$$K^D(r'', r', \phi'', \phi'; T)$$

$$= \int_0^\infty dp \sum_{l=-\infty}^{\infty} \exp\left[-iT\frac{p^2+1/4}{2m}\right] \frac{p \tanh \pi p}{2\pi} \Phi_{p,l}(r'')\, \Phi^*_{p,l}(r')\, e^{il(\phi''-\phi')}. \tag{11}$$

Thus the *wavefunctions and the energy spectrum on the Poincaré disc* are given by

$$\begin{aligned} \psi^D_{p,l}(r, \phi) &= \sqrt{\frac{p \tanh \pi p}{2\pi}}\, \Phi_{p,l}(r)\, e^{il\phi} \\ E^D_p &= \frac{1}{2m}\left(p^2 + \frac{1}{4}\right) \end{aligned} \tag{12}$$

($p>0$, $l\in\mathbf{Z}$) satisfying the orthogonality relation

$$\int_0^1 dr \int_0^{2\pi} d\phi \frac{4r}{(1-r^2)^2} \psi^D_{p',l'}(r,\phi)\, \psi^{D*}_{p,l}(r,\phi) = \delta_{l,l'}\, \delta(p-p'). \tag{13}$$

2. *The Poincaré Upper Half Plane*

The Poincaré disc D can be mapped onto the Poincaré upper half plane U by the transformation

$$\zeta = x+iy = \frac{-iz+i}{z+1}, \qquad z = \frac{-\zeta+i}{\zeta+i}. \tag{14}$$

The *classical Lagrangian and Hamiltonian* read, respectively,

$$\mathscr{L}_{\rm Cl} = \frac{m}{2}\frac{1}{y^2}(\dot{x}^2+\dot{y}^2), \qquad \mathscr{H}_{\rm Cl} = \frac{1}{2m} y^2(p_x^2+p_y^2). \tag{15}$$

The metric is $g_{ab}=(1/y^2)\,\delta_{ab}$. The Laplace–Beltrami operator or *quantum Hamiltonian* reads

$$H = -\frac{1}{2m} y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{16}$$

In a previous paper we have presented a complete path integral treatment on U [19], including its connection to Liouville quantum mechanics. So we state just the result for the *path integral on* U:[4]

$$\begin{aligned}
K^U(x'', y'', x', y'; T) &= \int \frac{Dx(t)\, Dy(t)}{y^2} \exp\left[\frac{im}{2}\int_{t'}^{t''} \frac{1}{y^2}(\dot{x}^2+\dot{y}^2)\, dt\right] \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^N \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{dx^{(j)}\, dy^{(j)}}{[y^{(j)}]^2} \\
&\quad \times \exp\left[\frac{im}{2\varepsilon}\sum_{j=1}^{N} \frac{(x^{(j)}-x^{(j-1)})^2 + (y^{(j)}-y^{(j-1)})^2}{y^{(j)}y^{(j-1)}}\right] \\
&= \frac{1}{\pi^3}\int_{-\infty}^{\infty} dk \int_0^{\infty} dp\, p \sinh \pi p \\
&\quad \times \exp\left(-iT\frac{p^2+1/4}{2m}\right) \sqrt{y'y''}\, K_{ip}(|k|\, y')\, K_{ip}(|k|\, y'')\, e^{ik(x''-x')}.
\end{aligned} \tag{17}$$

[4] Application of the Weyl-correspondence yields $\Delta V = 1/4m$. But using the "product rule" $\bar{y}^2_{(j)} \to y_{(j)} y_{(j-1)}$ in the lattice formulation of the path integral cancels ΔV, such that Eq. (17) is obtained. From its short-time kernel the correct Schrödinger equation can be deduced; see [19] for details.

The *wavefunctions and the energy spectrum on U* read

$$\begin{aligned}\psi^U_{p,k}(x,y)&=\sqrt{\frac{p\sinh\pi p}{\pi^3}}\,e^{ikx}\sqrt{y}\,K_{ip}(|k|\,y)\qquad(x\in\mathbf{R},\ y>0)\\ E^U_p&=\frac{1}{2m}\left(p^2+\frac{1}{4}\right)\end{aligned}\tag{18}$$

($p>0$, $k\in\mathbf{R}\backslash\{0\}$) satisfying the orthogonality relation

$$\int_{-\infty}^{\infty}dx\int_0^\infty\frac{dy}{y^2}\,\psi^{U*}_{p,k}(x,y)\,\psi^U_{p',k'}(x,y)=\delta(k-k')\,\delta(p-p').\tag{19}$$

The spectral representations (8) and (17) can be transformed into each other. In order to achieve this we use the integral [20, p. 732]

$$\int_0^\infty K_\nu(ax)\,K_\nu(bx)\cos cx\,dx=\frac{\pi^2}{4\sqrt{ab}\cos\nu\pi}\,\mathscr{P}_{-1/2+\nu}\left(\frac{a^2+b^2+c^2}{2ab}\right),\tag{20}$$

Eq. (II.33) for the case $d=3$, and the addition theorem for the associated Legendre functions [20, p. 1014],

$$\mathscr{P}_\nu(zz'-\sqrt{z^2-1}\sqrt{z'^2-1}\cos\phi)=\sum_{l=-\infty}^{\infty}(-1)^l e^{il\phi}\frac{\Gamma(\nu-l+1)}{\Gamma(\nu+l+1)}\mathscr{P}^l_\nu(z)\,\mathscr{P}^l_\nu(z').\tag{21}$$

This enables us to derive the identity,

$$\begin{aligned}&\frac{1}{\pi^3}\int_{-\infty}^{\infty}dk\int_0^\infty dp\,p\sinh\pi p\exp\left(-iT\frac{p^2+1/4}{2m}\right)\sqrt{y'y''}\,K_{ip}(|k|\,y')\,K_{ip}(|k|\,y'')\,e^{ik(x''-x')}\\ &\quad=\frac{1}{2\pi^2}\sum_{l=-\infty}^{\infty}\int_0^\infty dp\exp\left[-iT\frac{p^2+1/4}{2m}\right]p\sinh\pi p|\Gamma(\tfrac{1}{2}+ip-l)|^2\\ &\qquad\times e^{il(\phi''-\phi')}\mathscr{P}^l_{-1/2+ip}(\cosh\tau')\,\mathscr{P}^l_{-1/2+ip}(\cosh\tau''),\end{aligned}\tag{22}$$

where use has been made of the identity

$$\begin{aligned}&\frac{y''^2+y'^2+(x''-x')^2}{2y'y''}\\ &\quad=1+\frac{|\zeta''-\zeta'|^2}{2\,\mathrm{Im}(\zeta')\,\mathrm{Im}(\zeta'')}=1+\frac{2|z''-z'|^2}{(1-|z'|^2)(1-|z''|^2)}\\ &\quad=\cosh d(z'',z')=\cosh\tau''\cosh\tau'-\sinh\tau''\sinh\tau'\cos(\phi''-\phi').\end{aligned}\tag{23}$$

3. *The Hyperbolic Strip*

With the help of the transformation

$$\eta=X+iY=-\ln(-i\zeta)\qquad(=2\,\mathrm{artanh}\,z)\tag{24}$$

we can transform the Poincaré upper half plane (the Poincaré disc) onto the hyperbolic strip S. The *classical Lagrangian and Hamiltonian* read

$$\mathscr{L}_{\mathrm{Cl}}=\frac{m}{2}\frac{\dot{X}^2+\dot{Y}^2}{\cos^2 Y}, \qquad \mathscr{H}_{\mathrm{Cl}}=\frac{\cos^2 Y}{2m}(p_X^2+p_Y^2), \tag{25}$$

respectively. The *quantum Hamiltonian* reads

$$H=-\frac{\cos^2 Y}{2m}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right). \tag{26}$$

The metric is $g_{ab}=(1/\cos^2 Y)\,\delta_{ab}$. Therefore we get the *quantum correction* ΔV,

$$\Delta V=\frac{1}{4m}, \tag{27}$$

and the *effective Lagrangian* to be used in the path integral defined on midpoints reads

$$\mathscr{L}_{\mathrm{eff}}=\frac{m}{2}\frac{\dot{X}^2+\dot{Y}^2}{\cos^2 Y}-\frac{1}{4m}. \tag{28}$$

In a forthcoming paper we shall give a detailed path integral treatment on S, yielding the *wavefunctions and the energy spectrum*

$$\begin{aligned}\psi^S_{p,k}(X,Y)&=\left[\frac{p\sinh\pi p}{4\pi(\sinh^2\pi p+\cosh^2\pi k)}\right]^{1/2}\sqrt{\cos Y}\,P^{ip}_{ik-1/2}(\sin Y)\,e^{ikX}\\ E^S_p&=\frac{1}{2m}\left(p^2+\frac{1}{4}\right)\end{aligned} \tag{29}$$

$(p\in\mathbf{R},\, k\in\mathbf{R})$.

VI. SUMMARY

In this paper we have presented a complete path integral treatment on the pseudosphere Λ^{d-1}. The correct treatment of this Riemannian space is based on the Weyl-ordering rule for the quantum Hamiltonian which yields the well-defined quantum correction

$$\Delta V(\{\theta\})=\frac{1}{8mR^2}\left[(d-2)^2-\frac{1}{\sinh^2\tau}-\cdots-\frac{1}{\sinh^2\tau\cdots\sin^2\theta_2}\right] \tag{1}$$

to be used in the (Hamiltonian or Lagrangian) path integral. A crucial point in using the Weyl-ordering rule is that it leads in an unambiguous way to the prescription that all coordinate-dependent expressions in the path integrals must be taken at the midpoints. However, in our path integral formulation, we use a product form instead of the midpoints, because it simplifies our formulas considerably. Having the correct path integral on the pseudosphere, it turns out, however, that it is too complicated for explicit calculations. We can use, however, an identity in the path integral to cast it in a much simpler form, yielding the (constant!) quantum correction

$$V_c + \Delta V = \frac{(d-1)(d-3)}{8mR^2}. \tag{2}$$

The resulting path integral can be exactly calculated yielding the spectral representation of the Feynman kernel $K(T)$. From $K(T)$ we have obtained the normalised wavefunctions and the energy spectrum

$$E_p^{(d)} = \frac{1}{2mR^2}\left[p^2 + \left(\frac{d-2}{2}\right)^2\right] \qquad (p>0,\ d=2, 3, 4, \ldots), \tag{3}$$

showing a characteristic dependence on the dimension d.

We have discussed in some detail the even- and odd-dimensional cases. In even dimensions the Feynman kernel could be expressed in closed form, yielding simple expressions for $d=2, 4$ and finite sums for $d=6, 8, \ldots,$

$$K^{(d)}(r;\, T) = \sqrt{\frac{mR^2}{2\pi i T}}\, e^{-(iT/8mR^2)(d-2)^2}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{(d-2)/2} e^{(imR^2/2T)r^2}. \tag{4}$$

This, of course, is quite similar to the d-dimensional rotator, where in even dimensions the Feynman kernel can be expressed in terms of a θ_3-function and its derivatives.

To establish the connection of the exact expression (4) with the *semiclassical approximation* to the path integral, we reinsert $\hbar$ to obtain

$$K^{(d)}(r;\, T) = \left(\frac{mR^2}{2\pi i\hbar T}\right)^{(d-1)/2} \sqrt{D^{(d)}} \exp\left[\frac{i}{\hbar}\left(S - \hbar^2 \frac{(d-2)^2}{8mR^2} T\right)\right][1 + O(\hbar)]. \tag{5}$$

Here S denotes the "classical action" $S = mR^2 r^2/2T$ and $D^{(d)}$ the van Vleck–Pauli–Morette determinant which in our case is explicitly given by $(r/\sinh r)^{d-2}$. Note again the additional time-dependent phase factor which is due to the quantum correction (2). We have thus derived Gutzwiller's "mysterious phase factor" [21].

The Green's function $G(E)$ can be expressed in terms of an associated Legendre function of the second kind in all dimensions,

$$G^{(d)}(r; E) = \frac{mR^2}{\gamma_d \pi} \left(\frac{e^{i\pi}}{2\pi \sinh r} \right)^{(d-3)/2} \mathcal{Q}^{(d-3)/2}_{i\sqrt{2mER^2 - (d-2)^2/4} - 1/2}(\cosh r) \tag{6}$$

($\gamma_d = 1$; d, odd; $\gamma_d = i$, d, even).

The hyperboloid Λ^2 is of special interest, because it is analytically equivalent to three further Riemannian spaces, the Poincaré disc D, the Poincaré upper half plane U, and the hyperbolic strip S. The path integral solution on D can be found, once the solution for Λ^2 is known. One has only to perform a transformation of the variables τ, ϕ of Λ^2 to r, ϕ of D. The path integral solution on U has been presented in an earlier paper [19]. We have shown that the Feynman kernels on Λ and U can be transformed into each other. In all these cases the energy spectrum reads

$$E_p = \frac{1}{2mR^2} \left(p^2 + \frac{1}{4} \right) \qquad (p > 0). \tag{7}$$

We have thus added further examples to the short list of exactly solvable path integrals. The examples demonstrate once more the consistency as well as the universal utility and feasibility of our general method developed in [18].

APPENDIX A: The Associated Legendre Functions $\mathcal{P}_\nu^\mu$ and $\mathcal{Q}_\nu^\mu$

The functions $\mathcal{P}_\nu^\mu$ and $\mathcal{Q}_\nu^\mu$ are linearly independent solutions of the differential equation

$$(1 - z^2) \frac{d^2u(z)}{dz^2} - 2z \frac{du(z)}{dz} + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] u(z) = 0 \tag{1}$$

and are defined by means of the hypergeometric function ${}_2F_1$:

$$\mathcal{P}_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1} \right)^{\mu/2} {}_2F_1 \left(-\nu, \nu + 1; 1 - \mu; \frac{1-z}{2} \right) \tag{2}$$

$$\mathcal{Q}_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1) \sqrt{\pi}}{2^{\nu+1} \Gamma(\nu + 3/2)} (z^2 - 1)^{\mu/2} z^{-\nu-\mu-1}$$

$$\times {}_2F_1 \left(\frac{\nu + \mu + 2}{2}, \frac{\nu + \mu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{z^2} \right). \tag{3}$$

They are called associated Legendre functions (or spherical functions) of the first and second kind, respectively.[5] They are uniquely defined in the intervals $|1 - z| < 2$

[5] We use $\mathcal{P}_\nu^\mu(z)$, $\mathcal{Q}_\nu^\mu(z)$ for $z \in \mathbb{C} \backslash [-1, 1]$ and $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$ for $x \in (-1, 1)$ for the Legendre functions of the first and second kind, respectively.

and $|z| > 1$, respectively. They can be extended to the entire z-plane where a cut along the real axis from $-\infty$ to $+1$ must be made.

The so-called conical functions $\mathscr{P}^{\mu}_{-1/2+ip}$ have the special property

$$\mathscr{P}^{\mu}_{-1/2+ip}(z) = \mathscr{P}^{\mu}_{-1/2-ip}(z) \tag{4}$$

which is due to the general property $\mathscr{P}^{\mu}_{\nu} = \mathscr{P}^{\mu}_{-\nu-1}$. If $\mu = m \in \mathbf{Z}$,

$$\mathscr{P}^{m}_{\nu}(z) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} \mathscr{P}^{-m}_{\nu}(z). \tag{5}$$

The functions $\mathscr{P}^{\mu}_{-1/2+ip}$ form a complete set; that means a function g can be expanded (see [28, p. 202]),

$$g(y) = \int_0^{\infty} f(p)\, \mathscr{P}^{\mu}_{-1/2+ip}(y)\, dp \tag{6}$$

with

$$f(p) = \frac{p \sinh \pi p}{\pi} \Gamma\left(\frac{1}{2} - \mu + ip\right) \Gamma\left(\frac{1}{2} - \mu - ip\right) \int_1^{\infty} g(y)\, \mathscr{P}^{\mu}_{-1/2+ip}(y)\, dy. \tag{7}$$

Also,

$$g(p) = \int_1^{\infty} f(y)\, \mathscr{P}^{\mu}_{-1/2+ip}(y)\, dy \tag{8}$$

with

$$f(y) = \frac{1}{\pi} \int_0^{\infty} p \sinh \pi p \Gamma\left(\frac{1}{2} - \mu + ip\right) \Gamma\left(\frac{1}{2} - \mu - ip\right) g(p)\, \mathscr{P}^{\mu}_{-1/2+ip}(y)\, dp. \tag{9}$$

These properties follow from the orthogonality relation ($p, p' \in \mathbf{R}^+, \mu \in \mathbf{R}$)

$$\left|\frac{\Gamma(1/2+ip-\mu)}{\Gamma(ip)}\right|^2 \int_1^{\infty} \mathscr{P}^{\mu}_{-1/2+ip}(y)\, \mathscr{P}^{\mu}_{-1/2-ip'}(y)\, dy = \delta(p-p') \tag{10}$$

and, vice versa, from the completeness relation

$$\int_0^{\infty} \left|\frac{\Gamma(1/2+ip-\mu)}{\Gamma(ip)}\right|^2 \mathscr{P}^{\mu}_{-1/2+ip}(z)\, \mathscr{P}^{\mu}_{-1/2+ip}(y)\, dp = \delta(z-y). \tag{11}$$

These two relations are well known as generalised Mehler transformations.

The proof of the orthogonality relation is relatively easy. We use the general integral theorem (see [28, p. 191]),

$$\begin{aligned} &\int_a^b \left[(\nu-\sigma)(\nu+\sigma+1) + \frac{\rho^2-\mu^2}{1-z^2}\right] w^{\mu}_{\nu}(z)\, w^{\rho}_{\sigma}(z)\, dz \\ &\quad = \{\sqrt{1-z^2}\,[w^{1+\mu}_{\nu}(z)\, w^{\rho}_{\sigma}(z) - w^{\mu}_{\nu}(z)\, w^{1+\rho}_{\sigma}(z)] + (\mu-\rho)\, z w^{\mu}_{\nu}(z)\, w^{\rho}_{\sigma}(z)\}\big|_a^b, \end{aligned} \tag{12}$$

where w_ν^μ denotes any of the associated Legendre functions. Let us set $\mu=\rho$, $\nu=ip-\frac{1}{2}$, $\sigma=-ip'-\frac{1}{2}$, $a=1+\varepsilon$, and $b=1/\varepsilon$. Then we get

$$\int_{1+\varepsilon}^{1/\varepsilon} \mathscr{P}^{\mu}_{ip-1/2}(z)\,\mathscr{P}^{\mu}_{-ip'-1/2}(z)\,dz$$
$$=\frac{\sqrt{1-z^2}}{(p'-p)(p+p')}\left[\mathscr{P}^{1+\mu}_{ip-1/2}(z)\,\mathscr{P}^{\mu}_{-ip'-1/2}(z)-\mathscr{P}^{\mu}_{ip-1/2}(z)\,\mathscr{P}^{1+\mu}_{-ip'-1/2}(z)\right]\Big|_{1+\varepsilon}^{1/\varepsilon}. \quad (13)$$

With the expansion for $z\to 1$

$$\mathscr{P}^{\mu}_{\nu}(z)\simeq\frac{2^{\mu/2}}{(z-1)^{\mu/2}\Gamma(1-\mu)}, \quad (14)$$

we find, at $z=1+\varepsilon$ and $\mu\neq 0, 1, 2, 3$ in the limit $\varepsilon\to 0$,

$$\frac{\sqrt{1-z^2}}{p'^2-p^2}\left[\mathscr{P}^{1+\mu}_{ip-1/2}(z)\,\mathscr{P}^{\mu}_{-ip'-1/2}(z)-\mathscr{P}^{\mu}_{ip-1/2}(z)\,\mathscr{P}^{1+\mu}_{-ip'-1/2}(z)\right]\Big|_{z=1+\varepsilon}$$
$$=\frac{\sqrt{2\varepsilon}}{p'^2-p^2}\left(\frac{2}{\varepsilon}\right)^{(1+2\mu)/2}\left(\frac{1}{\Gamma(1-\mu)\,\Gamma(-\mu)}-\frac{1}{\Gamma(1-\mu)\,\Gamma(-\mu)}\right)+O(\varepsilon)=O(\varepsilon)\to 0. \quad (15)$$

At $z=1/\varepsilon$ we have the expansion [20, p. 1011]

$$\mathscr{P}^{\mu}_{\nu}(z)\simeq\frac{2^{\nu}}{\sqrt{\pi}}\frac{\Gamma(\nu+1/2)}{\Gamma(1+\nu-\mu)}z^{\nu}+\frac{1}{2^{\nu+1}\sqrt{\pi}}\frac{\Gamma(-\nu-1/2)}{\Gamma(-\nu-\mu)}z^{-\nu-1} \qquad (z\to\infty). \quad (16)$$

Inserting into Eqs. (13) yields

$$\frac{\sqrt{1-z^2}}{(p'-p)(p'+p)}\left[\mathscr{P}^{1+\mu}_{ip-1/2}(z)\,\mathscr{P}^{\mu}_{-ip'-1/2}(z)-\mathscr{P}^{\mu}_{ip-1/2}(z)\,\mathscr{P}^{1+\mu}_{-ip'-1/2}(z)\right]\Big|_{z=1/\varepsilon}$$
$$=\frac{1}{2\pi}\left\{\frac{(2/\varepsilon)^{i(p-p')}}{p'^2-p^2}\right.$$
$$\times\left[\frac{\Gamma(ip)\,\Gamma(-ip')}{\Gamma(ip-1/2-\mu)\,\Gamma(-ip'+1/2-\mu)}-\frac{\Gamma(ip)\,\Gamma(-ip')}{\Gamma(ip+1/2-\mu)\,\Gamma(-ip'-1/2-\mu)}\right]$$
$$+\text{h.c.}+\frac{(2/\varepsilon)^{i(p+p')}}{p'^2-p^2}$$
$$\left.\times\left[\frac{\Gamma(ip)\,\Gamma(ip')}{\Gamma(ip-1/2-\mu)\,\Gamma(ip'+1/2-\mu)}-\frac{\Gamma(ip)\,\Gamma(ip')}{\Gamma(ip+1/2-\mu)\,\Gamma(ip'-1/2-\mu)}\right]+\text{h.c.}\right\}. \quad (17)$$

The term $(2/\varepsilon)^{i(p-p')}/(p-p')$ yields, in the limit $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \frac{(2/\varepsilon)^{i(p-p')}}{p-p'} = \lim_{\varepsilon \to 0} \frac{\cos[(p-p')\ln(2/\varepsilon)] + i\sin[(p-p')\ln(2/\varepsilon)]}{p-p'} = i\pi\delta(p-p'). \tag{18}$$

So we can conclude

$$\begin{aligned} &\frac{1}{2\pi(p'-p)(p'+p)}\left(\frac{2}{\varepsilon}\right)^{i(p-p')} \\ &\quad\times\left[\frac{\Gamma(ip)\,\Gamma(-ip')}{\Gamma(ip-1/2-\mu)\,\Gamma(-ip'+1/2-\mu)} - \frac{\Gamma(ip)\,\Gamma(-ip')}{\Gamma(ip+1/2-\mu)\,\Gamma(-ip'-1/2-\mu)}\right] \\ &= \frac{1}{2}\frac{|\Gamma(ip)|^2}{|\Gamma(ip+1/2-\mu)|^2}\,\delta(p'-p) \end{aligned} \tag{19}$$

and similarly for the term proportional to $(2/\varepsilon)^{i(p+p')}$ and the h.c. terms. The contributions proportional to $\delta(p+p')$ actually are of the form $(p+p')\,\delta(p+p')$ and therefore vanish. Thus the orthogonality relation is proved. For the completeness relation see, e.g., [2].

In Section III we have used the relation ($v=\sqrt{2mR^2E-((d-2)/2)^2}$, $r>0$)

$$\frac{i\pi}{v}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^n e^{-ivr} = \left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{n-1/2} Q_{iv-1/2}^{n-1/2}(\cosh r) \tag{20}$$

for $n=(d-2)/2=0, 1, 2, \ldots$. Equation (20) states that the function Q_λ^μ, where μ is a half integer and $\lambda \in \mathbf{C}$, can be expressed by combinations of hyperbolics and an exponential, i.e., by elementary functions. We want to prove Eq. (20) by induction.

(i) Let $n=0$. With the help of [12, p. 150] $Q_\lambda^{-1/2}(z) = -i(1/(2\lambda+1))\sqrt{2\pi}\,(z^2-1)^{-1/4}\cdot[z+(z^2-1)^{1/2}]^{-\lambda-1/2}$ we see immediately that Eq. (20) holds for $n=0$.

(ii) Let $n \in \mathbf{N}_0$ such that Eq. (20) is true. We consider the right-hand side of (20) for $n \to n+1$. In order to reduce the upper index of $Q_{iv-1/2}^{n+1/2}$ by one unit we combine the relations $(z^2-1)(d/dz)\,Q_\lambda^\mu(z) = (\lambda-\mu+1)\,Q_{\lambda+1}^\mu(z) - (\lambda+1)\,zQ_\lambda^\mu(z)$ and $(\lambda+\mu+1)\,zQ_\lambda^\mu(z) + \sqrt{z^2-1}\,Q_\lambda^{\mu+1}(z) = (\lambda-\mu+1)\,Q_{\lambda+1}^\mu(z)$ [28, p. 1005] to get

$$Q_\lambda^{\mu+1}(z) = \sqrt{z^2-1}\,\frac{dQ_\lambda^\mu(z)}{dz} - \mu\frac{z}{\sqrt{z^2-1}}\,Q_\lambda^\mu(z). \tag{21}$$

This gives

$$\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{n+1/2} Q^{n+1/2}_{i\nu-1/2}(\cosh r)$$

$$=\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{n+1/2}\left[\sinh r\,\frac{dQ^{n-1/2}_{i\nu-1/2}(\cosh r)}{d\cosh r}-\left(n-\frac{1}{2}\right)\coth r\,Q^{n-1/2}_{i\nu-1/2}(\cosh r)\right]$$

$$=\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)\left\{\frac{d}{dr}\left[\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{n-1/2} Q^{n-1/2}_{i\nu-1/2}(\cosh r)\right]\right.$$

$$\left.+\left(n-\frac{1}{2}\right)\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{n-1/2}\coth r\,Q^{n-1/2}_{i\nu-1/2}(\cosh r)\right\}$$

$$-\left(n-\frac{1}{2}\right)\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{n+1/2}\coth r\,Q^{n-1/2}_{i\nu-1/2}(\cosh r)$$

$$=\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]\left(\frac{e^{i\pi}}{2\pi\sinh r}\right)^{n-1/2} Q^{n-1/2}_{i\nu-1/2}(\cosh r)$$

$$=\frac{i\pi}{\nu}\left[-\frac{1}{2\pi}\frac{d}{d\cosh r}\right]^{n+1} e^{-i\nu r}, \tag{22}$$

where we have used in the last step Eq. (20).

APPENDIX B: Proof of Eq. (II.25)

We shall derive the identity

$$\exp\{i\varepsilon\mathscr{L}^N_{\rm Cl}(\{\theta^{(j)}\},\{\theta^{(j-1)}\})\} \doteq \exp\left\{-\frac{im}{\varepsilon}R^2(1-\cosh l^{(j,j-1)})-i\varepsilon V_{\rm c}(\{\theta^{(j)}\})\right\}. \tag{1}$$

$\mathscr{L}^N_{\rm Cl}$ denotes the Lagrangian defined in Eq. (II.25). We start with the kinetic term $(x^{(j)}-x^{(j-1)})^2$ expressed in the pseudospherical polar coordinates (II.2) and expand it in terms of $\Delta r^{(j)}$ and $\Delta\theta^{(j)}_\nu$ ($\nu=1,\ldots,d-1$). In this procedure we follow the reasoning of Pak and Sökmen [33]):[6] If one has an expression like $\Delta f^{(j)}_l := f_l(u^{(j)}_1\cdots u^{(j)}_d)-f_l(u^{(j-1)}_1\cdots u^{(j-1)}_d)$, one gets the following for the expansion about the midpoint $\bar u^{(j)} := \frac{1}{2}(u^{(j)}+u^{(j-1)})$:

$$\Delta f^{(j)}_l=\sum_{m=1}^{d}\Delta u^{(j)}_m\left(\frac{\partial f^{(j)}_l}{\partial u^{(j)}_m}\right)_{\bar u^{(j)}}+\frac{1}{24}\sum_{m,n,k=1}^{d}\Delta u^{(j)}_m\,\Delta u^{(j)}_n\,\Delta u^{(j)}_k\left(\frac{\partial^3 f^{(j)}_l}{\partial u^{(j)}_m\,\partial u^{(j)}_n\,\partial u^{(j)}_k}\right)_{\bar u^{(j)}}+\cdots. \tag{2}$$

Here $f^{(j)}_l=x^{(j)}_l$ ($l=1,\ldots,d$), $u_m=\{u_d=r,\ u_k=\theta_k\ (k=1,\ldots,d-1)\}$. Calculating the various derivatives and inserting into Eq. (2) yield [G_{ab} is the metric tensor

[6] This method goes back to DeWitt [5], McLaughlin and Schulman [29], and Gervais and Jevicki [16]; we prefer the formulation of Ref. [33] because it seems more explicit to us.

in x-space, $g_{ab}=\mathrm{diag}(1, r^2, r^2\sinh^2\tau, ..., r^2\cdots\sin^2\theta_2)$ the metric tensor for the pseudospherical polar coordinates]

$$
\begin{aligned}
&(\Delta\mathbf{x}^{(j)})^2\\
&\quad=\sum_{l=1}^{d} G_{ll}\left[\sum_{m=1}^{d}\Delta u_m^{(j)}\left(\frac{\partial x_l^{(j)}}{\partial u_m^{(j)}}\right)_{\bar{u}^{(j)}}\right]^2+\frac{1}{12}\sum_{l=1}^{d} G_{ll}\left[\sum_{m=1}^{d}\Delta u_m^{(j)}\left(\frac{\partial x_l^{(j)}}{\partial u_m^{(j)}}\right)_{\bar{u}^{(j)}}\right]\\
&\qquad\times\left[\sum_{m,n,k=1}^{d}\Delta u_m^{(j)}\,\Delta u_n^{(j)}\,\Delta u_k^{(j)}\left(\frac{\partial^3 x_l^{(j)}}{\partial u_m^{(j)}\,\partial u_n^{(j)}\,\partial u_k^{(j)}}\right)_{\bar{u}^{(j)}}+\cdots\right.\\
&\quad=\frac{2\varepsilon^2}{m}\mathscr{L}_{\mathrm{Cl}}^{N}(\bar{r}^{(j)},\bar{\tau}^{(j)},\{\bar{\theta}^{(j)}\})-\frac{1}{4}A(\bar{r}^{(j)},\bar{\tau}^{(j)},\{\bar{\theta}^{(j)}\})-\frac{1}{12}B(\bar{r}^{(j)},\bar{\tau}^{(j)},\{\bar{\theta}^{(j)}\})+\cdots
\end{aligned}
\tag{3}
$$

with

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{Cl}}^{N}(\bar{r}^{(j)},\bar{\tau}^{(j)},\{\bar{\theta}^{(j)}\})\\
&\quad=\frac{m}{2\varepsilon^2}\{-\Delta^2 r^{(j)}+[\bar{r}^{(j)}]^2\,\Delta^2\tau^{(j)}+\cdots+[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}\cdots\sin\bar{\theta}_2^{(j)}]^2\,\Delta^2\phi^{(j)}]
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&A(\bar{r}^{(j)},\bar{\tau}^{(j)},\{\bar{\theta}^{(j)}\})\\
&\quad=\Delta^2 r^{(j)}\,\Delta^2\tau^{(j)}+\sinh^2\bar{\tau}^{(j)}\,\Delta^2 r^{(j)}\,\Delta^2\bar{\theta}_{d-2}^{(j)}+\cdots\\
&\qquad+[\sinh\bar{\tau}^{(j)}\cdots\sin\bar{\theta}_2^{(j)}]^2\,\Delta^2 r^{(j)}\,\Delta^2\phi^{(j)}\\
&\qquad+[\bar{r}^{(j)}]^2\,\Delta^2\tau^{(j)}\,\Delta^2\theta_{d-2}^{(j)}+[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}]^2\,\Delta^2\tau^{(j)}\,\Delta^2\theta_{d-3}^{(j)}+\cdots\\
&\qquad+[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}\cdots\sin\bar{\theta}_2^{(j)}]^2\,\Delta^2\tau^{(j)}\,\Delta^2\phi^{(j)}+\cdots\\
&\qquad+[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}\cdots\sin\bar{\theta}_2^{(j)}]^2\,\Delta^2\theta_2^{(j)}\,\Delta^2\phi^{(j)}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&B(\bar{r}^{(j)},\bar{\tau}^{(j)},\{\bar{\theta}^{(j)}\})\\
&\quad=-[\bar{r}^{(j)}]^2\,\Delta^4\tau^{(j)}+[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}]^2\,\Delta^4\theta_{d-2}^{(j)}+\cdots\\
&\qquad+[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}\cdots\sin\bar{\theta}_2^{(j)}]^2\,\Delta^4\phi^{(j)}.
\end{aligned}
\tag{6}
$$

In the next step we must change the integration measure. We get

$$
\begin{aligned}
&\prod_{j=1}^{N-1} dx_1^{(j)}\cdots dx_d^{(j)}\\
&\quad=\left(\prod_{j=1}^{N-1} dr^{(j)}\,d\tau^{(j)}\cdots d\phi^{(j)}\right)\prod_{j=1}^{N-1}\sqrt{g^{(j)}}\\
&\quad=[g^{(0)}g^{(N)}]^{-1/4}\left(\prod_{j=1}^{N-1} dr^{(j)}\,d\tau^{(j)}\cdots d\phi^{(j)}\right)\\
&\qquad\times\left\{\prod_{j=1}^{N}[r^{(j)}r^{(j-1)}]^{(d-1)/2}[\sinh\tau^{(j)}\sinh\tau^{(j-1)}]^{(d-2)/2}\cdots[\sin\theta_2^{(j)}\sin\theta_2^{(j-1)}]^{1/2}\right\}
\end{aligned}
\tag{7}
$$

($g^{(0)}$, $g^{(N)}$, and $g^{(j)}$ denote the determinant of the metric tensor taken at $j=0$, $j=N$, $j=1, ..., N-1$, respectively). Furthermore we have

$$[r^{(j)}r^{(j-1)}]^{(d-1)/2} \simeq [\bar{r}^{(j)}]^{d-1}\left(1-\frac{d-1}{8}\cdot\frac{\Delta^2 r^{(j)}}{[\bar{r}^{(j)}]^2}\right) \tag{8}$$

$$[\sinh\tau^{(j)}\sinh\tau^{(j-1)}]^{(d-2/2)} \simeq \sinh^{d-2}\bar{\tau}^{(j)}\left(1-\frac{d-2}{8}\cdot\frac{\Delta^2\tau^{(j)}}{\sinh^2\bar{\tau}^{(j)}}\right) \tag{9}$$

$$[\sinh\theta_\nu^{(j)}\sin\theta_\nu^{(j-2)}]^{\nu-1} \simeq \sin^{\nu-1}\bar{\theta}_\nu^{(j)}\left(1-\frac{\nu-1}{8}\cdot\frac{\Delta^2\theta_\nu^{(j)}}{\sin^2\bar{\theta}_\nu^{(j)}}\right)$$
$$(\nu=2, ..., d-2). \tag{10}$$

Combining Eqs. (7) to (10) we get for the measure

$$\prod_{j=1}^{N-1} dx_1^{(j)}\cdots dx_d^{(j)} = (g'g'')^{-1/4}\left(\prod_{j=1}^{N-1} dr^{(j)}d\tau^{(j)}\cdots d\phi^{(j)}\right)\left(\prod_{j=1}^{N}\sqrt{\bar{g}^{(j)}}\,e^{-C}\right) \tag{11}$$

with

$$C=\frac{1}{8}\left[(d-1)\frac{\Delta^2 r^{(j)}}{[\bar{r}^{(j)}]^2}+(d-2)\frac{\Delta^2\tau^{(j)}}{\sinh^2\bar{\tau}^{(j)}}+\cdots+\frac{\Delta^2\theta_2^{(j)}}{\sin^2\bar{\theta}_2^{(j)}}\right]; \tag{12}$$

g', g'', and $\bar{g}^{(j)}$ denote the determinant of the metric tensor taken at the points $j=0, N$ and the midpoint values for $j=1, ..., N$, respectively. Therefore we get for the jth ($j=1, ..., N$) integrand in the path integral (II.18) (without ΔV)

$$\sqrt{\bar{g}^{(j)}}\exp\{i\varepsilon[\mathscr{L}_{\mathrm{Cl}}^N(\bar{r}^{(j)}, \bar{\tau}^{(j)}, \{\bar{\theta}^{(j)}\})-\hat{V}(\bar{r}^{(j)}, \{\bar{\theta}^{(j)}\})]\}, \tag{13}$$

where

$$\hat{V}(\bar{r}^{(j)}, \{\bar{\theta}^{(j)}\})=\frac{m}{8\varepsilon^2}\left[A(\bar{r}^{(j)}, \bar{\tau}^{(j)}, \{\bar{\theta}^{(j)}\})+\frac{1}{3}B(\bar{r}^{(j)}, \bar{\tau}^{(j)}, \{\bar{\theta}^{(j)}\})\right]+\frac{1}{i\varepsilon}C(\bar{r}^{(j)}, \bar{\tau}^{(j)}, \{\bar{\theta}^{(j)}\}). \tag{14}$$

We want to transform the various $\Delta^2 u^{(j)}$-terms into potential-like terms. Following the general approach (see Feynman and Hibbs [14], McLaughlin and Schulman [29], or Pak and Sökmen [33]) we get the equivalence relations:

$$\Delta^2 r^{(j)} \doteq \frac{\varepsilon}{im}, \qquad \Delta^2\tau^{(j)} \doteq \frac{i\varepsilon}{m[\bar{r}^{(j)}]^2}$$

$$\Delta^2\theta_{d-2}^{(j)} \doteq \frac{i\varepsilon}{m[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}]^2}, \quad ..., \quad \Delta^2\phi^{(j)} \doteq \frac{i\varepsilon}{m[\bar{r}^{(j)}\cdots\sin\bar{\theta}_2^{(j)}}, $$

$$\Delta^4\tau^{(j)} \doteq 3\left(\frac{i\varepsilon}{m[\bar{r}^{(j)}]^2}\right)^2$$

$$\Delta^4\theta^{(j)}_{d-2} \doteq 3\left(\frac{i\varepsilon}{m[\bar{r}^{(j)}\sinh\bar{\tau}^{(j)}]^2}\right)^2, \quad ..., \quad \Delta^4\phi^{(j)} \doteq 3\left(\frac{i\varepsilon}{m[\bar{r}^{(j)}\cdots\sin\bar{\theta}^{(j)}_2]^2}\right)^2. \tag{15}$$

Inserting into (14) yields

$$\begin{aligned}
&\hat{V}(\bar{r}^{(j)}, \{\bar{\theta}^{(j)}\})\\
&\doteq \frac{1}{8m[\bar{r}^{(j)}]^2}\left[-d+\frac{d-1}{\sinh^2\bar{\tau}^{(j)}}+\frac{d-2}{\sinh^2\bar{\tau}^{(j)}\sin^2\bar{\theta}^{(j)}_{d-2}}+\cdots+\frac{2}{\sinh^2\bar{\tau}^{(j)}\cdots\sin^2\bar{\theta}^{(j)}_2}\right]\\
&\quad+(d-1)-\frac{d-2}{\sinh^2\bar{\tau}^{(j)}}-\frac{d-3}{\sinh^2\bar{\tau}^{(j)}\sinh^2\bar{\theta}^{(j)}_{d-2}}-\cdots-\frac{1}{\sinh^2\bar{\tau}^{(j)}\cdots\sin^2\bar{\theta}^{(j)}_2}\Bigg]\\
&=\frac{1}{8m[\bar{r}^{(j)}]^2}\left[-1+\frac{1}{\sinh^2\bar{\tau}^{(j)}}+\frac{1}{\sinh^2\bar{\tau}^{(j)}\sin^2\bar{\theta}^{(j)}_{d-2}}+\cdots+\frac{1}{\sinh^2\bar{\tau}^{(j)}\cdots\sin^2\bar{\theta}^{(j)}_2}\right]
\end{aligned} \tag{16}$$

which leads for $[\bar{r}^{(j)}]^2 = R^2$ to

$$\hat{V}(R, \{\bar{\theta}^{(j)}\}) \doteq V_c(\{\bar{\theta}^{(j)}\}). \tag{17}$$

We emphasize that V_c is the same whether or not $\Delta r^{(j)} = 0$.

In order to prove Eq. (1) we consider now on one hand

$$\begin{aligned}
&\prod_{j=1}^{N-1}\sqrt{g^{(j)}}\cdot\exp\left\{i\varepsilon\sum_{j=1}^{N}\mathscr{L}^N_{\mathrm{Cl}}(\{\theta^{(j)}, \theta^{(j-1)}\})\right\}\\
&\qquad\doteq (g'g'')^{-1/4}\prod_{j=1}^{N}\sqrt{\bar{g}^{(j)}}\cdot\exp\{i\varepsilon\mathscr{L}^N_{\mathrm{Cl}}(R, \bar{\tau}^{(j)}, \{\bar{\theta}^{(j)}\})\}.
\end{aligned} \tag{18}$$

On the other hand we have with Eqs. (3), (13), and (17)

$$\begin{aligned}
&\prod_{j=1}^{N-1}\sqrt{g^{(j)}}\cdot\exp\left\{-\frac{imR^2}{\varepsilon}\sum_{j=1}^{N}(1-\cosh l^{(j,j-1)})\right\}\\
&\qquad\doteq (g'g'')^{-1/4}\prod_{j=1}^{N}\sqrt{\bar{g}^{(j)}}\cdot\exp\{i\varepsilon[\mathscr{L}^N_{\mathrm{Cl}}(R, \bar{\tau}^{(j)}, \{\bar{\theta}^{(j)}\})+V_c(\{\bar{\theta}^{(j)}\})]\}.
\end{aligned} \tag{19}$$

Finally putting (18) and (19) together (the change from midpoints to postpoints in V_c does not matter, because it is of $O(\varepsilon^2)$ in the action) we get

$$\exp\{i\varepsilon\mathscr{L}^N_{\mathrm{Cl}}(\{\theta^{(j)}\}, \{\theta^{(j-1)}\})\} \doteq \exp\left\{-\frac{im}{\varepsilon}R^2(1-\cosh l^{(j,j-1)})-i\varepsilon V_c(\{\theta^{(j)}\})\right\} \tag{20}$$

and Eq. (1) is proved.

APPENDIX C: Derivation of the Schrödinger Equation from Eq. (II.18)

The derivation given below is similar to the d-dimensional rotator case which can be found in [18].

We want to prove that from the short-time kernel

$$\begin{aligned}
&K^{(d)}(\{\theta^{(j+1)}\}, \{\theta^{(j)}\}; \varepsilon)\\
&= \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2} \exp\left\{\frac{im}{2\varepsilon} R^2 \left[(\tau^{(j+1)}-\tau^{(j)})^2 + \sinh\tau^{(j+1)}\sinh\tau^{(j)}\right.\right.\\
&\quad \times (\theta_{d-2}^{(j+1)}-\theta_{d-2}^{(j)})^2 + \cdots + \left.(\sinh\tau^{(j+1)}\cdots\sin\theta_2^{(j)})(\phi^{(j+1)}-\phi^{(j)})^2\right]\\
&\quad \left.-\frac{i\varepsilon}{8mR^2}\left[(d-2)^2 - \frac{1}{\sinh\tau^{(j+1)}\sinh\tau^{(j)}} - \cdots - \frac{1}{\sinh\tau^{(j+1)}\cdots\sin\theta_2^{(j)}}\right]\right\} \qquad (1)
\end{aligned}$$

and the time-evolution equation

$$\psi(\{\theta^{(j+1)}\}, t+\varepsilon) = \int_{\Lambda^{d-1}} du^{(j)} K^{(d)}(\{\theta^{(j+1)}\}, \{\theta^{(j)}\}; \varepsilon)\, \psi(\{\theta^{(j)}\}, t) \qquad (2)$$

the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2mR^2} K_{(d)}^2 \psi \qquad (3)$$

can be derived. For this purpose, a Taylor expansion must be performed in (2) yielding ($\theta'_k := \theta_k^{(j)}$ and $\theta''_k := \theta_k^{(j+1)}$)

$$\begin{aligned}
&\psi(\{\theta''\}); t) + \varepsilon \frac{\partial\psi(\{\theta''\}; t)}{\partial t}\\
&= \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2} e^{-i\varepsilon\Delta V(\{\theta''\})} \left\{\psi(\{\theta''\}; t)\, B_0 + \sum_{\nu=1}^{d-1} \frac{\partial\psi(\{\theta''\}; t)}{\partial\theta''} (B_{\theta_\nu} - \theta''_\nu B_0)\right.\\
&\quad \left.+\frac{1}{2}\sum_{\substack{\mu,\nu=1\\ \mu\geqslant\nu}}^{d-1} \frac{\partial^2\psi(\{\theta''\}; t)}{\partial\theta''_\mu\,\partial\theta''_\nu} (B_{\theta_\mu\theta_\nu} - \theta''_\mu B_{\theta_\nu} - \theta''_\nu B_{\theta_\mu} + \theta''_\mu\theta''_\nu B_0)\right\}. \qquad (4)
\end{aligned}$$

We have used the abbreviations

$$B_0 = \int du' e^{i\varepsilon\mathscr{L}_{\text{Cl}}^N(\{\theta''\},\{\theta'\})} \simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-1)/2} e^{i\varepsilon\Delta V\{(\theta''\})}$$

$$B_\phi = \int du' \phi e^{i\varepsilon\mathscr{L}_{\text{Cl}}^N} \simeq \phi'' B_0$$

$$B_{\phi^2} = \int du' \phi'^2 e^{i\varepsilon\mathscr{L}_{\text{Cl}}^N} \simeq \phi''^2 B_0 + \frac{1}{\sinh^2\tau''\cdots\sin^2\theta''_?}\frac{i\varepsilon}{mR^2} B_0$$

$$B_{\theta_\nu} = \int du' \theta'_\nu e^{i\varepsilon \mathscr{L}^N_{\text{Cl}}}$$

$$\simeq \theta''_\nu B_0 + \frac{1}{2} \frac{d-\nu-1}{\sinh^2 \tau''_1 \cdots \sin^2 \theta''_{\nu-1}} \cot \theta''_\nu \frac{i\varepsilon}{mR^2} B_0 \tag{5}$$

$$B_{\phi\theta_\nu} = \int du' \phi' \theta'_\nu e^{i\varepsilon \mathscr{L}^N_{\text{Cl}}}$$

$$\simeq \phi'' \theta''_\nu B_0 + \frac{1}{2} \frac{d-\nu-1}{\sinh^2 \tau'' \cdots \sin^2 \theta''_{\nu-1}} \phi'' \cot \theta''_\nu \frac{i\varepsilon}{mR^2} B_0$$

$$B_{\theta_\nu \theta_\mu} = \int du' \theta'_\nu \theta'_\mu e^{i\varepsilon \mathscr{L}^N_{\text{Cl}}} \simeq \theta''_\nu \theta''_\mu B_0$$

$$+ \frac{1}{2} \left[\frac{d-\nu-1}{\sinh^2 \tau'' \cdots \sin^2 \theta''_{\nu-1}} \cot \theta''_\nu + \frac{d-\mu-1}{\sinh^2 \tau'' \cdots \sin^2 \theta''_{\mu-1}} \cot \theta''_\mu \right] \frac{i\varepsilon}{mR^2} B_0$$

$$B_{\theta_\nu^2} = \int du' \theta'^2_\nu e^{i\varepsilon \mathscr{L}^N_{\text{Cl}}} \simeq \theta''^2_\nu B_0 + \frac{1 + (d-\nu-1) \cot \theta''_\nu}{\sinh^2 \tau'' \cdots \sin^2 \theta''_{\nu-1}} \frac{i\varepsilon}{mR^2} B_0.$$

Here the equations are valid up to terms of $O(\varepsilon^{(d+1)/2})$, and

$$\mathscr{L}^N_{\text{Cl}}(\{\theta''\}, \{\theta'\})$$
$$= \frac{mR^2}{2\varepsilon^2} [(\tau' - \tau'')^2 + \cdots + (\sinh \tau' \sinh \tau'' \cdots \sin \theta'_2 \sin \theta''_2)(\phi' - \phi'')^2] \tag{6}$$

denotes the "classical Lagrangian" on the lattice—see Eq. (II.26). In order to make the calculation manageable, we have taken the ΔV-term at the argument $\{\theta''\}$ and have factored out this term in Eq. (4). This is legitimate, because changing $\sin \theta'_\nu$ (θ'_ν is an integration variable) to $\sin \theta''_\nu$ in ΔV gives a correction of $O(\varepsilon)$, hence of order ε^2 in the short-time kernel and, therefore, can be omitted.

We shall only illustrate how to calculate the integral B_0 in Eq. (5). All other integrals containing powers of $\theta'_\nu(\tau')$ are of similar type bacause they are of Gaussian form. For simplification we use the abbreviations ($\nu = 1, ..., d-1$)

$$E(\theta_\nu) = \exp\{-\alpha[(\tau' - \tau'')^2 + \cdots + (\sinh \tau' \sinh \tau'' \cdots \sin \theta'_{d-\nu-1} \sin \theta''_{d-\nu-1}) \times (\theta'_\nu - \theta''_\nu)^2]\} \tag{7}$$

and $\alpha = mR^2/2i\varepsilon$.

We consider the integral

$$B_0 = \int_0^\infty d\tau' \sinh^{d-2} \tau' \cdots \int_0^\pi d\theta'_2 \sin \theta'_2 \int_0^{2\pi} d\phi' E(\theta_1)$$

$$\simeq \int_0^\infty d\tau' \sinh^{d-2} \tau' \cdots \int_0^\pi d\theta'_2 \sin \theta'_2 E(\theta_2) \int_{-\infty}^\infty dx\, e^{-\alpha(\sinh \tau' \cdots \sin \theta''_2) x^2}, \tag{8}$$

where we have set $x := \phi' - \phi''$ which varies from $-\infty$ to $+\infty$, and $\simeq$ denotes that this is correct in the limit $\varepsilon \to 0$. The x-integration is of Gaussian form, and we get

$$B_0 \simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{1/2} \int_0^\infty d\tau' \frac{\sinh^{d-2}\tau'}{\sqrt{\sinh\tau' \sinh\tau''}} \cdots \int_0^\pi d\theta'_2 \sqrt{\frac{\sin\theta'_2}{\sin\theta''_2}}\, E(\theta_2)$$
$$\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{1/2} \int_0^\infty d\tau' \frac{\sinh^{d-2}\tau'}{\sqrt{\sinh\tau' \sinh\tau''}} \cdots \int_0^\pi d\theta'_3 \frac{\sin^2\theta'_3}{\sqrt{\sin\theta'_3 \sin\theta''_3}}\, E(\theta_3)$$
$$\times \int_{-\infty}^\infty dx \left[1 + \frac{1}{2}\cot\theta''_2 \cdot x - \frac{x^2}{8}\left(1 + \frac{1}{\sin^2\theta''_2}\right)\right] e^{-\alpha(\sinh\tau' \cdots \sin\theta''_3)x^2}, \tag{9}$$

where we have performed a Taylor expansion around θ''_2 in the last step. The integral is Gaussian again, the term linear in x vanishes,[7] and we get

$$B_0 \simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right) \int_0^\infty d\tau' \frac{\sinh^{d-2}\tau'}{\sinh\tau' \sinh\tau''} \cdots \int_0^\pi d\theta'_3 \frac{\sin^2\theta'_3}{\sin\theta'_3 \sin\theta''_3}\, E(\theta_3)$$
$$\times \left[1 - \frac{i\varepsilon}{8mR^2} \cdot \frac{1}{\sinh\tau' \cdots \sin\theta''_3}\left(1 + \frac{1}{\sin^2\theta''_2}\right)\right]$$
$$\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right) \int_0^\infty d\tau' \frac{\sinh^{d-2}\tau'}{\sinh\tau' \sinh\tau''} \cdots \int_0^\pi d\theta'_4 \frac{\sin^3\theta'_4}{\sin\theta'_4 \sin\theta''_4}\, E(\theta_4)$$
$$\times \int_{-\infty}^\infty dx \left(1 + \cot\theta''_3 \cdot x - \frac{x^2}{2}\right) e^{-\alpha(\sinh\tau' \cdots \sin\theta''_4)x^2}$$
$$- \frac{i\varepsilon}{8mR^2}\left(\frac{2\pi i\varepsilon}{mR^2}\right) \int_0^\infty d\theta'_1 \frac{\sinh^{d-2}\tau'}{(\sinh\tau' \sinh\tau'')^2} \cdots \int_0^\pi d\theta'_4 \frac{\sin^3\theta'_4}{(\sin\theta'_4 \sin\theta''_4)^2}\, E(\theta_4)$$
$$\times \frac{1}{\sin^2\theta''_3}\left(1 + \frac{1}{\sin^2\theta''_2}\right) \int_{-\infty}^\infty dx\, e^{-\alpha(\sinh\tau' \cdots \sin\theta''_4)x^2}$$
$$\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{3/2} \int_0^\infty d\tau' \frac{\sinh^{d-2}\tau'}{(\sinh\tau' \sinh\tau'')^{3/2}} \cdots \int_0^\pi d\theta'_4 \frac{\sin^3\theta'_4}{(\sin\theta'_4 \sin\theta''_4)^{3/2}}\, E(\theta_4)$$
$$\times \left[1 - \frac{i\varepsilon}{8mR^2} \cdot \frac{1}{\sinh\tau' \cdots \sin\theta''_4} \cdot \left(4 + \frac{1}{\sin^2\theta''_3} + \frac{1}{\sin^2\theta''_3 \sin^2\theta''_2}\right)\right], \tag{10}$$

and so on up to the kth step,

$$B_0 \simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{k/2} \int_0^\infty d\tau' \frac{\sinh^{d-2}\tau'}{(\sinh\tau' \sinh\tau'')^{k/2}} \cdots \int_0^\pi d\theta'_{k+1} \frac{\sin^k\theta'_{k+1}}{(\sin\theta'_{k+1} \sin\theta''_{k+1})^{k/2}}\, E(\theta_{k+1})$$
$$\times \left\{1 - \frac{i\varepsilon}{8mR^2} \cdot \frac{1}{\sinh\tau' \cdots \sin\theta''_{k+1}}\left[(k-1)^2 + \frac{1}{\sin^2\theta''_k} + \frac{1}{\sin^2\theta''_k \cdots \sin^2\theta''_2}\right]\right\}, \tag{11}$$

[7] The linear term will become important in the calculation of the other integrals, e.g., in B_{θ_ν}, where it generates the term proportional to $\cot\theta_\nu$.

and finally after $d-1$ steps,

$$B_0 \simeq \left(\frac{2\pi i\varepsilon}{8mR^2}\right)^{(d-1)/2} \left\{1 + \frac{i\varepsilon}{8mR^2}\left[(d-2)^2 - \frac{1}{\sinh^2\tau''} - \cdots - \frac{1}{\sinh^2\tau'' \cdots \sin^2\theta_2''}\right]\right\}, \tag{12}$$

which gives in the required approximation the result quoted in (5). Note that in the last step several minus signs appear, which are due to the hyperbolics of τ. As a simple example, consider the case $d=3$, then

$$\begin{aligned} B_0^{d=3} &\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{1/2} \int_0^\infty d\tau' \sqrt{\frac{\sinh\tau'}{\sinh\tau''}}\, e^{-\alpha(\tau''-\tau')^2} \\ &\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{1/2} \int_{-\infty}^\infty dx \left[1 + \left(1 - \frac{1}{\sinh^2\tau''}\right)\frac{x^2}{8}\right] e^{-\alpha x^2} \\ &= \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{1/2} \left[1 + \frac{i\varepsilon}{8mR^2}\left(1 - \frac{1}{\sinh^2\tau''}\right)\right]. \end{aligned} \tag{13}$$

Substituting the expressions (5) in the Taylor expansion (4), one obtains in the limit $\varepsilon \to 0$ the correct Schrödinger equation (3).

APPENDIX D: Derivation of the Schrödinger Equation from Eq. (II.29)

We must consider the short-time kernel (see Eq. (II.30)):

$$K^{(d)}(l^{(j,j-1)}; \varepsilon) = \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2} e^{(mR^2/i\varepsilon)(1-\cosh l^{(j,j-1)}) - (i\varepsilon/8mR^2)(d-1)(d-3)}. \tag{1}$$

In order to derive the Schrödinger equation one must perform a Taylor expansion in the time evolution equation (C.2) once again, but now with the short-time kernel (1), yielding

$$\begin{aligned} &\psi(\{\theta''\}; t) + \varepsilon \frac{\partial\psi(\{\theta''\}; t)}{\partial t} \\ &\quad = \left(\frac{mR^2}{2\pi i\varepsilon}\right)^{(d-1)/2} e^{-(i\varepsilon/8mR^2)(d-1)(d-3)} \\ &\qquad \times \left\{\psi(\{\theta''\}; t)\, B_0 + \sum_{\nu=1}^{d-1} \frac{\partial\psi(\{\theta''\}; t)}{\partial\theta''_\nu}(B_{\theta_\nu} - \theta''_\nu B_0) \right. \\ &\qquad \left. + \frac{1}{2} \sum_{\substack{\mu,\nu=1 \\ \mu \geqslant \nu}}^{d-1} \frac{\partial^2\psi(\{\theta''\}; t)}{\partial\theta''_\mu\, \partial\theta''_\nu}(B_{\theta_\mu\theta_\nu} - \theta''_\mu B_{\theta_\nu} - \theta''_\nu B_{\theta_\mu} + \theta''_\mu\theta''_\nu B_0)\right\}. \end{aligned} \tag{2}$$

The abbreviations in (2) are the same as those in (C.5) except for B_0 which reads

$$B_0 = 2\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-2)/2} e^{mR^2/i\varepsilon} K_{(d-2)/2}\left(\frac{mR^2}{i\varepsilon}\right)$$
$$\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-1)/2} \exp\left[\frac{i\varepsilon}{8mR^2}(d-1)(d-3)\right]. \tag{3}$$

We shall illustrate how to calculate the integrals B_0 and B_τ in Eq. (2). All the others are calculated in a manner similar to that of B_τ,

$$B_0 = \int_{\Lambda^{d-1}} du'\, e^{(mR^2/i\varepsilon)(1-\cosh l^{(\cdot,\cdot)})}$$
$$= e^{mR^2/i\varepsilon}\int_0^\infty \sinh^{d-2}\tau'\, d\tau'\, e^{-(mR^2/i\varepsilon)\cosh\tau'\cosh\tau''}\left(\prod_{\nu=2}^{d-2}\int_0^\pi \sin^{\nu-1}\theta'_\nu\, d\theta'\right)\int_0^{2\pi} d\phi'$$
$$\times \exp\left(-\frac{imR^2}{\varepsilon}\sinh\tau^{(1)}\sinh\tau^{(2)}\right.$$
$$\left.\times\left[\sum_{m=2}^{d-2}\cos\theta^{(1)}_{m+1}\cos\theta^{(2)}_{m+1}\prod_{n=2}^{m}\sin\theta^{(1)}_n\sin\theta^{(2)}_n + \prod_{n=2}^{d-1}\sin\theta^{(1)}_n\sin\theta^{(2)}_n\right]\right)$$
$$= 2\pi\left(\frac{2\pi\varepsilon}{mR^2\sinh\tau''}\right)^{(d-3)/2} e^{mR^2/i\varepsilon}$$
$$\times\int_1^\infty (z^2-1)^{(d-3)/4}\, e^{-(mR^2/i\varepsilon)(\cosh\tau''\cdot z}\, J_{(d-3)/2}\left(\frac{mR^2}{\varepsilon}\sinh\tau''\cdot\sqrt{z^2-1}\right), \tag{4}$$

where we have used in the last step the integral

$$\int_0^\pi d\theta'_{d-2}\sin^{d-2}\theta'_{d-2}\cdots\int_0^\pi d\theta'_2\sin\theta'_2\int_0^{2\pi} d\phi'$$
$$\times\exp\left(-iz\left[\cos\theta^{(1)}_{d-2}\cos\theta^{(2)}_{d-2} + \sum_{m=1}^{d-3}\cos\theta^{(1)}_m\cos\theta^{(2)}_m\right.\right.$$
$$\left.\left.\times\prod_{n=m+1}^{d-2}\sin\theta^{(1)}_n\sin\theta^{(2)}_n + \prod_{n-1}^{d-2}\sin\theta^{(2)}_n\sin\theta^{(2)}_n\right]\right)$$
$$= 2\pi\left(\frac{2\pi}{z}\right)^{(d-3)/2} J_{(d-3)/2}(z) \tag{5}$$

which we have calculated in [18]. To perform the integral in (4) we use [20, p. 721]

$$\int_1^\infty (x^2-1)^{\nu/2}\, e^{-\alpha x} J_\nu(\beta\sqrt{x^2-1})\, dx$$
$$= \sqrt{\frac{2}{\pi}}\,\beta^\nu(\alpha^2+\beta^2)^{-(1/2)(\nu+1/2)} K_{\nu+1/2}(\sqrt{\alpha^2+\beta^2}) \tag{6}$$

and the asymptotic expansion (II.40) of the K_ν-Bessel functions to get

$$\begin{aligned} B_0 &= 2\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-2)/2} e^{mR^2/i\varepsilon} K_{(d-2)/2}\left(\frac{mR^2}{i\varepsilon}\right) \\ &\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-1)/2} e^{(i\varepsilon/8mR^2)(d-1)(d-3)}. \end{aligned} \tag{7}$$

This proves B_0. In order to calculate B_τ we consider

$$\begin{aligned} \tilde{B}_\tau &= \int_{\Lambda^{d-1}} du'(\tau''-\tau')\, e^{(mR^2/i\varepsilon)(1-\cosh l(',''))} \\ &= 2\pi\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-3)/2} \exp\left(\frac{mR^2}{i\varepsilon}\right) \sin^{(3-d)/2}\tau'' \\ &\quad\times \int_0^\infty d\tau'(\tau''-\tau') \sinh^{(d-1)/2}\tau' \exp\left(-\frac{mR^2}{i\varepsilon}\cosh\tau'\cosh\tau''\right) \\ &\quad\times I_{(d-3)/2}\left(\frac{mR^2}{i\varepsilon}\sinh\tau'\sinh\tau''\right) \\ &\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-2)/2} \sinh^{(3-d)/2}\tau'' \exp\left[\frac{mR^2}{i\varepsilon} - i\varepsilon\frac{(d-3)^2-1}{8mR^2\sinh^2\tau''}\right] \\ &\quad\times \int_{-\infty}^{\infty} d\tau'(\tau''-\tau') \sinh^{(d-2)/2}\tau' \exp\left[-\frac{mR^2}{2i\varepsilon}(\tau''-\tau')^2\right] \\ &\simeq \left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-2)/2} \exp\left(\frac{mR^2}{i\varepsilon}\right) \int_{-\infty}^{\infty} dx\, x\left(1+\frac{d-2}{2}\coth\tau'' x\right) e^{-(mR^2/2i\varepsilon)x^2} \\ &= \frac{i\varepsilon}{mR^2}\left(\frac{2\pi i\varepsilon}{mR^2}\right)^{(d-2)/2} \frac{d-2}{2}\coth\tau'', \end{aligned} \tag{8}$$

where we have used the asymptotic expansion of the modified Bessel function in the limit $\varepsilon \to 0$ (after having performed a Wick rotation) and neglect all terms which are of higher order in ε. With $B_\tau = \tau'' B_0 + \tilde{B}_\tau$ Eq. (C.5d) for $\nu = d-1$ is proven. Substituting the expression B_0 as well as the other expressions of (B.5) in the Taylor expansion (2), one obtains the Schrödinger equation (II.1).

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