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Optimized perturbation theory applied to jet cross sections in e^+e^- annihilation

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Abstract. The optimized perturbation theory proposed by Stevenson to deal with coupling constant scheme dependence is applied to the calculation of σ_{tot} and jet multiplicities in e^+e^- annihilation. The results are compared with those of simple perturbation theory and with recent experimental cluster multiplicities.

1 Introduction

Physical quantities, for example cross sections, calculated in QCD are independent of the particular scale μ used to renormalize the theory. However, this is true only for the infinite perturbative series. Any expansion truncated at some finite order (in practice at the second order) does depend on the renormalization scale μ. Then the question arises what scale should be used in these truncated series. This is a well-known problem, which cannot be solved in mathematical terms, since perturbation theory at some finite order does not specify, which is the right scale to produce the best approximation to the complete series. Some years ago, Stevenson proposed the following solution to this dilemma: Since the true result is completely independent of the scale μ, the best approximation is the one which is least sensitive to small changes in μ [1]. This leads to the requirement that for the best scale the nth order (in practice the second order) result $\sigma^{(n)}$ for a cross section obeys

$$\frac{d\sigma^{(n)}}{d\ln\mu^2}=0. \tag{1.1}$$

This equation yields the optimal scale $\mu=\mu_{\text{opt}}$, which, if introduced in the nth order result, gives the optimized prediction for $\sigma^{(n)}$. That (1.1) is the best approximation to the true result, is of course impossible to prove. But Stevenson has tested his procedure with examples and found it successful.

This principle of minimal sensitivity or scale optimization procedure has been applied recently with sensible results to massive lepton pair production [2]* and to processes involving photons in hard collisions [3]*. So far it has not been applied to e^+e^- annihilation cross sections, although second order results are available for several of these cross sections. In this paper we want to fill this gap and wish to see, whether optimized perturbation theory yields reasonable results. We use our recently calculated $O(\alpha_s^2)$ cross sections for the production of 2 and 3 jets as a function of the resolution cut y [4], to obtain the perturbative 2- and 3-jet multiplicities up to $O(\alpha_s^2)$ by dividing by σ_{tot} whose expansion up to $O(\alpha_s^2)$ has been known for some time [5]. We apply the optimization procedure to the three physical quantities σ_{tot}, $\sigma_{2\text{-jet}}(y)/\sigma_{\text{tot}}$ and $\sigma_{3\text{-jet}}(y)/\sigma_{\text{tot}}$, with y values ranging between 0.01 and 0.05. $\sigma_{4\text{-jet}}(y)/\sigma_{\text{tot}}$ is obtained from the rule that 2-, 3- and 4-jet multiplicity sum up to 1. In principle the predictions for the jet multiplicities could be compared to experimental data and the value of the QCD scale $\Lambda_{\overline{\text{MS}}}$ could be derived. However, these experimental jet multiplicities are not available yet. The JADE-collaboration at PETRA has published cluster multiplicities as a function of y [6]. The cluster multiplicities still contain fragmentation effects. We expect these fragmentation corrections to be moderate. Therefore we feel free to compare our results with the cluster data, to see whether they are approximately consistent with our results. In particular it is of interest whether the 4-jet rate as a result of the optimization procedure is higher than the lowest order prediction based on the coupling α_s with scale q^2. In [6] it was found that the

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* In these processes one has the additional problem that the finite order predictions also depend on the factorization scale.

4-jet cluster rate is much larger than the prediction based on lowest order perturbation theory including fragmentation effects.

In Sect. 2 we present the results of the perturbative expansion up to $O(\alpha_s^2)$ for σ_{tot}, $\sigma_{\text{2-jet}}(y)$, $\sigma_{\text{3-jet}}(y)$ and $\sigma_{\text{4-jet}}(y)$ and collect the numerical second order coefficients of $\sigma_{\text{2-jet}}(y)$, $\sigma_{\text{3-jet}}(y)$ and $\sigma_{\text{4-jet}}(y)$ for various y values in several tables. This is based on our earlier work [4]. Section 3 contains a short review of the optimization procedure. Here we derive the equations from which the optimized scale is determined. In Sect. 4 we present our results and conclusions.

2 Cross sections up to $O(\alpha_s^2)$

In this section we collect the information on the perturbative calculation of σ_{tot}, $\sigma_{\text{2-jet}}$, $\sigma_{\text{3-jet}}$ and $\sigma_{\text{4-jet}}$ in the $\overline{\text{MS}}$ renormalization scheme and with $\mu^2 = q^2$ (q^2 is the total c.m. energy squared) as the scale in α_s.

The total inclusive e^+e^- annihilation cross section σ_{tot} up to $O(\alpha_s^2)$ has been calculated from the imaginary part of the vacuum polarization by several groups already some time ago. In the $\overline{\text{MS}}$ renormalization scheme the result is [5]

$$\sigma_{\text{tot}} = \sigma_0\{1 + \tfrac{3}{2}C_F\lambda(q^2) + C_F[-\tfrac{3}{8}C_F + (\tfrac{123}{8} - 11\zeta_3)N_c + (4\zeta_3 - \tfrac{11}{2})T_R]\lambda^2(q^2)\}. \tag{2.1}$$

Here and also for the other cross sections we write the second order term as a sum of the three colour factors C_F^2, $C_F N_c$ and $C_F T_R$ ($C_F = \frac{4}{3}$, $N_c = 3$, $T_R = N_f/2$, N_f = number of flavours). ζ_i are the usual zeta functions. σ_0 is the zero-order cross section for the production of five flavours. We have introduced $\lambda = \alpha_s/2\pi$. In the $\overline{\text{MS}}$ scheme with scale q^2 the second order term proportional to λ^2 is small, the coefficient is equal to 5.66 for five flavours. λ is typically of order 0.02 at the highest PETRA energies, so that the second order term is a very small correction.

In our earlier work we have calculated the cross sections for the production of 2, 3 and 4 jets in e^+e^- annihilation up to α_s^2 [4]. These cross sections depend on a resolution parameter which was chosen to be the scaled invariant mass $y_{ij} = (p_i + p_j)^2/q^2$. These resolution parameters are needed to define infrared finite jet cross sections in perturbative QCD. In our definition two partons i and j were considered to be irresolvable if $y_{ij} \leqq y$ with some fixed chosen resolution parameter y. So, for example, if in the four-parton cross section all possible $y_{ij} \geqq y$ the contribution is considered to be 4-jet. Similarly one defines 2- and 3-jet cross sections. For more details we refer to [4]. Unfortunately the theoretical predictions for the jet cross section are not unique. Of course, $\sigma_{\text{4-jet}}$, which has been calculated in lowest order proportional to α_s^2, is given uniquely. But $\sigma_{\text{3-jet}}$ and $\sigma_{\text{2-jet}}$, since $\sigma_{\text{2-jet}} = \sigma_{\text{tot}} - \sigma_{\text{3-jet}} - \sigma_{\text{4-jet}}$, depend how the 3-jet variables are defined in terms of the original four-parton variables. In our previous work we studied two schemes, the KL and the KL' scheme. Here we shall make use only of the results in the KL' scheme, where the higher order corrections to $\sigma_{\text{2-jet}}$ and $\sigma_{\text{3-jet}}$ are more moderate and therefore lead to a more consistent result if combined with experimental data on σ_{tot}. The KL' scheme is characterized by the fact that in the configuration $e^+e^- \to q(1)\bar{q}(2)g(3)g(4)$ with gluon 3 considered soft or collinear with the quark the 3-jet variables are "y_{13}" $= y_{134}$, "y_{23}" $= y_{24} - y_{13}$ and "y_{12}" $= y_{123}$, where $y_{ijk} = (p_i + p_j + p_k)^2/q^2$. The choice of "$y_{23}$" in this form has bearing on the separation of 2 jets from 3 jets and 4 jets [4, 7]. Our numerical results for $\sigma_{\text{2-jet}}(y)$, $\sigma_{\text{3-jet}}(y)$ and $\sigma_{\text{4-jet}}(y)$ are represented in the form

$$\sigma_{\text{2-jet}}(y)/\sigma_0 = 1 + C_F Z_1^{(2)}\lambda(q^2) + C_F(C_F Z_C^{(2)} + N_c Z_N^{(2)} + T_R Z_T^{(2)})\lambda^2(q^2) \tag{2.2}$$

$$\sigma_{\text{3-jet}}(y)/\sigma_0 = C_F Z_1^{(3)}\lambda(q^2) + C_F(C_F Z_C^{(3)} + N_c Z_N^{(3)} + T_R Z_T^{(3)})\lambda^2(q^2) \tag{2.3}$$

Table 1. $O(\lambda^2)$ coefficients for 2-, 3- and 4-jet cross sections as defined in (2.2), (2.3) and (2.4) for $y = 0.05$

	$Z_C^{(i)}$	$Z_N^{(i)}$	$Z_T^{(i)}$
$i=2$	34.71	−102.13	30.03
$i=3$	−41.44	103.94	−30.98
$i=4$	6.36	0.34	0.26

Table 2. Same as Table 1 for $y = 0.04$

	$Z_C^{(i)}$	$Z_N^{(i)}$	$Z_T^{(i)}$
$i=2$	53.96	−131.61	38.87
$i=3$	−66.60	133.15	−40.04
$i=4$	12.26	0.62	0.48

Table 3. Same as Table 1 for $y = 0.03$

	$Z_C^{(i)}$	$Z_N^{(i)}$	$Z_T^{(i)}$
$i=2$	88.9	−175.0	51.5
$i=3$	−112.2	175.8	−53.2
$i=4$	22.9	1.2	1.0

Table 4. Same as Table 1 for $y = 0.02$

	$Z_C^{(i)}$	$Z_N^{(i)}$	$Z_T^{(i)}$
$i=2$	161.05	−253.81	75.85
$i=3$	−217.23	253.21	−78.47
$i=4$	55.80	2.75	1.93

Table 5. Same as Table 1 for $y = 0.01$

	$Z_C^{(i)}$	$Z_N^{(i)}$	$Z_T^{(i)}$
$i=2$	377.39	−427.17	129.39
$i=3$	−538.25	421.22	−134.92
$i=4$	160.49	8.10	4.83

$\sigma_{4\text{-jet}}(y)/\sigma_0$
$$= C_F(C_F Z_C{}^{(4)} + N_c Z_N{}^{(4)} + T_R Z_T{}^{(4)})\lambda^2(q^2). \quad (2.4)$$

Complete formulas for $Z_1{}^{(2)}$ and $Z_1{}^{(3)}$ are found in [4]. The higher order coefficients $Z_C{}^{(i)}$, $Z_N{}^{(i)}$ and $Z_T{}^{(i)}$ are tabulated in Tables 1–5 for $y = 0.05, 0.04, 0.03, 0.02$ and 0.01 and were taken from our work [4].

Actually the calculations in [4] were done only for $y = 0.05$, 0.04 and 0.02. Therefore the values in table 3 had to be obtained by interpolation. The values for the $Z_C{}^{(i)}$, $Z_N{}^{(i)}$ and $Z_T{}^{(i)}$ $(i = 2, 3, 4)$ fulfil the relations $Z_C{}^{(2)} + Z_C{}^{(3)} + Z_C{}^{(4)} = -\frac{3}{8}$, $Z_N{}^{(2)} + Z_N{}^{(3)} + Z_N{}^{(4)} = \frac{123}{8} - 11\zeta_3$ and $Z_T{}^{(2)} + Z_T{}^{(3)} + Z_T{}^{(4)} = 4\zeta_3 - \frac{11}{2}$, i.e. the contributions of 2, 3 and 4 jets sum up to the higher order correction term in σ_{tot}.

We observe from Tables 1–4 that the λ^2 correction term for $\sigma_{2\text{-jet}}$ is negative and for $\sigma_{3\text{-jet}}$ is positive. Both increase in absolute value with decreasing y. Then there exists a small y for which $\sigma_{2\text{-jet}}$ reaches the unitarity limit zero and $\sigma_{3\text{-jet}}$ reaches the unitarity limit σ_{tot}, respectively, the actual value depends for fixed y on the value of λ. Therefore it is clear that we can apply the perturbative results only down to some finite y value which is near 0.02. The coefficient of $\sigma_{4\text{-jet}}$ is rather small compared to the coefficients in $\sigma_{2\text{-jet}}$ and $\sigma_{3\text{-jet}}$ and it increases only moderately with decreasing y. $\sigma_{4\text{-jet}}$ is determined approximately by $Z_C{}^{(4)}$, the contributions of $Z_N{}^{(4)}$ and $Z_T{}^{(4)}$ are small. Therefore in an abelean theory the y dependence of $\sigma_{4\text{-jet}}$ remains unchanged as compared to QCD, whereas the behaviour of the λ^2 coefficients in $\sigma_{2\text{-jet}}$ and $\sigma_{3\text{-jet}}$ are completely different. In the abelean theory the coefficient in $\sigma_{2\text{-jet}}$ is positive $(Z_N{}^{(2)} = 0)$ and increases with decreasing y, whereas the coefficient in $\sigma_{3\text{-jet}}$ is negative and decreases with decreasing y.

3 Review of the optimization procedure

The prototype problem to explain Stevenson's optimization procedure [1] is the total annihilation cross section $\sigma_{\text{tot}}(q^2)$ at total c.m. energy $q = \sqrt{q^2}$ in QCD with massless quarks. We write this cross section in terms of $R = \sigma_{\text{tot}}(q^2)/\sigma_0$ where σ_0 is the zero-order annihilation cross section. In QCD perturbation theory R has the following expansion in λ

$$R = 1 + \Delta R \quad (3.1)$$

$$\Delta R = \lambda(r_1 + r_2\lambda + r_3\lambda^2 + \cdots) \quad (3.2)$$

where λ is $\lambda(\mu^2)$, i.e. the coupling in some renormalization scheme is renormalized at scale μ. The expansion coefficients $r_i (i = 2, \ldots)$ depend on μ and q and are also renormalization scheme (RS) dependent. $r_1 = \frac{3}{2}C_F$ is a constant. The μ dependence of λ satisfies the well-known differential equation ($\partial := \partial/\partial \ln \mu^2$)

$$\partial\lambda = -b_0\lambda^2(1 + b_1\lambda + b_2\lambda^2 + \cdots) \quad (3.3)$$

in which b_0 and b_1 are scheme independent while the b_2, b_3 etc. are not. The constants b_0 and b_1 are

$$b_0 = \tfrac{11}{6}N_c - \tfrac{2}{3}T_R \quad (3.4)$$

$$b_1 = (\tfrac{17}{3}N_c^2 - \tfrac{10}{3}N_c T_R - 2C_F T_R)/2b_0. \quad (3.5)$$

Since we intend to consider all possible schemes by varying μ it is not useful to present results in terms of the coupling λ. Instead we introduce the scale Λ, the theory's one free physical parameter by

$$\lambda(\mu^2) = \frac{1}{b_0 \ln\dfrac{\mu^2}{\Lambda^2} + b_1 \ln\ln\dfrac{\mu^2}{\Lambda^2}} \quad (3.6)$$

Λ will be held fixed when varying over all possible schemes. Equation (3.6) is a solution of (3.3) up to terms $O(\lambda^3)$ with the usual boundary conditions. The definition of Λ through (3.6) is RS-dependent, since λ and the right-hand side is RS dependent through b_2, b_3 etc. However, Λ's in different RS's can be related exactly by a one-loop calculation as shown by Celmaster and Gonsalves [8].

Since the expansion of ΔR has been calculated only up to terms $O(\lambda^2)$ [5] we consider the optimization only for

$$\Delta R_2 = r_1\lambda + r_2\lambda^2 \quad (3.7)$$

and require the μ optimization condition on ΔR_2 to be satisfied exactly

$$\partial \Delta R_2|_{\mu=\mu_{\text{opt}}} = 0. \quad (3.8)$$

This gives a transcendental equation for the optimum μ, denoted by μ_{opt}. The value of μ_{opt} depends on q, Λ and the scheme originally used to calculate r_2. The value of R_2 at μ_{opt} for a given q^2 and Λ is scheme independent.

r_2 has been calculated for the $\overline{\text{MS}}$ renormalization scheme at the scale q^2. The explicit value was given in the last section in (2.1). Let us denote it by $\bar{r}_2$. Then r_2 at the scale μ^2 is related to $\bar{r}_2$ by

$$r_2 = \bar{r}_2 + b_0 \ln\left(\frac{\mu^2}{q^2}\right) r_1 \quad (3.9)$$

and we have the following equation for ΔR_2 at scale μ^2 in terms of r_1 and $\bar{r}_2$

$$\Delta R_2 = r_1\left(\lambda + b_0 \ln\left(\frac{\mu^2}{q^2}\right)\lambda^2\right) + \bar{r}_2\lambda^2. \quad (3.10)$$

This form for ΔR_2 serves for determining μ_{opt}^2. Indeed from

$$\partial\left(r_1\left(\lambda + b_0 \ln\left(\frac{\mu^2}{q^2}\right)\lambda^2\right) + \bar{r}_2\lambda^2\right)\Bigg|_{\mu=\mu_{\text{opt}}} = 0 \quad (3.11)$$

and

$$\partial\lambda = -b_0\lambda^2(1 + b_1\lambda) \quad (3.12)$$

which is the truncation of (3.3) to two-loop terms, we obtain

$$\left(b_1 + 2(1+b_1\lambda_{\text{opt}})b_0 \ln\left(\frac{\mu_{\text{opt}}^2}{q^2}\right)\right)r_1 + 2(1+b_1\lambda_{\text{opt}})\bar{r}_2 = 0 \tag{3.13}$$

with

$$\lambda_{\text{opt}} = \left(b_0 \ln\left(\frac{\mu_{\text{opt}}^2}{\Lambda^2}\right) + b_1 \ln\ln\left(\frac{\mu_{\text{opt}}^2}{\Lambda^2}\right)\right)^{-1}. \tag{3.14}$$

From (3.13) with (3.14) μ_{opt}^2/q^2 is calculated numerically for a fixed Λ. The corresponding optimal ΔR_2 is

$$(\Delta R_2)_{\text{opt}} = r_1 \lambda_{\text{opt}} + r_{2\,\text{opt}} \lambda_{\text{opt}}^2 \tag{3.15}$$

$r_{2\,\text{opt}}$ follows from (3.13)

$$r_{2\,\text{opt}} = -\frac{b_1 r_1}{2(1+b_1\lambda_{\text{opt}})} \tag{3.16}$$

so that

$$(\Delta R_2)_{\text{opt}} = r_1 \lambda_{\text{opt}} \frac{1+b_1\lambda_{\text{opt}}/2}{1+b_1\lambda_{\text{opt}}}. \tag{3.17}$$

Equation (3.17) is the result for the optimal R_2 which can be calculated as soon as λ_{opt} is known from (3.14) and (3.13).

The optimization of $\sigma_{2\text{-jet}}/\sigma_{\text{tot}}$ and $\sigma_{3\text{-jet}}/\sigma_{\text{tot}}$ is performed in an analogous way. We start from

$$\sigma_{2\text{-jet}}/\sigma_0 = 1 + \bar{r}_{SW}\lambda(q^2) + \bar{r}_{KL}\lambda^2(q^2) \tag{3.18}$$

where $\bar{r}_{SW}$ and $\bar{r}_{KL}$ are given in Sect. 2. They are the expansion coefficients of the 2-jet cross section with scale q^2 in the $\overline{\text{MS}}$ scheme. From this we calculate the expansion terms for $\sigma_{2\text{-jet}}/\sigma_{\text{tot}}$ by dividing (3.18) by R in (3.1). Then

$$\sigma_{2\text{-jet}}/\sigma_{\text{tot}} = 1 + (\bar{r}_{SW} - r_1)\lambda(q^2) + (\bar{r}_{KL} - r_1(\bar{r}_{SW} - r_1) - \bar{r}_2)\lambda^2(q^2) \tag{3.19}$$

so that $\sigma_{2\text{-jet}}/\sigma_{\text{tot}} - 1$ has a similar expansion as ΔR_2. Therefore the equation for determining μ_{opt}^2/q^2 is the same as (3.13) if we substitute $r_1 \to \bar{r}_{SW} - r_1$ and $\bar{r}_2 \to \bar{r}_{KL} - r_1(\bar{r}_{SW} - r_1) - \bar{r}_2$. The optimized value for $\sigma_{2\text{-jet}}/\sigma_{\text{tot}} - 1$ follows from (3.17) with the replacement $r_1 \to \bar{r}_{SW} - r_1$.

The expansion of $\sigma_{3\text{-jet}}/\sigma_0$ starts with a term $O(\lambda)$. In the $\overline{\text{MS}}$-Scheme and with scale q^2 we have

$$\sigma_{3\text{-jet}}/\sigma_0 = \bar{s}_{SW}\lambda(q^2) + \bar{s}_{KL}\lambda^2(q^2) \tag{3.20}$$

and

$$\sigma_{3\text{-jet}}/\sigma_{\text{tot}} = \bar{s}_{SW}\lambda(q^2) + (\bar{s}_{KL} - r_1\bar{s}_{SW})\lambda^2(q^2). \tag{3.21}$$

Then μ_{opt}^2/q^2 for the 3-jet multiplicity follows from (3.13) if $r_1 \to \bar{s}_{SW}$ and $\bar{r}_2 \to \bar{s}_{KL} - r_1\bar{s}_{SW}$ and the optimized 3-jet multiplicity is calculated from (3.17) with r_1 replaced by $\bar{s}_{SW}$. We have $\bar{s}_{SW} = -(\bar{r}_{SW} - r_1)$ as it should be because $(\sigma_{3\text{-jet}} + \sigma_{2\text{-jet}})/\sigma_{\text{tot}} = 1$ in $O(\lambda)$.

In our earlier work [4] we found that $\bar{r}_{KL}$ contains the term $2C_F b_0 \ln^3 y$. This term proportional to $\ln^3 y$ has a rather large numerical coefficient and is in leading order of $\ln y$ equal to $(-b_0 \ln y \bar{r}_{SW})$. This term can easily be absorbed into $\lambda(q^2)$ by changing the scale of λ into yq^2 since up to $O(\lambda^2)$ we have

$$\lambda(q^2) = \lambda(yq^2)(1 + b_0 \ln y\, \lambda(yq^2)). \tag{3.22}$$

Then instead of (3.19) we get for the 2-jet multiplicity:

$$\sigma_{2\text{-jet}}/\sigma_{\text{tot}} - 1 = (\bar{r}_{SW} - r_1)\lambda(yq^2) + (\bar{r}_{KL} - r_1(\bar{r}_{SW} - r_1) - \bar{r}_2 + (\bar{r}_{SW} - r_1)b_0 \ln y)\lambda^2(yq^2). \tag{3.23}$$

The same term with opposite sign appears in the $O(\lambda^2)$ term $\bar{s}_{KL}$ of the 3-jet cross section, where it can be absorbed into λ as well. If we do this we have instead of (3.21):

$$\sigma_{3\text{-jet}}/\sigma_{\text{tot}} = \bar{s}_{SW}\lambda(yq^2) + (\bar{s}_{KL} - r_1\bar{s}_{SW} + \bar{s}_{SW} b_0 \ln y)\lambda^2(yq^2). \tag{3.24}$$

The formula (3.7) for $\sigma_{\text{tot}}/\sigma_0 = R$, the excess of the total annihilation cross section over the lowest order point cross section (with $r_2 \to \bar{r}_2$) and the formulas (3.23) and (3.24) for the two- and three-jet multiplicities respectively are the formulas on which our optimization procedure as described in detail above for R is based. The results of the optimization will be described in the next section.

4 Results and conclusions

In this section we present the results of the optimization of σ_{tot}, $\sigma_{2\text{-jet}}/\sigma_{\text{tot}}$ and $\sigma_{3\text{-jet}}/\sigma_{\text{tot}}$. Then $\sigma_{4\text{-jet}}/\sigma_{\text{tot}}$ is calculated from

$$\sigma_{4\text{-jet}}/\sigma_{\text{tot}} = 1 - \sigma_{2\text{-jet}}/\sigma_{\text{tot}} - \sigma_{3\text{-jet}}/\sigma_{\text{tot}}. \tag{4.1}$$

All four quantities are compared to experimental data. The total annihilation cross section σ_{tot} has been measured by the PETRA and PEP experiments. The CELLO collaboration at PETRA has made a fit to all these data [9]. In the fitting procedure they took the correlations between measurements into account and determined the electroweak mixing angle $\sin^2\Theta_W$ and the strong coupling constant $\alpha_s(q^2)$ using the second order formula (2.1). The fit to the combined data yielded $\alpha_s((34\,\text{GeV})^2) = 0.145 \pm 0.019$ [9].* From this value of α_s we infer $\sigma_{\text{tot}}/\sigma_0 = 1.049 \pm 0.007$, which we take as the experimental value of σ_{tot} at $q^2 = (34\,\text{GeV})^2$ in our comparison.

The results for $\sigma_{n\text{-jet}}/\sigma_{\text{tot}}$ $(n = 2, 3, 4)$ are compared with the n-cluster event rates measured by the JADE collaboration at PETRA [6]. These cluster event rates for up to 5 clusters were obtained as a function of

* A more recent analysis which includes data from TRISTAN and data below $\sqrt{q^2} = 10\,\text{GeV}$ gives $\alpha_s((34\,\text{GeV})^2) = 0.145 \pm 0.019$ (W. de Boer, private communication). We use this new value instead of the published value $\alpha_s((34\,\text{GeV})^2) = 0.165 \pm 0.030$

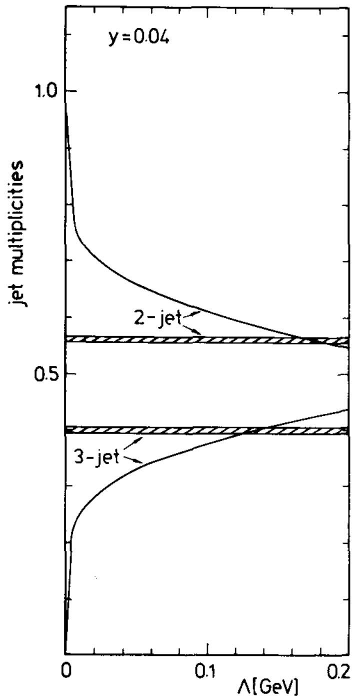

Fig. 1. Jet multiplicities as a function of Λ for $y=0.04$ with scale q^2 compared to cluster multiplicities of [6]

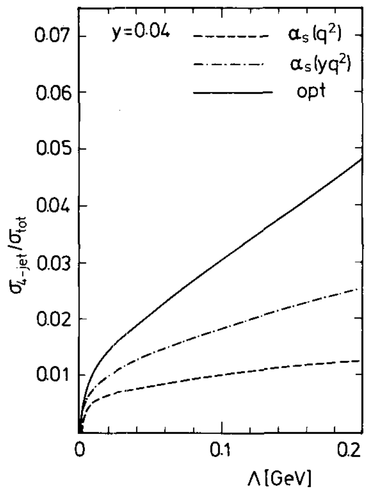

Fig. 2. 4-jet fraction as a function of Λ for $y=0.04$ for the three cases: scale q^2, scale yq^2 and optimized scale

invariant mass cut y between $y=0.015$ and $y=0.08$ for $q^2=(34\,\mathrm{GeV})^2$. Unfortunately these n-cluster multiplicities are not equal to the n-jet multiplicities as calculated from QCD perturbation theory. Due to fragmentation effects of quarks and gluons into hadrons not all events with n clusters with a fixed y cut originate from a perturbative n-jet production with the same y cut. The fragmentation produces fluctuations which might for example cause a primary 2-jet process to be classified as a 3-cluster event. To unfold these effects from the measured cluster event rates one must do calculations with fragmentation models on top of the perturbative QCD predictions which have not been done yet. In an earlier study of 3-jet production at $y=0.04$ it was found that these corrections are fairly small for the invariant mass method [10]. For 4-jet production and for 2- and 3-jet production at small y's we must expect larger corrections [6]. As long as these corrections are not known we shall not draw any conclusion concerning $\Lambda_{\overline{\mathrm{MS}}}$ from a comparison of optimized n-jet rates with the empirical n-cluster rates from JADE. But the comparison with the cluster rates will help us to see more clearly the change of the jet multiplicities due to optimization as compared to the non-optimized values.

In Fig. 1 we show the 2- and 3-jet multiplicities as a function of $\Lambda_{\overline{\mathrm{MS}}}$ for $y=0.04$ calculated directly from the expansion of $\sigma_{2\text{-jet}}/\sigma_{\mathrm{tot}}$ and $\sigma_{3\text{-jet}}/\sigma_{\mathrm{tot}}$ obtained from (2.1), (2.2) and (2.3) with $\lambda((34\,\mathrm{GeV})^2)$ due to (3.6) and compared to the 2- and 3-cluster multiplicities of [6]. Of course, the 2-jet multiplicity decreases and the 3-jet multiplicity increases as a function of Λ. The theoretical curves cross the empirical bands for 2- and 3-cluster rates at two different Λ's, 0.18 GeV and 0.14 GeV, which need not disturb us, since the cluster rates have corrections if compared to QCD jet rates. For later comparison we note that the Λ values to fit the cluster rates are around 0.15 GeV. The corresponding 4-jet rate, i.e. calculated from (2.4) with $\lambda(q^2)=\alpha_s(q^2)/2\pi$, $q^2=(34\,\mathrm{GeV})^2$, i.e. for non-optimized α_s can be seen in Fig. 2 as a function of Λ. Up to $\Lambda=0.2\,\mathrm{GeV}$ the 4-jet multiplicity is small, approximately 1%, and is roughly a factor 4 smaller than the measured 4-cluster rate at the same energy and the same y. This means that the 4-cluster rate is larger as one expects from lowest order QCD with α_s evaluated at scale q^2. Since we do not expect that the fragmentation corrections for 4 clusters are so large, as to cause a change of a factor of 4 compared to the 4-jet rate, we conclude that the 4-jet rate comes out too small in lowest order QCD and scale q^2 in α_s. This agrees with the conclusion of [6]. In [6] the cluster rates were calculated from a model based on perturbation theory up to $O(\alpha_s^2)$ and the hadronization of 2-, 3- and 4-jets built in. The scale of α_s was equal to q^2. α_s was determined in such a way that the 2- and 3-cluster rate was in agreement with the data.

Since the higher order coefficients in $\sigma_{2\text{-jet}}$ and $\sigma_{3\text{-jet}}$ (see (2.2) and (2.3) together with Tables 1–5) are large we expect appreciable changes in our predictions by changing the scale of α_s. As was mentioned in the last section the analytical calculations of the higher order terms of $\sigma_{2\text{-jet}}$ suggest to absorb large terms $\sim \ln^3 y$ into the coupling constant α_s. This brings us to the scale yq^2 instead of q^2. The results for jet multiplicities based on $\alpha_s(yq^2)$ are shown in Fig. 3 for $y=0.04$. Comparing these predictions with the results in Fig. 1 we notice some change. Now $\sigma_{2\text{-jet}}/\sigma_{\mathrm{tot}}$ decreases and $\sigma_{3\text{-jet}}/\sigma_{\mathrm{tot}}$ increases

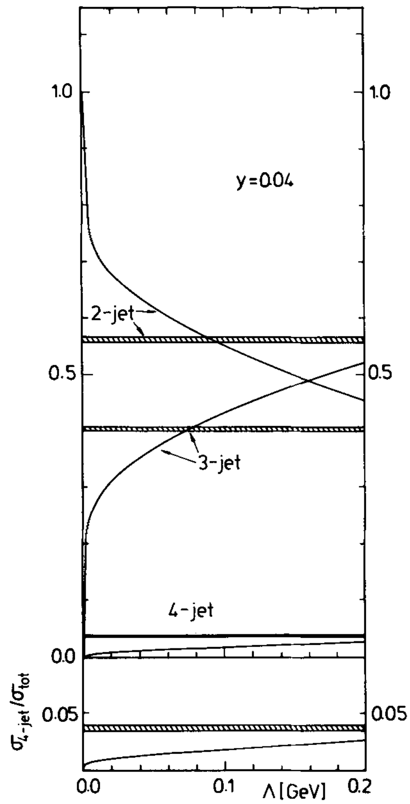

Fig. 3. Same as Fig. 1 with scale yq^2

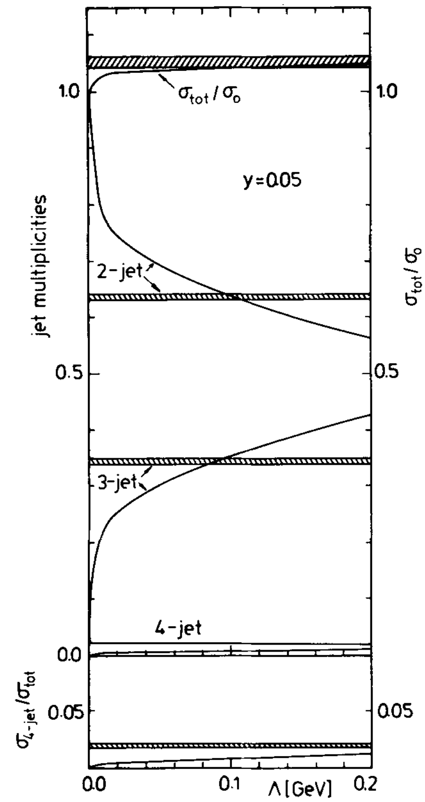

Fig. 4. Jet multiplicities as a function of Λ for $y=0.05$ with scale yq^2 and σ_{tot}/σ_0 as a function of Λ with scale q^2 compared to cluster multiplicities of [6] and σ_{tot} data from [9]

stronger with increasing Λ. They fit the experimental 2- and 3-cluster rates for $\Lambda \simeq 0.1$ GeV. $\sigma_{4\text{-jet}}/\sigma_{tot}$ (see Fig. 2) changes roughly by a factor of 2, since α_s is now evaluated at a much smaller scale. It is still smaller than the experimental 4-cluster rate for the Λ's of interest. Results for $y=0.05$ are exhibited in Fig. 4. The curves fit the experimental 2- and 3-cluster rates also for $\Lambda \simeq 0.1$ GeV and the 4-jet rate is still smaller than the 4-cluster rate. We also show σ_{tot}/σ_0 as a function of Λ together with the experimental data from the CELLO analysis [9]. σ_{tot} is calculated with $\alpha_s(q^2)$ from (2.1). The CELLO data require

$$\Lambda = \left(0.24 \pm {0.24 \atop 0.13}\right) \text{GeV}.$$

The same calculations for $y=0.03$ and 0.02 give similar results, except that the Λ values obtained from fitting 2- and 3-jets to the corresponding cluster rates are different which indicates either a breakdown of perturbation theory or different fragmentation corrections than for $y=0.05$ and $y=0.04$. For $y=0.01$ the higher order corrections are so large that $\sigma_{2\text{-jet}}$ becomes unphysical for $\Lambda \gtrsim 0.08$ GeV. One should note that the 2- and 3-jet rates are much more suitable to determine Λ since the variation with Λ is much larger than in the case of σ_{tot} and the 4-jet rate.

The results with optimization are shown in Figs. 5, 6 and 7. In Fig. 5 the optimized curve for σ_{tot}/σ_0 is very similar to the non-optimized result in Fig. 4. This is to be expected since the higher order coefficient in σ_{tot} is small so that the optimized scale is near the original scale. The scale comes out as $\mu_{opt}^2/q^2 = 0.3509$ for $\Lambda = 0.1$ GeV. For the other Λ's μ_{opt} is roughly the same. Furthermore in Fig. 5 we see the optimized curves for the 2-, 3- and 4-jet multiplicities with $y=0.05$, where the 4-jet multiplicity is calculated from (4.1). Since $\sigma_{4\text{-jet}}$ is available only in lowest order it cannot be optimized. We see that the theoretical curves fit the experimental cluster multiplicities for $\Lambda = 0.08$ GeV which is somewhat smaller than the Λ in Fig. 4. We also observe that $\sigma_{4\text{-jet}}$ is now even larger than with $\alpha_s(yq^2)$ in Fig. 4. It almost fits the 4-cluster rate for $\Lambda = 0.08$ GeV. The lower bound on the experimental σ_{tot}/σ_0 crosses the optimized curve approximately at the same Λ value. The values for μ_{opt} for 2- and 3-jet rates are collected in Table 6, always for $\Lambda = 0.1$ GeV. The results for $y=0.04$ are in Fig. 6. The conclusions are similar as for $y=0.05$. The Λ value which fits the cluster rates is a little smaller than for $y=0.05$. $\sigma_{4\text{-jet}}$ is increased compared to the result in Fig. 3. In Fig. 2 we have the comparison of $\sigma_{4\text{-jet}}$ with $y=0.04$ for the three cases (i) coupling $\alpha_s(q^2)$, (ii) coupling $\alpha_s(yq^2)$ and (iii) optimized coupling. In case (iii) the 4-jet rate is the largest. The values for μ_{opt} are again in Table 6. Finally in Fig. 7 the results for $y=0.03$ are shown. Here 2- and 3-jet rates cannot be fitted to the corresponding cluster rates with the

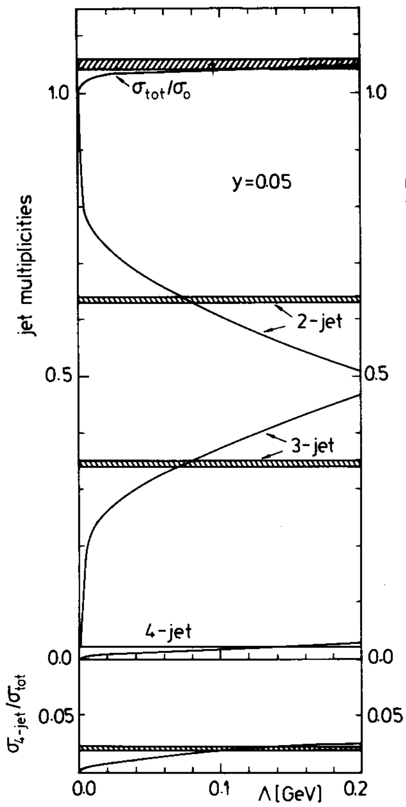

Fig. 5. Jet multiplicities for $y = 0.05$ and σ_{tot}/σ_0 as a function of Λ with optimized scale compared to cluster multiplicities from [6] and σ_{tot} data from [9]

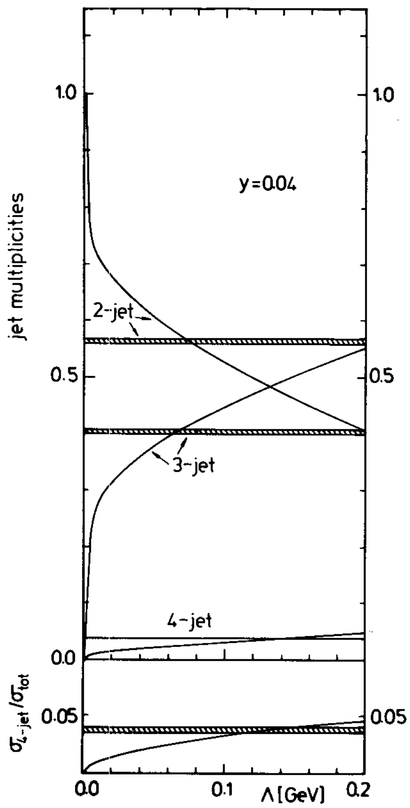

Fig. 6. Jet multiplicities for $y = 0.04$ as a function of Λ compared to cluster data from [6]

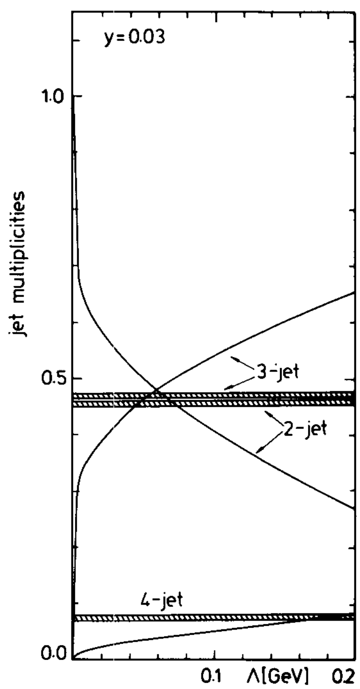

Fig. 7. Same as Fig. 6 for $y = 0.03$

same Λ and the 4-cluster rate is still larger than the 4-jet multiplicity. Whether this can be improved after correcting the cluster rates due to fragmentation will be seen in the future.

The optimization scales μ_{opt} for 2- and 3-jet multiplicities for $y = 0.05$ to 0.01 are all collected in Table 6. They are not equal for 2- and 3-jet cross sections, since they have appreciable different higher order corrections. Therefore they differ more for $y = 0.01$ than for $y = 0.05$. For $y = 0.02$ and $y = 0.01$ the optimization scale μ_{opt} is approximately equal to yq^2. Therefore the optimization does not change the 2-, 3- and 4-jet rates as compared to the prediction with coupling $\alpha_s(yq^2)$. The values of the optimization scale μ_{opt} for $q^2 = (34\,\text{GeV})^2$ and $y = 0.04$ is equal to 2.28 GeV for the 2-jet multiplicity and equal to 2.90 GeV for the 3-jet multiplicity. These values are still much larger than the confinement scale where perturbation theory definitely breaks down.

By comparing the results in Fig. 1 with those in Fig. 3 and Fig. 6 we get an overview about the effect of changing $\alpha_s(q^2)$ into $\alpha_s(yq^2)$ and into $\alpha_s(\mu^2_{opt})$. Whereas in Fig. 1 the 2- and 3-jet rates are equal for $\Lambda = 0.28\,\text{GeV}$ they cross in Fig. 3 for $\Lambda = 0.16\,\text{GeV}$ and in Fig. 6 for $\Lambda = 0.13\,\text{GeV}$. So the dependence on the scale changes appreciably through the optimization as compared to simple perturbation theory with coupling $\alpha_s(q^2)$. In a first approximation the curves in

Table 6. Optimization scales μ_{opt} for 2- and 3-jet cross sections for $y = 0.05$, 0.04, 0.03, 0.02 and 0.01

y	μ^2_{opt}/yq^2 2-jet	μ^2_{opt}/yq^2 3-jet
0.05	0.08250	0.1121
0.04	0.1121	0.1814
0.03	0.1866	0.3730
0.02	0.3618	1.254
0.01	1.706	17.59

Fig. 6 are similar to those in Fig. 3. Therefore a first step would be to use the scale yq^2 instead of q^2*. This improves already the 4-jet rate to a large extent. This procedure could also easily be incorporated into models based on perturbation theory up to $O(\alpha_s{}^2)$ augmented with hadronization of quarks and gluons. Of course, it would also be no problem to introduce one of the optimized scales from Table 6, i.e. either the 2-jet or the 3-jet scale μ_{opt} into these models with the effect that the 4-cluster rate might be described even better.

We emphasize that the 4-jet rate has not been optimized. This is not possible since it has been calculated only in lowest order. But we have optimized the scales of the 2-jet and the 3-jet multiplicity and have determined the 4-jet multiplicity from (4.1). According to (3.17) the procedure of optimization has the effect that in our case the 2- and 3-jet rate is replaced by an infinite series with coefficients given by powers of b_1, which is the second coefficient of the β-function. By calculating $\sigma_{4\text{-jet}}/\sigma_{tot}$ from (4.1) we derive it from a similar series with the only difference, that the coupling λ_{opt} is not determined from the higher order calculation of $\sigma_{4\text{-jet}}$, which is not available, but instead from higher order calculations of $\sigma_{2\text{-jet}}$ and $\sigma_{3\text{-jet}}$. We expect that this way we should come near to the result of what an optimization of $\sigma_{4\text{-jet}}$ would give. Another way to justify that we get a better result for $\sigma_{4\text{-jet}}$ than in lowest order, is to say the following. The optimization has quite generally the effect that higher order contributions are reduced by changing the coupling constant. μ_{opt} gives the scale where this happens in the most reasonable way. If the scale determined in $\sigma_{2\text{-jet}}$ and $\sigma_{3\text{-jet}}$ reduces the higher order terms in $\sigma_{4\text{-jet}}$ by the same amount as in $\sigma_{2\text{-jet}}$ and $\sigma_{3\text{-jet}}$ we can approximately neglect the higher order terms in $\sigma_{4\text{-jet}}$. But then we must evaluate $\sigma_{4\text{-jet}}$ at a scale as determined in $\sigma_{2\text{-jet}}$ and $\sigma_{3\text{-jet}}$.

In conclusion we state that the optimization of scale yields different predictions for 2- and 3-jet multiplicities as a function of the mass cut y as compared to simple perturbation theory with scale q^2. This will give a different $\Lambda_{\overline{MS}}$ parameter than the usual perturbation prediction with scale q^2 if a comparison with experimental jet multiplicities becomes available. The 4-jet multiplicity determined from the fact that the sum of all jet rates is equal to one comes out much larger than from lowest order perturbation theory in $\alpha_s(q^2)$ and $\Lambda_{\overline{MS}}$ determined from 3 jets.

* To introduce the scale yq^2 instead of q^2 to absorb large terms also in differential 3-jet cross sections was emphasized already in [11]

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