A CHIRAL $SU(2)_L \otimes SU(2)_R$ GAUGE MODEL ON THE LATTICE

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An explicitly chiral symmetric $SU(2)_L \otimes SU(2)_R$ model with a scalar doublet and a mirror pair of fermion doublets is investigated in lattice perturbation theory.

The basic matter fields in the standard $SU(3)\otimes SU(2)\otimes U(1)$ theory of elementary particles are chiral quarks and leptons. The masses of these (and other) fields are produced by spontaneous symmetry breaking via the Higgs mechanism. A non-perturbative investigation of the Higgs mechanism would obviously be important, but the inclusion of chiral fermions in a non-perturbative lattice regularization scheme is notoriously difficult (for an incomplete list of recent attempts see ref. [1]).

A simple manifestation of the chiral fermion problem is species doubling: putting the first-order Dirac equation on the lattice by a naive transcription of the derivative in lattice differences the propagator has 15 superfluous poles at the corners of the Brillouin zone. The general reason behind this is the cancellation of triangle anomalies for finite lattice spacings [2]. Moreover, a no-go theorem can also be proven [3], implying that under some mild assumptions, for finite lattice spacings, there always has to be an equal number of left- and right-handed particles. In the continuum limit the superfluous lattice fermion states can be removed from the spectrum following the prescription given by Wilson [4]: after the introduction of an appropriate higher-dimensional term in the action the masses of the additional fermions become proportional to the cutoff. The Wilson term in the lattice fermion action, however, breaks chiral symmetry explicitly. Nonetheless, the correct current algebra and chiral Ward–Takahashi identities seem to be reproduced in the continuum limit [5], therefore the Wilson fermions probably provide a correct lattice formulation of QCD.

The chiral symmetry breaking of Wilson fermions is, however, an obstacle for a lattice formulation of chiral gauge theories describing the electroweak interactions. A first step for a lattice formulation of the standard electroweak theory would be to find a chiral formulation of some prototype models. In the present paper a simple chiral model is defined on the lattice by an explicitly chiral symmetric extension of the Wilson fermion method. This model can be considered as a lattice version of the gauged $SU(2)_L \otimes SU(2)_R$-symmetric Gell-Mann–Lévy σ-model [6], which includes, besides the usual scalar- and fermion-doublet fields, also a "mirror-fermion" doublet field with exchanged left–right transformation properties.

Before considering the gauged version of the model, let us begin by investigating its pure matter sector. This contains a doublet fermion field $\psi_x = \psi_{Lx} + \psi_{Rx}$ and a doublet mirror-fermion field $\chi_x = \chi_{Lx} + \chi_{Rx}$, interacting with a scalar doublet field φ_x. (The left- and right-handed components of the fermion fields have an index L, respectively R.) The transformation properties of the fields under $SU(2)_L \otimes SU(2)_R$ are

$$\psi'_{Lx} = U_L \psi_{Lx}, \quad \bar\psi'_{Lx} = \bar\psi_{Lx} U_L^{-1}, \quad \psi'_{Rx} = U_R \psi_{Rx}, \quad \bar\psi'_{Rx} = \bar\psi_{Rx} U_R^{-1},$$

$$\chi'_{Lx} = U_R \chi_{Lx}, \quad \bar\chi'_{Lx} = \bar\chi_{Lx} U_R^{-1}, \quad \chi'_{Rx} = U_L \chi_{Rx}, \quad \bar\chi'_{Rx} = \bar\chi_{Rx} U_L^{-1}, \quad \varphi'_x = U_L \varphi_x U_R^{-1}. \qquad (1)$$

Instead of the scalar doublet φ_x, it is sometimes advantageous to consider the equivalent real O(4) fields ϕ_{Sx} ($S = 0, 1, 2, 3$) defined by

$$\varphi_x \equiv \phi_{0x} + i\phi_{sx}\tau_s \tag{2}$$

Here τ_s ($s=1, 2, 3$) denotes Pauli matrices. (An automatic summation over the O(3) and O(4) indices will be understood.) In terms of these fields let us consider the following euclidean lattice action

$$S=\sum_x \Big(\mu\phi_{Sx}\phi_{Sx}+\lambda(\phi_{Sx}\phi_{Sx})^2-\kappa\sum_\mu \phi_{Sx+\mu}\phi_{Sx}+\mu_{\psi\chi}[(\tilde\chi_x\psi_x)+(\tilde\psi_x\chi_x)]+\mu_\psi(\tilde\psi_x\psi_x)+\mu_\chi(\tilde\chi_x\chi_x)$$
$$-\sum_\mu [K_\psi(\tilde\psi_{x+\mu}\gamma_\mu\psi_x)+K_\chi(\tilde\chi_{x+\mu}\gamma_\mu\chi_x)]+r\sum_\mu[(\tilde\chi_x\psi_x)-(\tilde\chi_{x+\mu}\psi_x)+(\tilde\psi_x\chi_x)-(\tilde\psi_{x+\mu}\chi_x)]$$
$$+G_\psi\phi_{Sx}(\tilde\psi_x\Gamma_S\psi_x)+G_\chi\phi_{Sx}(\tilde\chi_x\Gamma_S^+\chi_x)\Big) \tag{3}$$

Here $\sum_x$ runs over the lattice points and $\sum_\mu$ over eight directions of the neighbours $\mu=\pm1, \pm2, \pm3, \pm4$. In the Yukawa couplings the $8\otimes8$ matrices Γ_S ($S=0, ..., 3$) are defined as $\Gamma_S=(1, -i\gamma_5\tau_s)$. The normalization of the fields is left free. By rescaling it is possible to achieve some convenient normalization: for the scalar field ϕ_{Sx} in perturbation theory it is convenient to choose the hopping parameter $\kappa=\frac{1}{2}$, whereas for the limit of very strong quartic scalar coupling $\lambda\to\infty$ the best choice is $\mu=1-2\lambda$. Similarly, the fermion fields can be rescaled according to

$$\psi'_x=z_\psi\psi_x,\quad \tilde\psi'_x=z_{\tilde\psi}\tilde\psi_x,\quad \chi'_x=z_\chi\chi_x,\quad \tilde\chi'_x=z_{\tilde\chi}\tilde\chi_x \tag{4}$$

In perturbation theory a convenient rescaling is defined by $K_\psi=K_\chi=\frac{1}{2}$. For a numerical study one can choose $\mu_\psi=\mu_\chi=1$, whereas for strong bare Yukawa couplings or small mass parameters $\mu_\psi, \mu_\chi\simeq0$ it is natural to put $G_\psi=G_\chi=1$.

According to eq. (1), the above lattice action is chiral invariant if and only if $\mu_\psi=\mu_\chi=0$ (the mass mixing proportional to $\mu_{\psi\chi}$, however, does not break chiral symmetry). Let us now consider the general massive case $\mu_\psi, \mu_\chi\neq0$ in perturbation theory. For this we shall use the convenient normalization $2\kappa=2K_\psi=2K_\chi=1$. The scalar lattice propagator is, as usual,

$$\Delta^\varphi_{xy}=\frac{1}{N}\sum_k \exp[-i(k, x-y)]\,\tilde G^\varphi_k, \tag{5}$$

where N is the number of lattice points and the momentum sum is performed over the Brillouin zone corresponding to the periodic boundary conditions. The momentum space propagator is given by

$$\tilde G^\varphi_k=(\mu_0^2+\hat k^2)^{-1},\quad \mu_0^2\equiv2\mu-8,\quad \hat k^2\equiv4\sum_{\rho>0}\sin^2(\tfrac{1}{2}k_\rho) \tag{6}$$

The fermion propagator is a block matrix in $\psi-\chi$:

$$\Delta_{xy}=\frac{1}{N}\sum_k \exp[-i(k, x-y)]\,\tilde G_k,\quad \tilde G_k\equiv\begin{pmatrix}\tilde G_k^{\psi\psi} & \tilde G_k^{\psi\chi}\\ \tilde G_k^{\chi\psi} & \tilde G_k^{\chi\chi}\end{pmatrix} \tag{7}$$

The inverse fermion propagator in momentum space is

$$\tilde G_k^{-1}=\begin{pmatrix}\mu_\psi+i\gamma\cdot\bar k & \mu_{\psi\chi}+r\hat k^2\\ \mu_{\psi\chi}+r\hat k^2 & \mu_\chi+i\gamma\cdot\bar k\end{pmatrix} \tag{8}$$

implying

$$\tilde{G}_k^{\psi\psi} = \{\mu_\psi(\mu_\chi^2 + \bar{k}^2) - \mu_\chi(\mu_{\psi\chi} + r\hat{k}^2)^2 - i\gamma\,\bar{k}[\mu_\chi^2 + \bar{k}^2 + (\mu_{\psi\chi} + r\hat{k}^2)^2]\}\ \tilde{D}_k,$$

$$\tilde{G}_k^{\psi\chi} = (\mu_{\psi\chi} + r\hat{k}^2)\ [\bar{k}^2 + (\mu_{\psi\chi} + r\hat{k}^2)^2 - \mu_\psi\mu_\chi + i\gamma\,\bar{k}(\mu_\psi + \mu_\chi)]\ \tilde{D}_k,$$

$$\tilde{D}_k \equiv \{[\bar{k}^2 + (\mu_{\psi\chi} + r\hat{k}^2)^2 - \mu_\psi\mu_\chi]^2 + \bar{k}^2(\mu_\psi + \mu_\chi)^2\}^{-1} \tag{9}$$

Here we also introduced the notation $\bar{k}_\mu \equiv \sin k_\mu$. The other matrix elements are obtained from these by exchanging $\psi \leftrightarrow \chi$. As it can be seen, the fermion-doubling species at $k_\mu = \pi$ are removed here for $r > 0$ in a similar way to Wilson fermions.

The eigenvalues of the fermion mass matrix (i.e., of the zero momentum inverse propagator $\tilde{G}_0^{-1}$) are in general given by

$$\mu_1 = \tfrac{1}{2}\ [\mu_\psi + \mu_\chi - \sqrt{(\mu_\psi - \mu_\chi)^2 + 4\mu_{\psi\chi}^2}\,],\quad \mu_2 = \tfrac{1}{2}\ [\mu_\psi + \mu_\chi + \sqrt{(\mu_\psi - \mu_\chi)^2 + 4\mu_{\psi\chi}^2}\,] \tag{10}$$

The corresponding eigenvectors in the $\psi - \chi$ basis are, respectively,

$$e_1 = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix},\quad e_2 = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}, \tag{11}$$

where the mixing angle α is determined by

$$\sin\alpha = \mu_{\psi\chi}\sqrt{2}\ [(\mu_\psi - \mu_\chi)^2 + 4\mu_{\psi\chi}^2 - (\mu_\psi - \mu_\chi)\sqrt{(\mu_\psi - \mu_\chi)^2 + 4\mu_{\psi\chi}^2}\,]^{-1/2} \tag{12}$$

In the special case of $\mu_{\psi\chi}^2 = \mu_\psi\mu_\chi$ there is an exactly zero mass $\mu_1 = 0$, and the other mass is $\mu_2 = (\mu_\psi + \mu_\chi)$. The mixing angle becomes

$$\sin\alpha_0 = \sqrt{\mu_\psi/(\mu_\psi + \mu_\chi)} \tag{13}$$

Therefore, if $\mu_\psi \ll \mu_\chi$ then the zero mass belongs to the dominantly ψ component.

The possibility of an exactly zero eigenvalue in the fermion mass matrix is an important property of the model. We shall see below, that the chiral symmetry breaking masses μ_ψ, μ_χ are produced in the chiral gauge model by spontaneous symmetry breaking, hence they are tied up to the scale of the vacuum expectation value. The ψ–χ mass mixing parameter $\mu_{\psi\chi}$, however, is a chiral symmetry conserving free parameter. For given $\mu_\psi\mu_\chi$, the deviation

$$\bar{\mu} \equiv \sqrt{\mu_\psi\mu_\chi} - \mu_{\psi\chi} \tag{14}$$

can have any scale. In particular it can also be zero. In other words, there is a possibility for an arbitrarily large fermion mass hierarchy for one of the mirror-fermion states, *downwards* from the spontaneous symmetry breaking scale. In the case of a small deviation $\bar{\mu}$ the small mass eigenvalue is

$$\mu_1 = \bar{\mu}\sin(2\alpha_0) - [\bar{\mu}^2/(\mu_\psi + \mu_\chi)]\ \cos^2(2\alpha_0) + O(\bar{\mu}^3) \tag{15}$$

The consequence of this relation is that in the case of spontaneous symmetry breaking the rigid relation between the small fermion masses and the corresponding Yukawa couplings is resolved. The ratio of Yukawa couplings G_ψ and G_χ is, however, directly responsible for the value of the mixing angle α_0 in eq. (13).

Using the propagators and the vertices in the action (3), it is straightforward to calculate, for instance, the one-loop graphs in perturbation theory. The fermion tadpole contributions to the expectation value of the scalar field deserve special attention. The one-loop contribution to the expectation value of ϕ_{0x} is

$$\frac{8}{N}\sum_k \tilde{D}_k\{(\mu_\psi G_\psi + \mu_\chi G_\chi)\,\bar{k}^2 - (\mu_\chi G_\psi + \mu_\psi G_\chi)\ [(\mu_{\psi\chi} + r\hat{k}^2)^2 - \mu_\psi\mu_\chi]\} \tag{16}$$

This vanishes for zero masses (i.e., for exact chiral symmetry). Therefore, a compensation in the action with a counterterm linear in ϕ_{0x} is not necessary. This is different for Wilson fermions, where the fermion tadpole contribution is non-zero, and a tuning in the "external magnetic field" is necessary The critical set of bare parameters, where the physical masses in lattice units vanish, can be determined order by order in perturbation theory. For instance, in the case of the fermion mass matrix the chiral symmetry point $\mu_\psi = \mu_\chi = 0$ is not renormalized, but there are non-zero loop contributions to the mixing parameter $\mu_{\psi\chi}$ The one-loop equation for vanishing $\bar{\mu}$ is

$$0=\sqrt{\mu_\psi \mu_\chi}-\mu_{\psi\chi}+4rG_\psi G_\chi \frac{1}{N}\sum_k (\bar{k}^2+r^2\hat{k}^4)^{-1}=\sqrt{\mu_\psi\mu_\chi}-\mu_{\psi\chi}+G_\psi G_\chi\ 0\ 121 \qquad (17)$$

The numerical value here corresponds to an infinite lattice and $r=1$ From the logarithmically divergent contributions to the renormalized couplings one can obtain the Callan–Symanzik β-functions In one-loop order we have

$$\beta_{G\psi}=\frac{1}{16\pi^2}4G_\psi(G_\psi^2+G_\chi^2),\quad \beta_{G\chi}=\frac{1}{16\pi^2}4G_\chi(G_\psi^2+G_\chi^2),\quad \beta_\lambda=\frac{1}{16\pi^2}(96\lambda^2+16G_\psi^2\lambda+16G_\chi^2\lambda-4G_\psi^4-4G_\chi^4) \qquad (18)$$

After this preparation we can write down the lattice action for the chiral $SU(2)_L \otimes SU(2)_R$ gauge theory

$$S=S_g+S_m \qquad (19)$$

The pure gauge piece with $\beta_{L\,R} \equiv 4/g^2_{L\,R}$ and $U_{L\,R} \in SU(2)_{L\,R}$ is the usual sum over plaquettes

$$S_g=\beta_L \sum_P [1-\tfrac{1}{2}\,\mathrm{Tr}\,U_L(P)]+\beta_R \sum_P [1-\tfrac{1}{2}\,\mathrm{Tr}\,U_R(P)] \qquad (20)$$

The piece containing the physical matter fields is, by using the doublet scalar field φ_x and writing out the L- and R-components separately,

$$\begin{aligned} S_m=\sum_x \Big(& \tfrac{1}{2}\mu\,\mathrm{Tr}(\varphi_x^+\varphi_x)+\tfrac{1}{4}\lambda[\mathrm{Tr}(\varphi_x^+\varphi_x)]^2-\tfrac{1}{2}\kappa\sum_\mu \mathrm{Tr}[\varphi^+_{x+\mu}U_L(x,\mu)\,\varphi_x\,U_R^+(x,\mu)] \\ & +\mu_{\psi\chi}[(\bar{\chi}_{Rx}\psi_{Lx})+(\bar{\chi}_{Lx}\psi_{Rx})+(\bar{\psi}_{Rx}\chi_{Lx})+(\bar{\psi}_{Lx}\chi_{Rx})] \\ & -K_\psi \sum_\mu [(\bar{\psi}_{L\,x+\mu}U_L(x,\mu)\,\gamma_\mu\psi_{Lx})+(\bar{\psi}_{R\,x+\mu}U_R(x,\mu)\,\gamma_\mu\psi_{Rx})] \\ & -K_\chi \sum_\mu [(\bar{\chi}_{L\,x+\mu}U_R(x,\mu)\,\gamma_\mu\chi_{Lx})+(\bar{\chi}_{R\,x+\mu}U_L(x,\mu)\,\gamma_\mu\chi_{Rx})] \\ & +r\sum_\mu [(\bar{\chi}_{Rx}\psi_{Lx})-(\bar{\chi}_{R\,x+\mu}U_L(x,\mu)\,\psi_{Lx})+(\bar{\chi}_{Lx}\psi_{Rx})-(\bar{\chi}_{L\,x+\mu}U_R(x,\mu)\,\psi_{Rx}) \\ & +(\bar{\psi}_{Rx}\chi_{Lx})-(\bar{\psi}_{R\,x+\mu}U_R(x,\mu)\,\chi_{Lx})+(\bar{\psi}_{Lx}\chi_{Rx})-(\bar{\psi}_{L\,x+\mu}U_L(x,\mu)\,\chi_{Rx})] \\ & +G_\psi[(\bar{\psi}_{Rx}\varphi_x^+\psi_{Lx})+(\bar{\psi}_{Lx}\varphi_x\psi_{Rx})]+G_\chi[(\bar{\chi}_{Rx}\varphi_x\chi_{Lx})+(\bar{\chi}_{Lx}\varphi_x^+\chi_{Rx})]\Big) \end{aligned} \qquad (21)$$

The fermion masses are now provided by the vacuum expectation value of the scalar field

$$\langle\varphi_x\rangle=\langle\varphi_x^+\rangle\equiv v \qquad (22)$$

and by the Yukawa couplings

$$\mu_\psi=vG_\psi/2K_\psi\sqrt{2\kappa},\quad \mu_\chi=vG_\chi/2K_\chi\sqrt{2\kappa} \qquad (23)$$

It is assumed that in the continuum limit $\mu_{\psi\chi}\to 0$ $v\to 0$ and therefore μ_ψ, μ_χ, μ_1, $\mu_2 \to 0$

In general, the continuum limit can be defined at some critical point of the pure matter theory (in thid limit the gauge couplings g_L^2 and g_R^2 tend to zero) However the mathematical continuum limit of the models defined by the above actions is probably trivial (for a recent discussion see ref [7] and the references therein) In order to have a non-trivial quasi-continuum theory the lattice spacing has to be kept finite (but small compared to the physical distances) Thinking of a physical theory, the cut-off is naturally always finite

The two gauge group factors $SU(2)_L$ and $SU(2)_R$ need not be treated symmetrically One can, for instance, gauge only $SU(2)_L$ (put $g_R=0$ and $U_R=1$) When both group factors are gauged [as it stands in eq (21)] the spontaneous symmetry breaking leaves the diagonal SU(2) subgroup unbroken For a complete symmetry breaking one can introduce, for instance, two further scalar fields φ_L and φ_R which are doublets with respect to one of the SU(2) subgroups, but are scalar with respect to the other, as is usually done in left–right symmetric models (For early references on $SU(2)_L \otimes SU(2)_R$-models see ref [8]) In this way it is possible to split the gauge field masses independently of the value of g_L and g_R

The $SU(2)_L \otimes SU(2)_R$ chiral model considered here is a representative for a large class of lattice chiral models with mirror fermions The above discussion of small fermion masses shows that, by an appropriate extension, models can also be constructed which look below ~ 100 GeV similar to the standard electroweak model by chosing Yukawa couplings like G_ψ small one can make the mirror-fermion components of the light fermions small (the mirror-fermion components have V+A coupling to the W-boson) At higher energies there is, however, a marked difference due to the occurrence of the physical mirror-fermion states In simple models with only one spontaneous symmetry breaking scale the masses of the mirro fermions have to be roughly below ~ 500 GeV, corresponding to the unitarity limit for Yukawa couplings [9] In a more complicated model with several spontaneous symmetry breaking scales this may, however, be different

The question naturally arises, whether it would be possible to arrange the bare parameters of the model in such a way that the mirror-fermion partners get a mass proportional to the cut-off (and hence are removed from the physical spectrum) In particular, one can try to define the continuum limit in some critical point where only a subset of the masses in lattice units tends to zero The difficulty which occurred in this case in all my attempts, also after the introduction of additional scalar fields, is that if the mirror partners are removed then the gauge bosons go with them Apparently, one cannot arrange a spontaneous symmetry breaking pattern with finite W-boson, Higgs-boson and fermion masses and a very large mirror-fermion mass A general reason behind this is that in the unbroken phase the chiral symmetry has to be realized by a parity doublet pair of degenerate fermion doublets Indeed, as one can see from the fermion mass matrix $\tilde{G}_0^{-1}$, in the case of $\mu_\psi = \mu_\chi = 0$ and $\mu_{\psi\chi} \neq 0$ there is a degenerate pair of fermion doublets, and the eigenvectors $(\psi \pm \chi)/\sqrt{2}$ have opposite parity

It would be interesting to study the models defined by the above actions also non-perturbatively The first question, which is important for the understanding of the physical content, would be to investigate the critical set and the phase structure in the space of bare parameters

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