Determination of quark and gluon vacuum condensates from τ-lepton decay data

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Abstract. The values of the gluon and four-quark vacuum condensates are estimated using recent experimental data on the semileptonic τ-lepton decays \( \tau \rightarrow \nu + n \pi \) which determine the vector and axial-vector hadronic spectral functions. An optimal estimate is achieved through a systematic combined use of Finite Energy, Laplace and Gaussian transform QCD sum rules. As a byproduct, the values of the dimension \( d = 8 \) vacuum condensates in the vector and axial-vector channels are also estimated.

1 Introduction

The method of QCD sum rules, first introduced by Shifman et al. [1], has become a popular and powerful technique to study hadronic physics in the low energy resonance region [2]. As is well known, this method relates through dispersion relations low energy parameters, e.g. particle masses and coupling constants, to the Operator Product Expansion (OPE) of current correlators at short distances. The basic underlying assumption is that this OPE remains valid in the presence of non-perturbative effects which are parameterized by a set of vacuum expectation values of quark and gluon fields. These vacuum condensates induce power corrections to asymptotic freedom and are supposed to be responsible for the rich resonance structure observed at low energies [3]. To be more specific let us consider the following two-point function

\[
\Pi(q^2) = i \int d^4 x e^{i q x} \langle 0 | T(J(x)J(0)) | 0 \rangle
\]

where \( J(x) \) stands for any local current built from the quark and gluon fields appearing in the fundamental QCD Lagrangian. The corresponding OPE reads

\[
i \int d^4 x e^{i q x} T(J(x)J(0)) = C_0 \hat{1} + \sum_N C_N(q) \hat{O}_N
\]

where the Wilson coefficients in this expansion depend on the Lorentz indices and quantum numbers of \( J(x) \) and also on the local gauge invariant operators \( \hat{O}_N \) built from quark and gluon fields. These operators are ordered by increasing dimensionality and the Wilson coefficients, calculable in perturbation theory, fall off by corresponding powers of \( q^2 \). The unit operator in (2) has dimension \( d = 0 \) and \( C_0 \hat{1} \) stands for the purely perturbative contribution. Examples of \( d = 4 \) operators are \( m_q \bar{q} q \) and \( G_{\mu \nu} G^{\mu \nu} \). To use the OPE (2) in (1) one assumes that short and long distance effects factorize. The former are buried in the Wilson coefficients and the latter in the non-vanishing vacuum expectation values \( \langle 0 | \hat{O}_N | 0 \rangle \). After this the rest follows from analyticity, viz. \( \Pi(q) \) satisfies a dispersion relation and thus one relates the hadronic spectral function appearing there to the OPE. Different choices of the weight in the dispersion relation lead to different kinds of QCD sum rules, e.g. Hilbert, Laplace or Gaussian transforms, Finite Energy Sum Rules (FESR), etc.

Concerning the actual numerical values of the various quark and gluon condensates, in general they cannot be computed from first principles as this would be tantamount to solving QCD exactly. Hence, the standard procedure has been to extract these condensates by using QCD sum rules in a few channels where experimental data on the hadronic spectral functions is available, e.g. the charmonium system [1, 4–7], \( e^+ e^- \) cross sections [8–12] etc. Once the values of the various vacuum condensates are known one may proceed to apply the QCD sum rules to other channels and make predictions for particle masses, couplings constants, etc. Using experimental information on charmonium as well as on \( e^+ e^- \) cross sections, some authors have found [1, 4–5, 8–9]
which we shall refer to as the standard value of the gluon condensate. However, this result is somewhat controversial, to wit: it has been conjectured in the framework of potential models [13], as well as in two-dimensional QCD [14], that the value (1) may be an underestimate. Also, a value about 30% higher has been advocated in [2], and recent charmonium [6–7] and $e^+e^-$ [11–12] analyses suggest even larger deviations. A separate controversy exists in connection with the vacuum saturation approximation [1] often used to estimate the magnitude of the dimension-six four-quark condensates, e.g.

$$\langle \bar{q} q, \lambda^a \rangle^2 \big|_{V, S} = -\frac{16}{9} \langle q\bar{q} \rangle^2$$

where $\lambda^a$ are the Gell-Mann matrices in colour space. In fact, several authors have noticed that this approximation leads to inconsistencies [9, 15–17] a view supported by some $e^+e^-$ analyses [11, 12].

Given the impact of the gluon and the four-quark condensates on current phenomenology it becomes quite important to attempt a resolution of the above discrepancies. This should be done e.g. by using as many independent sets of experimental data as possible and, at the same time, employing different kinds of sum rules in a systematic fashion. In fact, depending on the choice of kernel in the dispersion relations the extraction of the vacuum condensates may be more or less sensitive to a given parametrization of the data, as well as to the influence of higher dimensional condensates.

In this paper we wish to present an attempt in this direction by exploiting the unique opportunity being offered by the availability of $\tau$-lepton semileptonic decay data from the ARGUS collaboration at DORIS [18], namely $\tau \to v, 2\pi$ and $\tau \to v, 4\pi$ which determine, essentially, the vector isovector hadronic spectral function, and $\tau \to v, \rho\pi$ which relates to the axial-vector isovector channel. These data have been used recently [19] to study the saturation of the Weinberg sum rules and to estimate the $\pi^+ - \pi^0$ mass difference in QCD.

Here we shall consider the vector and the axial-vector channels separately and extract the vacuum condensates from the corresponding sum rules. Since the gluon condensate contributes equally to both cases we shall be able to perform valuable consistency checks on our analysis. In order to optimize the determination of the vacuum condensates we shall make a systematic combined use of FESR, Laplace and Gaussian transforms QCD sum rules.

The paper is organized as follows. In Sect. 2 we define the two-point functions for the vector and axial-vector channels and give their QCD expressions. The various types of QCD sum rules are introduced in Sect. 3, where we also discuss briefly their advantages and shortcomings. In Sect. 4 we present the various numerical fits to the experimental data. Sections 5, 6 and 7 contain the results obtained from the FESR, Laplace and Gaussian transforms, respectively. In Sect. 8 we discuss the impact of higher dimensional condensates concentrating on the dimension $d = 8$. Finally, in Sect. 9 we summarize our results.

### 2 Two-point functions and the OPE in QCD

We start by defining the two-point function

$$\Pi_{\mu\nu}(q^2) = \int d^4 x d^4 y \langle 0 | T(V_\mu(x) V_\nu(y)) | 0 \rangle$$

where the current $V_\mu(x)$ has the quantum numbers of the $\rho$-meson, i.e.

$$V_\mu(x) = \frac{1}{2} [\bar{u}(x)\gamma_\mu u(x) - \bar{d}(x)\gamma_\mu d(x)]$$

From the conservation of the vector current one can write

$$\Pi_{\mu\nu}(q^2) = - (g_{\mu\nu} q^2 - q_{\mu} q_{\nu})$$

The function $\Pi_{\mu\nu}(q^2)$ in (10) has been calculated in perturbation theory, in the MS renormalization scheme [20], up to three loops [21], and including the leading non-perturbative corrections [1], with the result

$$Q^2 \equiv - q^2 > 0$$

$$8\pi^2 \Pi_{\nu}(q^2) = - \ln \frac{Q^2}{\lambda^2} + \frac{3}{2} \frac{Q^2}{\lambda^2} [m_\pi^2(v^2) + m_\rho^2(v^2)]$$

$$- \frac{\alpha_s(v^2)}{\pi} \left[ \ln \frac{Q^2}{\lambda^2} + \frac{\alpha_s(v^2)}{\pi} \right]$$

$$- \frac{\beta_1}{4} \ln^2 \frac{Q^2}{\lambda^2} - F_3 \ln \frac{Q^2}{\lambda^2}$$

$$+ C_4 (\langle O_4 \rangle) \nu + C_6 (\langle O_6 \rangle) \nu + C_8 (\langle O_8 \rangle) \nu$$

$$+ \cdots \left[ \frac{\alpha_s}{\pi} \right]^3 + \cdots \left( \frac{1}{Q^{10}} \right)$$

where $\beta_1 = 29/6$ and $F_3 = 1.756...$ for three colours and two flavours.

The non-perturbative term of dimension $d = 4$ is given by

$$C_4 (\langle O_4 \rangle) \nu = \frac{\pi}{3} \langle \alpha_s G^2 \rangle + 4\pi^2 [m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle]$$

In principle there are various operators contributing to $C_6 (\langle O_6 \rangle)$, namely the quark-gluon, the three gluon and the four-quark condensates. The first is suppressed by a factor of $m_\tau^2$ [1–2], while the second gives no contribution in the case of light quarks [22–23], thus leaving only the four-quark condensate contribution

$$C_6 (\langle O_6 \rangle) \nu = - 8\pi^3 \alpha_s \left[ \langle \bar{q} q, \lambda^a \rangle^2 \right]$$

$$+ \frac{3}{8} \langle \bar{q} q, \lambda^a \rangle^2 \right]$$

where

$$\langle \bar{q} q, \lambda^a \rangle^2 \right]$$

There are quite a few quark and gluon condensates
contributing to \( C_8 \langle O_8 \rangle \). The Wilson coefficients of the various condensates for light quark currents have been computed in [24]. We defer a discussion on \( C_8 \langle O_8 \rangle \) till Sect. 8 and proceed with our analysis neglecting this term. We shall also neglect the quark mass loop insertion term proportional to \( m_q^2 \) in eq. (8) as it has no impact on our results. We shall retain, however, the second term in (9) which may be estimated using the well known current algebra low energy theorem [25]

\[
-(m_u + m_d) \langle \bar{u} u + \bar{d} d \rangle = 2 f_\pi^2 \mu^2 (1 - \delta_x) \tag{11}
\]

where \( \delta_x \) stands for the (analytic) corrections to pion-PCAC. These corrections have been estimated through QCD sum rules in the pionic channel [26–27]:

\[
\delta_x = (4 \pm 1)\%.
\]

Assuming \( \langle \bar{u} u > \approx \langle \bar{d} d > \) one finds from (11)

\[
4\pi^2 [m_u <\bar{u} u > + m_d <\bar{d} d >] \simeq -0.0064 \text{ GeV}^4. \tag{12}
\]

In this way and with \( A_{QCD} \) fixed, e.g. \( A_{QCD} = 100 \text{ MeV} \), (8) contains as free parameters the gluon condensate and the four-quark condensate (eventually also the \( d = 8 \) condensates) which we shall determine by confronting (8) with the \( \bar{q} q \rightarrow \gamma \gamma, Z \gamma \) data. The reference value for \( G_{\pi}^2 = \langle q \gamma \gamma q \rangle \) was given in (3). In the case of \( C_6 \langle O_6 \rangle \) we could take as reference value the one following from the vacuum saturation approximation [1]. Making use of the relation (11) \n
\[
\langle \bar{q} \gamma_{\mu} \gamma_5 q^2 \rangle \mid_{V.S.} = -\langle \bar{q} \gamma_{\mu} \gamma_5 q^2 \rangle \mid_{V.S.}
\]

(13)

together with (4) in (10) one finds [1]

\[
C_6 \langle O_6 \rangle \mid_{V.S.} = -8\pi \pi^3 \alpha_s \langle \bar{q} q \rangle^2 \simeq -0.06 \text{ GeV}^6. \tag{14}
\]

Turning to the axial-vector channel we define

\[
\Pi_{A}^{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0 | T(A_\mu(x) A^\nu_\nu(0)) | 0 \rangle
\]

\[= -g_{\mu\nu} \tilde{A}_\nu(q^2) + q_\mu q_\nu \Pi_{A}(q^2) \tag{15}\]

where the current \( A_\mu(x) \) has the quantum numbers of the \( A_1 \)-meson, i.e.

\[
A_\mu(x) = \frac{1}{2} [\bar{u}(x) \gamma_{\mu} \gamma_5 u(x) - \bar{d}(x) \gamma_{\mu} \gamma_5 d(x)]. \tag{16}
\]

Since \( A_\mu(x) \) is not conserved (15) contains two independent structure functions. However, at large \( Q^2 \equiv -q^2 \) and to leading order in the quark masses one has the relation [1]

\[
\tilde{H}_A(Q^2) + Q^2 \Pi_{A}(Q^2) = \frac{(m_u + m_d) \langle \bar{u} u + \bar{d} d \rangle}{Q^2}. \tag{17}\]

Concentrating on e.g. \( \Pi_{A}(q^2) \), its perturbative QCD expression is the same as that for \( \Pi_{V}(q^2) \) in (8). However, the non-perturbative coefficients are different, viz.

\[
C_4 \langle O_4 \rangle_A = \frac{\pi}{3} \langle \alpha_s G^2 \rangle - 4\pi^2 [m_u <\bar{u} u > + m_d <\bar{d} d >] \tag{18}\]

\[
C_6 \langle O_6 \rangle_A = -\frac{88}{9} \pi^3 \alpha_s \langle \bar{q} \gamma_{\mu} \gamma_5 q \rangle^2. \tag{19}\]

Notice that \( C_4 \langle O_4 \rangle_A \) should be larger than \( C_4 \langle O_4 \rangle_V \) by precisely twice the value (12), i.e.

\[
C_4 \langle O_4 \rangle_A - C_4 \langle O_4 \rangle_V \simeq 0.013 \text{ GeV}^4 \tag{20}\]

On the other hand, comparing eqs. (10) and (19) and using (13) one finds

\[
\begin{align*}
\mathcal{C}_6 \langle O_6 \rangle_A \mid_{V.S.} &= -\frac{11}{7}, \\
\mathcal{C}_6 \langle O_6 \rangle_V \mid_{V.S.} \end{align*} \tag{21}\]

These two relations will become useful later on to perform valuable consistency checks on the results of our analysis. Concerning \( C_8 \langle O_8 \rangle_A \), at least the contribution from the gluonic operators should be the same as in \( C_8 \langle O_8 \rangle_V \) as calculated in [24].

3 QCD sum rules

3.1 Finite energy sum rules

Finite Energy Sum Rules (FESR), including radiative corrections to order \( \mathcal{O}(\epsilon^2) \), have been derived in [10]. For the vector or the axial-vector channel they read

\[
C_{4N+2} \langle O_{4N+2} \rangle_{V,A} = 8\pi^2 \int_0 \frac{d\zeta}{\zeta^{N+2}} \frac{1}{\pi} \operatorname{Im} \Pi_{V,A} (s) \tag{22}\]

\[
\begin{align*}
-\frac{s_0^{N+1}}{2N+1} [1 + F_{4N+2}(s_0)] \\
-\frac{s_0^{N+2}}{2N+2} [1 + F_{4N+4}(s_0)] \end{align*} \tag{23}\]

where \( N = 0, 1, 2, \ldots \), and the radiative corrections \( F_p(s_0) \), with \( p = 4N + 2 \) or \( p = 4N + 4 \), are given by

\[
F_p(s_0) = \frac{\alpha_s(s_0)}{4} + \left[ \frac{\alpha_s(s_0)}{\pi} \right] \left[ \frac{F_3 - \beta_2}{\beta_1 - \beta_2} \ln \frac{s_0 - \Lambda_{QCD}^2}{\Lambda_{QCD}^2} \right] \tag{24}\]

where \( F_3 \) and \( \beta_1 \) were defined previously (8), \( \beta_2 = -115/12 \) for three colours and two flavours, and

\[
\left. \frac{\alpha_s(s_0)}{\pi} = -\frac{\beta_1}{\beta_2} \ln \frac{s_0 - \Lambda_{QCD}^2}{\Lambda_{QCD}^2}. \right) \tag{25}\]

In the above equations \( s_0 \) is the threshold for asymptotic freedom, i.e. for \( s > s_0 \) the spectral function behaves as predicted by perturbative QCD. Since the quark mass loop insertion term in eq. (8), formally equivalent to a \( C_2 \langle O_2 \rangle \) effective contribution, is negligible one finds from (22) with \( N = 0 \)

\[
1 + F_2(s_0) \equiv I_0(s_0)_{V,A} = \frac{8\pi^2 \alpha_s}{s_0} \int_0 \frac{1}{\pi} \operatorname{Im} \Pi_{V,A} (s). \tag{26}\]

Hence, in this case \( s_0 \) is not a free parameter to be determined from additional physical assumptions outside the sum rule framework, but rather it is a solution of the above eigenvalue equation. Once \( s_0 \) deter-
mined from (26) one may proceed to determine the values of $C_N \langle O_N \rangle$ using the remaining (22)-(23).

There is an important issue, often overlooked in past applications of these sum rules, which has to do with the stability of the eigenvalue solutions to the FESR. As first pointed out in [28] one should trust predictions from FESR provided they are stable against reasonable changes of $s_0$ inside some duality region; only then there will be duality between the experimental data and a given set of QCD parameters.

One expects that the more accurate the parametrization of the data the wider this duality region where e.g. the rhs of (26) will be approximately equal to its lhs. A wide duality region seems to guarantee the stability of the vacuum condensates [10, 11, 27-29], at least for those of lowest dimension. Clearly, the power-like weight of the spectral function in the FESR will eventually be the cause of their failure in making reliable predictions for high dimensional condensates. That this stability issue is not trivial may be seen by parametrizing e.g. the vector or the axial-vector spectral function with a delta function. In this case the slowly varying lhs of (26) will intercept the rapidly varying rhs at only one point. The duality region then reduces to a single value of $s_0$ and, furthermore, the vacuum condensates extracted from (22)-(23) turn out to be quite unstable against small changes around this eigenvalue. As shown in [11], in connection with the vector case, this instability problem does not get any better by attempting a finite-width single resonance parametrization of the spectral function. It is only when a fit to the actual $e^+e^-$ data is used in the FESR that one finds stable eigenvalue solutions.

3.2 Laplace transform sum rules

Another useful set of QCD sum rules is that based on the Laplace transform of the spectral function [1]

$$L(\sigma) = \int_0^\infty ds e^{-s\sigma} - \frac{1}{\pi} \text{Im} P(s)$$

(27)

where the Laplace variable $\sigma$ plays now the role of the short distance expansion parameter analogous to $1/Q^2$ in e.g. (8). In the case of vector or axial-vector current correlators the QCD expression for $L(\sigma)$ reads [30]

$$8\pi^2 \sigma L_{VA}(\sigma) = 1 + \frac{\alpha_s(1/\sigma)}{\pi} + \left[ \frac{\alpha_s(1/\sigma)}{\pi} \right]^2 \left[ F_3 - \frac{1}{2} \beta_1 \gamma_E + \frac{1}{\beta_1} \ln(\sigma \Lambda^2) \right] + C_4 \langle O_4 \rangle_{VA} \sigma^2 + C_6 \langle O_6 \rangle_{VA} \sigma^4$$

$$+ \frac{\sigma^3}{2!} + C_8 \langle O_8 \rangle_{VA} \frac{\sigma^4}{3!} + \mathcal{O}(m^2 \sigma^2) + \mathcal{O}\left( \frac{\sigma}{\pi} \right)^3 + \mathcal{O}(\sigma^4)$$

(28)

where $\gamma_E = 0.5772\ldots$ is the Euler constant.

Because of its exponential weight, the Laplace transform (27) places more emphasis on the low energy part of the hadronic spectral function, a clear advantage over e.g. the FESR. At the same time, however, the influence of $s_0$ becomes exponentially suppressed and, hence, for all practical purposes $s_0$ ends up as an adjustable parameter not fixed by the Laplace sum rules themselves. On the QCD side, higher dimensional condensates become factorially suppressed. Since our knowledge of the Wilson coefficients does not usually go beyond $d = 8$, this feature appears as an advantage. Nevertheless, notice from (28) that now all of the condensates appear in the sum rules; it might happen in some cases that the factorial suppression may not be enough to quench some of the higher dimensional condensates. As we shall discuss in more detail later (cf. Sect. 6) this fact can make it difficult to extract the values of $C_N \langle O_N \rangle$ in an unbiased way. In contrast, vacuum condensates of different dimensionality obey uncoupled FESR, at least when no radiative corrections are taken into account in the Wilson coefficients (for a generalization to higher orders in $\alpha_s$, see [31]).

In practical applications of Laplace transform QCD sum rules it has been customary to consider the ratio

$$R(\sigma) = \frac{L(\sigma)}{L(\sigma)} = \frac{\int_0^\infty ds e^{-s\sigma} - \frac{1}{\pi} \text{Im} P(s)}{\int_0^\infty ds e^{-s\sigma} - \frac{1}{\pi} \text{Im} P(s)}$$

(29)

where $L(\sigma)$ stands for $dL(\sigma)/d\sigma$. On the hadronic side the ratio (29) tends to minimize the experimental errors in the spectral function while on the QCD side it tends to emphasize the non-perturbative contributions over the purely perturbative ones, viz.

$$R_{\text{QCD}}(\sigma) = \frac{1}{\sigma} \left[ 1 + F_4(\sigma) - C_4 \langle O_4 \rangle \sigma^2 + C_6 \langle O_6 \rangle \sigma^4 - \cdots \right]$$

$$\times \left( 1 + F_4(\sigma) + C_4 \langle O_4 \rangle \sigma^2 + C_6 \langle O_6 \rangle \sigma^4 + \cdots \right)$$

(30)

where $F_4(\sigma)$ stands for the perturbative terms in eq. (28). We shall adhere to this established practice and use the ratio (29) in our analysis.

3.3 Gaussian sum rules

The final set of sum rules we will consider is that based on the Gaussian transform of the spectral function [10]

$$G(s, t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty ds e^{-(s^2 + t^2)/4t} - \frac{1}{\pi} \text{Im} P(s)$$

(31)

i.e. the convolution of the spectral function with a Gaussian kernel centered at an arbitrary point $\sigma$ with a finite-width resolution $\sqrt{4t}$. The short distance expansion parameter in this case is $1/\sqrt{4t}$, while $\sqrt{4t}$ determines the resolution with which the physical spectrum is to be sampled at the various points $\sigma$. In
particular, for \( z = 0 \) one would obtain strict local duality, i.e.

\[
G(\delta, 0) = \frac{1}{\pi} \text{Im} \Pi(\delta)
\]  

(32)

However, this is only a hypothetical limit as it would imply a complete solution to the bound state problem in QCD. Therefore, in practice \( \tau \) must be kept finite, typically \( \tau \approx 0.5-1.5 \text{GeV}^2 \) \cite{10-11, 27-29}. A very important property of \( G(\delta, \tau) \) is that it obeys the partial differential equation

\[
\frac{\partial^2 G(\delta, \tau)}{(\partial \delta)^2} = \frac{\partial G(\delta, \tau)}{\partial \tau}
\]  

(33)

which is nothing but the heat equation if one reinterprets \( \delta \) as a 'position' variable and \( \tau \) as a 'time' variable. In this analogy the hadronic spectral function corresponds to the initial heat distribution in a semi-infinite rod \( 0 \leq \delta \leq \infty \) and \( G(\delta, \tau) \) measures the evolution in time of the heat distribution in this rod. This provides a convenient framework to check the consistency between given data on the spectral function and a specific choice of QCD parameters, e.g. \( \Lambda_{\text{QCD}}, s_0 \) and the values of the vacuum condensates in the OPE. In fact, after a certain 'time' \( \tau \) sufficiently large so that the uncalculated QCD corrections become relatively small, the predicted QCD heat distribution should match the evolution of the data. This is the heat evolution test first proposed in \cite{10} and which serves as a quantitative formulation of the notion of local duality.

The heat equation (33) for the semi-infinite rod \( 0 \leq \delta \leq \infty \) admits two general solutions corresponding to the two possible choices of boundary conditions: vanishing function or vanishing first partial derivative with respect to \( \delta \) at the origin, both for \( \tau > 0 \), subject to the initial condition (32). Denoting these two solutions by \( U^+(\delta, \tau) \) and \( U^- (\delta, \tau) \), respectively, one has

\[
U^+(\delta, \tau) = G(\delta, \tau) + G(-\delta, \tau)
\]  

(34)

\[
U^-(\delta, \tau) = G(\delta, \tau) - G(-\delta, \tau)
\]  

(35)

The calculation of \( U^\pm(\delta, \tau) \) in QCD has been discussed in \cite{10} and the result is

\[
8\pi^2 U^-_{\nu,A}(\delta, \tau) = \text{erf}(\hat{x}) \left\{ 1 + \frac{\alpha_4(\sqrt{\tau})}{\pi} + \left[ \frac{\alpha_4(\sqrt{\tau})}{\pi} + \frac{\beta_3(\sqrt{\tau})}{\beta_1 A_{\text{QCD}}^2} \right]^2 \left[ F_3 + \frac{\beta_1}{2} \ln \frac{\sqrt{\tau}}{A_{\text{QCD}}} \right] \right\} + \beta_1 \left[ \int_0^\delta d\zeta D(\zeta) \right] + \beta_2 \left[ \frac{\alpha_4(\sqrt{\tau})}{\pi} + \frac{\beta_3(\sqrt{\tau})}{\beta_1 A_{\text{QCD}}^2} \right]^2 
\]

\[
+ \beta_1 \left[ \int_0^\delta d\zeta \right] + \beta_2 \left[ \frac{\alpha_4(\sqrt{\tau})}{\pi} + \frac{\beta_3(\sqrt{\tau})}{\beta_1 A_{\text{QCD}}^2} \right]^2 
\]

\[
+ \frac{2}{\sqrt{\pi}} e^{-\tau} \left[ \theta(m^2_\pi/\sqrt{\tau}) \right] + \frac{1}{2} \frac{C_6 \langle O_6 \rangle_{\nu,A}}{(2\sqrt{\tau})^2} H_2(\hat{x}) + \ldots
\]  

(36)

where

\[
\hat{x} = \frac{\delta}{2\sqrt{\tau}},
\]

(38)

\( \text{erf}(\hat{x}) \) is the error function, \( H_n(\hat{x}) \) are Hermite polynomials,

\[
D(\zeta) = e^{-\zeta^2} \int_0^\zeta dt e^{t^2}
\]  

(39)

is the Dawson function,

\[
\text{Der} f(\zeta) = e^{-\zeta^2} \int_0^\zeta dt e^{t^2} \text{erf}(t),
\]  

(40)

and \( \alpha_n(\sqrt{\tau}) \) is given as in (25).

Notice that \( U^+(\delta, \tau) \) contains the dimension \( d = 6 \) vacuum condensate (and eventually \( d = 10 \), etc. which we are not considering here), while \( U^-(\delta, \tau) \) contains \( C_6 \langle O_6 \rangle \) and \( C_8 \langle O_8 \rangle \). This will allow for two independent heat evolution tests to be performed on the eigenvalue solutions to the FESR, as well as on the results from the Laplace transform analysis.

3.4 Comparative discussion of the QCD sum rules

It should be clear from our short review of the various QCD sum rules above that each one of them has advantages as well as shortcomings. A brief summary follows.

(i) Because of their exponential kernel the Laplace transform QCD sum rules emphasize the low energy, ground state, part of the hadronic spectral function. Therefore, an accurate knowledge of the data in the intermediate and high energy region may not be necessary. In contrast, FESR are rather sensitive to this region and hence call for a much more accurate parametrization of the data.

(ii) In spite of the Laplace factorial suppression of higher dimensional non-perturbative terms in the OPE, this might not be enough to avoid a biased determination of the lowest dimensional condensates through a confrontation between (29) and (30). To
elaborate on this point suppose we assume that $C_6^0 = C_6^4 = C_6^{10} = \ldots = 0$ and then proceed to determine $C_4^0$ by fitting $R(\sigma)_{\text{QCD}}$ in (30) to $R(\sigma)_{\text{DATA}}$ in (29). Let us call this result $C_4^0_{\text{I}}$. In a second step we take into account $C_4^0$ and $C_6^0$ but continue to neglect $C_8^0$, etc., and call these results $C_4^0_{\text{I2}}$ and $C_6^0_{\text{I1}}$. If $C_6^0$ is not negligible we would expect in general $C_4^0_{\text{I1}} \neq C_4^0_{\text{I2}}$. In fact, as will be shown in Sect. 6 even the sign of $C_4^0_{\text{I2}}$ turns out to be different from that of $C_4^0_{\text{I1}}$. The established practice has been to stop the analysis at this second stage, assuming the influence of $C_8^0$ on $C_4^0$ and $C_6^0$ to be negligible. We shall argue later (cf. Sect. 8) that this may not be the case. At any rate, this Laplace correlation among the various vacuum condensates is absent in the FESR as they obey uncoupled eigenvalue equations, provided the corresponding Wilson coefficients are computed to lowest order in $\alpha_s$ (see [31]).

(iii) The value of the asymptotic freedom threshold $s_0$ is essentially a free parameter not fixed by the Laplace sum rules. In contrast, $s_0$ is one of the eigenvalue solutions to the FESR. However, the dependence of the vacuum condensates on $s_0$ is power-like in the FESR while it is exponentially quenched in Laplace sum rules.

(iv) Having fixed $s_0$ by means of some external arguments, predictions from Laplace sum rules follow from the criterion that there should exist some ‘window’ in $\alpha_s$ such that perturbative QCD remains valid and, at the same time, only the leading power corrections are needed. This criterion should be thoroughly tested and checked by comparing with the results from the FESR. Nevertheless, the latter may only be trusted if the eigenvalue solutions are stable against changes in $s_0$ inside the duality window. The ultimate overall consistency check of whether the values of $s_0$ and of $C_6^0$ so determined are dual to the data or not, should be performed using the heat evolution tests which follow from the Gaussian sum rules.

Our conclusion is that Laplace transform and Finite Energy QCD sum rules are complementary tools to be used in the determination of the vacuum condensates. A systematic and reliable analysis should be based on both approaches together with the heat evolution tests. This is, in fact, the procedure we follow in the sequel to extract the QCD vacuum condensates from the $\tau$-lepton decay data.

### 4 Fits to the data

The amplitude for the semileptonic decay of the $\tau$-lepton into a non-strange ($S = 0$) hadronic final state is given, in the four-Fermi approximation, by

$$A(\tau^+ \rightarrow \nu_\tau, + \text{ hadrons}(S = 0))$$

$$= \sqrt{2} G_F \cos \theta_W \beta(p_\tau)\gamma^\mu(1 - \gamma_5)\bar{u}(p_\nu)$$

$$\times \langle \text{hadrons}(S = 0)|J_\mu^{1+2}|0\rangle,$$  \hspace{1cm} (41)
where \( J^u_v = V^u_v - A^u_v \). If the hadronic final state consists of an even (odd) number of pions then only the vector (axial-vector) piece of \( J^u_v \) contributes to the matrix element in the rhs of (41). Therefore, the differential transition probabilities per invariant mass squared \( d\Gamma /ds \) for the processes \( \tau \rightarrow \nu_\tau + n\pi \) are related to the vector or axial-vector spectral functions, viz.

\[
\sum_n \frac{d\Gamma (\tau \rightarrow \nu_\tau + n\pi)}{ds} = \frac{G_F^2 \cos^2 \theta}{8\pi} \left( M_\pi^2 + 2s \right) \left( M_\pi^2 - s \right)^2 
\]

\[
\begin{cases} 
1 - 1\text{Im}\Pi_v(s) & (n = \text{even}) \\
1 - 1\text{Im}\Pi_A(s) & (n = \text{odd}). 
\end{cases}
\] (42)

In the axial-vector case the spectral function contains in addition the pion-pole which we treat separately.

Our analysis is based mainly on the recent data obtained by the ARGUS collaboration at DORIS [18]. The experimental invariant mass distributions for \( \tau^+ \rightarrow \nu_\tau \pi^+ \pi^0 \) and \( \tau^+ \rightarrow \nu_\tau \pi^+ \pi^- \pi^0 \) are shown in Fig. 1a and b, respectively. We also show there (solid curves) the result of the fits done in [19] using the product of a Breit–Wigner function with a high order polynomial (Fig. 1a) and a single high order polynomial (Fig. 1b). These fits were performed using the numerical program MINUIT and their quality is quite good as measured by the resulting values of the chi-squared per degree of freedom \( \chi^2 \), namely: \( \chi^2 = 51 = 0.89 \) for Fig. 1a and \( \chi^2 = 81 = 0.76 \) for Fig. 1b. Concerning the decay mode \( \tau \rightarrow \nu_\tau \pi^+ n^0 \pi^0 \pi^0 \), we use the theoretical estimate of [32] based on CVC:

\[
\frac{\Gamma (\tau^+ \rightarrow \nu_\tau \pi^+ n^0 \pi^0 \pi^0)}{\Gamma (\tau^+ \rightarrow \nu_\tau \pi^+ \pi^0 \pi^- \pi^0)} = 5. \] (43)

As to higher multiplicity final states, e.g. \( \tau \rightarrow \nu_\tau + 6\pi \), although there is no precise data available their contribution is thought to be negligible [32–33].

Finally, in order to obtain the vector spectral function through (42) we have used, as in [19],

\[
\text{BR}(\tau^+ \rightarrow \nu_\tau \pi^+ \pi^0) \approx \text{BR}(\tau^+ \rightarrow \nu_\tau \rho^+),
\]

\[
= (22.3 \pm 0.6 \pm 1.4)\%
\] (44)

and

\[
\text{BR}(\tau^+ \rightarrow \nu_\tau \pi^+ \pi^- \pi^0) = (4.5 \pm 0.4 \pm 1.5)\%. \] (45)

The final result for \( 1/\pi \text{Im}\Pi_v(s) \) is shown in Fig. 2 (curve a). We have found that the area under this spectral function is in excellent agreement with the area under the corresponding vector spectral function obtained from a fit to the \( e^+e^-\pi \) data [11], as expected from CVC considerations. This agreement holds over a wide region of \( s \) extending up to \( s = 2.5 \text{GeV}^2 \), which is quite enough for our purposes.

Turning to the axial-vector channel, the experimental invariant mass distribution for \( \tau^+ \rightarrow \nu_\tau \rho^0 \pi^\pm \) is shown in Fig. 1c. The solid curve in this figure is the result of the best chi-squared fit performed with a Breit–Wigner function times a high order polynomial [19]. The resulting value of \( \chi^2 \) is \( \chi^2 = 69 = 0.97 \). There is no experimental information available in the channel \( \tau^+ \rightarrow \nu_\tau \rho^0 \pi^0 \). However, given the rather strong \( A_1 \)-resonance dominance in \( \tau^+ \rightarrow \nu_\tau \rho^0 \pi^\pm \) (cf. [18, 34]) we can safely estimate the contribution of the former using isospin symmetry which simply entails \( \text{BR}(\tau^+ \rightarrow \nu_\tau \rho^0 \pi^\pm) = \text{BR}(\tau^+ \rightarrow \nu_\tau \rho^0 \pi^0) \). As with \( \tau \rightarrow \nu_\tau 6\pi \) we shall neglect here the channel \( \tau \rightarrow \nu_\tau 5\pi \).

A source of uncertainty in the extraction of \( \text{Im}\Pi_A(s) \) from \( d\Gamma /ds \) is due to the wide spread of experimental values for the branching ratio \( \text{BR}(\tau^+ \rightarrow \nu_\tau \rho^0 \pi^\pm) \) [33, 35]. We shall use here the world average [33, 35]

\[
\text{BR}(\tau^+ \rightarrow \nu_\tau \rho^0 \pi^\pm) = (6.6 \pm 0.6)\%. \] (46)

In addition, there may be a non-negligible background contribution to the decay \( \tau \rightarrow \nu_\tau + 3\pi \), i.e. [33–35]

\[
\text{BR}(\tau \rightarrow \nu_\tau + 3\pi)_{\text{Non-Res.}} \leq 1.4\%. \] (47)

We have used here a very simple parametrization of this background in terms of \( e^+e^- \) non-resonant phase space normalized to the value (47). The final result for the axial-vector spectral function is shown in Fig. 2 (curve b). Not shown there, of course, is the pion-pole contribution which in the chiral limit is given by

\[
\frac{1}{\pi} \text{Im}\Pi_A(s)_{\pi} = f_{\pi}^2 \delta(s), \] (48)

where \( f_{\pi} \approx 93.2 \text{MeV} \).

Before closing this section we point out that to the hadronic parametrizations of the spectral functions discussed above one should add the perturbative QCD piece starting at some threshold \( s_0 \), viz.

\[
\frac{1}{\pi} \text{Im}\Pi_A(s) = \frac{1}{\pi} \text{Im}\Pi_A(s)_{\text{had.}} + \theta(s - s_0) \frac{1}{8\pi^2} \left[ \frac{\alpha_s}{\pi} + \cdots \right]. \] (49)

5 Vacuum condensates from FESR

Using the fit to the vector spectral function discussed in Sect. 4 we have solved first the eigenvalue equation (26) to find the duality region. In Fig. 3 we plot the behaviour of the lhs of (26) (curve b) versus \( s_0 \). The solid curve (a) shows the corresponding behaviour of the rhs of (26). The duality region lies in the range

\[
1.44 \text{GeV}^2 \leq s_0 \leq 1.75 \text{GeV}^2 \] (50)

Solving next the FESR (22)–(23) for values of \( s_0 \) inside this region we find for \( C_4 \langle O_4 \rangle_V \) and \( C_6 \langle O_6 \rangle_V \) the results shown in Fig. 4 as curves (a) and (b), res-
Fig. 3. Behaviour of the hadronic integral in the rhs of the FESR (26) (curve a) together with the behaviour of the QCD lhs (curve b), in the vector channel.

Fig. 4. Behaviour of $C_4 \langle 0_4 \rangle_V$ (curve a), $C_6 \langle 0_6 \rangle_V$ (curve b), and $C_8 \langle 0_8 \rangle_V$ (curve c) as obtained from the FESR (22)-(23) in the vector channel.

Fig. 5. Behaviour of the hadronic integral in the rhs of the FESR (26) (curve a) together with the behaviour of the QCD lhs (curve b), in the axial-vector channel.

Fig. 6. Behaviour of $C_4 \langle 0_4 \rangle_A$ (curve a), $C_6 \langle 0_6 \rangle_A$ (curve b) and $C_8 \langle 0_8 \rangle_A$ (curve c) as obtained from the FESR (22)-(23) in the axial-vector channel.

respectively. Also shown there (curve (c)) is the calculated behaviour of $C_8 \langle 0_8 \rangle$, but we defer the discussion on this condensate till Sect. 8. As it may be appreciated from Fig. 4 the vacuum condensates are reasonably stable against changes in $s_0$ within the duality region. Numerically the results are

$$C_4 \langle 0_4 \rangle_V = (0.025 - 0.11) \text{GeV}^4$$

$$C_6 \langle 0_6 \rangle_V = -(0.16 - 0.32) \text{GeV}^6.$$  

Using (9) and (12) the above value of $C_4 \langle 0_4 \rangle_V$ translates into

$$\frac{\pi}{3} \langle \alpha_s G^2 \rangle = (0.03 - 0.12) \text{GeV}^4.$$  

This result is consistent with the standard value (3) as well as with higher values from other analysis [11-12]. However, it is barely consistent with a recent s-wave charmonium determination (cf. [7]) giving $\langle \alpha_s G^2 \rangle = (0.14 - 0.23) \text{GeV}^4$. Concerning the dimension $d = 6$ four-quark vacuum condensate our result (52) shows a clear deviation from the vacuum saturation approximation value (14), and is in agreement with $e^+ e^-$ results [9, 11-12].

Turning to the axial-vector channel and repeating the above analysis we find first that the duality region in this case is somewhat wider, as may be appreciated from Fig. 5, i.e.

$$1.75 \text{GeV}^2 \leq s_0 \leq 2.25 \text{GeV}^2.$$  

One may understand the result $s_0 \vert_A > s_0 \vert_V$ on account of $M_{A_1} > M_{A'}$. With a wider and better duality region one would expect the vacuum condensates to be more stable in $s_0$, and thus to be affected by a lesser uncertainty. This is indeed the case as shown in Fig. 6. The numerical results now read

$$C_4 \langle 0_4 \rangle_A = (0.045 - 0.10) \text{GeV}^4$$

$$C_6 \langle 0_6 \rangle_A = (0.16 - 0.28) \text{GeV}^6$$

and using (12) and (18)

$$\frac{\pi}{3} \langle \alpha_s G^2 \rangle = (0.04 - 0.09) \text{GeV}^4.$$  

It is rewarding that the use of two independent sets of experimental data lead to essentially the same value for the gluon condensate. This is due to the fact that the results we obtain for $C_4 \langle 0_4 \rangle_V$ and $C_4 \langle 0_4 \rangle_A$ satisfy (20). Notice furthermore that $C_8 \langle 0_8 \rangle_A$ has the same sign as expected from theoretical considerations (V.S.), cf. (21), although its value is a factor 2-3 larger than expected from the vacuum saturation approximation.
6. Vacuum condensates from Laplace transform sum rules

First of all we wish to elaborate on the point raised in Sect. 3.4, (ii), concerning the potential bias of Laplace transform determinations of vacuum condensates. Let us concentrate, for instance, on the vector channel and compute $R(a)$ in (29) using the data on the spectral function complemented with the continuum contribution starting at $s_0$ (49). The result is shown in Fig. 7 (solid curve) for values of $a$ inside the sum rule window $a \simeq (0.5-1.25) \text{GeV}^{-2}$. Suppose now that we were to set $C_6 \langle O_6 \rangle_V = C_8 \langle O_8 \rangle_V = \cdots = 0$ and fit $C_4 \langle O_4 \rangle_V$ in (30) so that $R(\sigma)_{\text{QCD}} \simeq R(\sigma)_{\text{DATA}}$ e.g. in a chi-squared sense. A very reasonable fit would then be obtained with $C_4 \langle O_4 \rangle_V = -(0.02 - 0.04) \text{GeV}^4$ as shown in Fig. 7 (broken curves). Since on very general grounds [1] $C_4 \langle O_4 \rangle_V > 0$ this result is meaningless, in spite of being the best fit to the Laplace ratio (29). Including next $C_6 \langle O_6 \rangle_V$ in the fit will change $C_4 \langle O_4 \rangle_V$ and lead to a pair of 'effective' values for these two condensates. These 'effective' values could be the true values if and only if the additional presence of $C_8 \langle O_8 \rangle_V$ would induce no sizable changes. As we shall argue in Sect. 8 this may well not be the case here. A similar observation has been made recently in [12]. Therefore, in our opinion Laplace transform sum rules should not be used in isolation but rather, in conjunction with other methods (e.g. FESR) to search for overall consistency. In short: they are a necessary but in general not a sufficient condition for consistency.

Following this philosophy, we have used the values of the condensates determined with FESR to compute $R(\sigma)_{\text{QCD}}$ (30), and confronted it to $R(\sigma)_{\text{DATA}}$ obtained from the data through (29). In Fig. 8 we show $R(\sigma)_{\text{QCD}}$ in the vector channel for $C_4 \langle O_4 \rangle_V = 0.025 \text{GeV}^4$, $C_6 \langle O_6 \rangle_V = -0.16 \text{GeV}^6$ (upper broken curve) and $C_4 \langle O_4 \rangle_V = 0.025 \text{GeV}^4$ (lower broken curve), both for $C_8 \langle O_8 \rangle_V = \cdots = 0$. The solid curve in this figure corresponds to $R(\sigma)_{\text{DATA}}$. For $s_0$ we have used values inside the duality region (50). Other combinations of eigenvalue solutions to the FESR lead to equally acceptable results for $R(\sigma)_{\text{QCD}}$ with the exception of $C_6 \langle O_6 \rangle_V$ values near the high end of (52). In view of our previous discussion, and since we are neglecting $C_8 \langle O_8 \rangle_V$, we do not ascribe much significance to this discrepancy.

In Fig. 9a we show $R(\sigma)_{\text{QCD}}$ in the axial-vector channel for $C_4 \langle O_4 \rangle_A = 0.045 \text{GeV}^4$, $C_6 \langle O_6 \rangle_A = 0.1 \text{GeV}^6$ (upper broken curve) and $C_6 \langle O_6 \rangle_A = 0.2 \text{GeV}^6$ (lower broken curve) together with $R(\sigma)_{\text{DATA}}$ (solid curve). Using the higher value in (55), i.e. $C_4 \langle O_4 \rangle_A = 0.1 \text{GeV}^4$, would require that $C_6 \langle O_6 \rangle_A \simeq (0 - 0.1) \text{GeV}^6$ in order to get agreement with $R(\sigma)_{\text{DATA}}$ as shown in Fig. 9b. In fact, an almost perfect fit would be obtained if $C_6 \langle O_6 \rangle_A = 0.05 \text{GeV}^6$. 

---

![Fig. 7](image1.png)

**Fig. 7.** The solid curve shows the behaviour of the vector Laplace ratio $R(\sigma)_{\text{DATA}}$. The upper (lower) broken curves correspond to $R(\sigma)_{\text{QCD}}$ for $C_4 \langle O_4 \rangle_V = C_8 \langle O_8 \rangle_V = 0$ and $C_4 \langle O_4 \rangle_V = -0.04 \text{GeV}^4(-0.02 \text{GeV}^4)$.

![Fig. 8](image2.png)

**Fig. 8.** The solid curve is the same as in Fig. 7. The upper (lower) broken curves correspond to $R(\sigma)_{\text{QCD}}$ for $C_4 \langle O_4 \rangle_V = 0$, and $C_4 \langle O_4 \rangle_V = -0.04 \text{GeV}^4 (-0.02 \text{GeV}^4)$.

![Fig. 9a](image3.png)

**Fig. 9a.** The solid curve shows the behaviour of the axial-vector Laplace ratio $R(\sigma)_{\text{DATA}}$. The upper (lower) broken curves correspond to $R(\sigma)_{\text{QCD}}$ for $C_6 \langle O_6 \rangle_A = 0.1 \text{GeV}^6$, $C_8 \langle O_8 \rangle_A = 0$ and $C_6 \langle O_6 \rangle_A = 0.025 \text{GeV}^4$.

![Fig. 9b](image4.png)

**Fig. 9b.** The solid curve is the same as in Fig. 7. The broken curves correspond to $R(\sigma)_{\text{QCD}}$ for $C_4 \langle O_4 \rangle_A = 0.045 \text{GeV}^4$, $C_6 \langle O_6 \rangle_A = 0$ and $C_6 \langle O_6 \rangle_A = 0.025 \text{GeV}^4$. 

---
However, such a value is well outside the duality range (56).

In summary, with some exceptions the eigenvalue solutions to the FESR satisfy also the Laplace transform QCD sum rules. As noted above, these exceptions correspond to values of $C_6\langle O_6\rangle_V$ near the high end of (52), and to the value $C_4\langle O_4\rangle_A \approx 0.1$ GeV$^4$ which appears to require a $C_6\langle O_6\rangle_A$ outside the duality range (56). If one had sound theoretical arguments to support the assumption that $C_6\langle O_6\rangle_{V,A}$ is negligible, then this confrontation between FESR and Laplace transform results could be useful to narrow down the range of the predictions. However, we shall argue in Sect. 8 that this does not seem to be the case and, furthermore, that the above discrepancies disappear almost entirely once $C_8\langle O_8\rangle_{V,A}$ is included in $R(\sigma)_{QCD}$.

7 Heat evolution tests

In this section we perform the heat evolution tests [10] in order to check if there is duality between the hadronic data and the set of QCD parameters determined previously from the sum rules. To this end we have computed, from (31), (34)–(35) and (49), the Gaussian transforms $U_+(t,z)_{DATA}$ (upper solid curve) and $U_-(t,z)_{DATA}$ (lower solid curve), in the vector channel, at a 'time' $t=0.5$ GeV$^4$. The upper broken curve corresponds to $U_+(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = -0.16$ GeV$^6$ and the lower broken curve to $U_-(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = 0$. Same as a but at a later 'time' $t=1.5$ GeV$^4$. For longer 'times' the corresponding solid and broken curves overlap completely.

Fig. 10. a The Gaussian transforms $U_+(t,z)_{DATA}$ (upper solid curve) and $U_-(t,z)_{DATA}$ (lower solid curve), in the vector channel, at a 'time' $t=0.5$ GeV$^4$. The upper broken curve corresponds to $U_+(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = -0.16$ GeV$^6$ and the lower broken curve to $U_-(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = 0$. b Same as a but at a later 'time' $t=1.5$ GeV$^4$. For longer 'times' the corresponding solid and broken curves overlap completely.

Fig. 11. A comparison between $U_+(t,z)_{DATA}$ in the vector channel (solid curve) and $U_+(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = 0$, at $t = 1.5$ GeV$^4$.

Fig. 12. a The Gaussian transforms $U_+(t,z)_{DATA}$ (upper solid curve) and $U_-(t,z)_{DATA}$ (lower solid curve) in the axial-vector channel, at a 'time' $t=0.5$ GeV$^4$. The upper broken curve corresponds to $U_+(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = 0.16$ GeV$^6$ and the lower broken curve to $U_-(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = -0.045$ GeV$^6$. b Same as a but at a later 'time' $t=1.5$ GeV$^4$. For longer 'times' the corresponding solid and broken curves overlap completely.

Fig. 10. a The Gaussian transforms $U_+(t,z)_{DATA}$ (upper solid curve) and $U_-(t,z)_{DATA}$ (lower solid curve), in the vector channel, at a 'time' $t=0.5$ GeV$^4$. The upper broken curve corresponds to $U_+(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = -0.16$ GeV$^6$ and the lower broken curve to $U_-(t,z)_{QCD}$ for $C_6\langle O_6\rangle_\nu = 0$. b Same as a but at a later 'time' $t=1.5$ GeV$^4$. For longer 'times' the corresponding solid and broken curves overlap completely.

Gaussian transforms $U_+(t,z)_{DATA}$ using the parametrization of the vector and axial-vector spectral functions discussed in Sect. 4, as well as $U_+(t,z)_{QCD}$ (36)–(37), for values of $s_0$ inside the duality regions (50), (54), and vacuum condensates as in (51), (52), (55) and (56). In Fig. 10a and b we show $U_+(t,z)_{DATA}$ (solid curves) together with $U_+(t,z)_{QCD}$ (broken curves) for $C_6\langle O_6\rangle_\nu = 0.11$ GeV$^4$ and $C_6\langle O_6\rangle_\nu = -0.16$ GeV$^6$, at the two different 'times' $t=0.5$ GeV$^4$ and $t=1.5$ GeV$^4$, respectively. As it may be appreciated from these figures, with increasing 'time' the heat

Gaussian transforms $U_+(t,z)_{DATA}$ using the parametrization of the vector and axial-vector spectral functions discussed in Sect. 4, as well as $U_+(t,z)_{QCD}$ (36)–(37), for values of $s_0$ inside the duality regions (50), (54), and vacuum condensates as in (51), (52), (55) and (56). In Fig. 10a and b we show $U_+(t,z)_{DATA}$ (solid curves) together with $U_+(t,z)_{QCD}$ (broken curves) for $C_6\langle O_6\rangle_\nu = 0.11$ GeV$^4$ and $C_6\langle O_6\rangle_\nu = -0.16$ GeV$^6$, at the two different 'times' $t=0.5$ GeV$^4$ and $t=1.5$ GeV$^4$, respectively. As it may be appreciated from these figures, with increasing 'time' the heat...
distribution of the data matches the theoretical heat distribution computed with the above values of the QCD parameters. For later 'times', i.e. \( \tau > 1.5 \text{GeV}^4 \), this matching improves to the point that the two curves cannot be distinguished. As an example of a set of QCD parameters which do not pass this test we show in Fig. 11 the function \( U^\prime(\tilde{s}, \tau)_{\text{QCD}} \) for \( C_6 <O_6>_V = 0 \) at \( \tau = 1.5 \text{GeV}^4 \). Recall that this case gave a good fit to the Laplace ratio (29) with a negative value of \( C_8 <O_8>_V \) (cf. Sect. 6). Repeating the procedure for other values of the vacuum condensates inside the duality region we find equally good heat evolutions with the exception of \( C_6 <O_6>_V \) values near the high end of (52). A relative qualitative comparison among all possibilities confirms the result (51) but narrows down the range (52) to

\[
C_6 <O_6>_V = -(0.16 - 0.24) \text{GeV}^6.
\] (58)

Notice that for \( \tilde{x} = \tilde{s}/2\sqrt{\tilde{\tau}} \) large, the vacuum condensates have no practical impact on \( U^\prime(\tilde{s}, \tau) \) due to their exponential suppression (cf. eqs. (36)-(37)). This region of large \( \tilde{x} \) one is then testing duality between \( s_0 \) and perturbative QCD. The rate of approach to the asymptotic limit

\[
\lim_{\tilde{s} \to \infty} U^\prime(\tilde{s}, \tau) = \frac{1}{8\pi^2}
\] (59)

depends, of course, on the value of \( \tau \) as may be appreciated e.g. from Fig. 10a and b.

Turning to the axial-vector channel, we show in Fig. 12a and b \( U^\prime(\tilde{s}, \tau)_{\text{DATA}} \) together with the corresponding QCD transforms for \( C_4 <O_4>_A = 0.045 \text{GeV}^4 \) and \( C_6 <O_6>_A = 0.16 \text{GeV}^6 \) at the 'times' \( \tau = 0.5 \text{GeV}^4 \) and \( \tau = 1.5 \text{GeV}^4 \), respectively. Heat evolutions for other values of the condensates in the duality range (55)-(56) were also carried out. They are of comparable good quality and thus confirm the sum rule results.

8 On the value of \( C_8 <O_8> \)

In principle the FESR (23) with \( N = 1 \) may be used to determine the numerical value of \( C_8 <O_8>_V,A \). However, given the rather high weight of the spectral function in this case one would expect the accuracy of the results to be somewhat limited. With this due reservation in mind we have solved eq. (23) in the vector and axial-vector channels for values of \( s_0 \) inside the duality regions (50) and (54). The results for \( C_8 <O_8>_V \) and \( C_8 <O_8>_A \) are exhibited in Figs. 4 and 6 (curve (c)), respectively. Numerically they read

\[
C_8 <O_8>_V = (0.28 - 0.55) \text{GeV}^8
\] (60)
\[
C_8 <O_8>_A = -(0.36 - 0.54) \text{GeV}^8.
\] (61)

As with \( C_4 <O_4>_A \) and \( C_6 <O_6>_V \), the result for \( C_8 <O_8>_A \) in the axial-vector channel is more stable and less inaccurate than that in the vector channel. This is a reflection of the quality of the corresponding duality regions as may be appreciated from Figs. 3 and 5. To remain on the safe side we proceed with our discussion choosing the lowest eigenvalue in (60)-(61).

In order to show the influence of \( C_8 <O_8>_A \) on the Laplace analysis we plot in Fig. 13 \( R(\sigma)_{\text{DATA}} \) (solid curve) and \( R(\sigma)_{\text{QCD}} \) for \( C_4 <O_4>_V = 0.025 \text{GeV}^4 \), \( C_6 <O_6>_V = -0.16 \text{GeV}^6 \), \( C_8 <O_8>_V = 0 \) (upper broken curve) and \( C_8 <O_8>_V = 0.25 \text{GeV}^8 \) (lower broken curve)
proved by the presence of $C_8(O_8)_V$. Concerning the heat evolution tests, a remarkable amelioration of the matching between $U_{V}(\xi, \tau)|_{OATA}$ and $U_{V}(\xi, \tau)|_{QCD}$ is achieved, already at early 'times', when one includes the contribution from $C_8(O_8)_V$, as seen from Fig. 14 for $\tau = 0.5 \text{GeV}^4$. The good agreement found earlier between $U_{V}(\xi, \tau)|_{OATA}$ and $U_{V}(\xi, \tau)|_{QCD}$ (with $C_8(O_8)_V = 0$) at later 'times', e.g. $\tau = 1.5 \text{GeV}^4$, is less affected by $C_8(O_8)_V$ on account of the $\tau$-suppression (see (37)).

Similar improvements take place in the axial-vector channel as may be appreciated from Fig. 15 where $R_0(\xi)|_{OATA}$ has been computed using $C_4(O_4)_A = 0.045 \text{GeV}^4$, $C_6(O_6)_A = 0.28 \text{GeV}^6$, and $C_8(O_8)_A = 0$ (lower broken curve) and $C_8(O_8)_A = -0.36 \text{GeV}^8$ (upper broken curve). In Fig. 16 we show $U_{A}(\xi, \tau)|_{DATA}$ at $\tau = 0.5 \text{GeV}^4$ (solid curve) compared with $U_{A}(\xi, \tau)|_{QCD}$ for $C_4(O_4)_A$ as above and $C_8(O_8)_A = -0.36 \text{GeV}^8$. This overall improvement caused by the inclusion of $C_8(O_8)_V$, as calculated from FESR, in the Laplace and Gaussian transforms should not come as a surprise if one keeps in mind that FESR follow from the latter.

In addition to what has just been pointed out, the rather large values of $C_8(O_8)_V,A$ derived through FESR may have important phenomenological consequences in applications of QCD sum rules to other channels. To see how much the results (60)–(61) deviate from current theoretical estimates let us concentrate first on the vector-channel. The pure gluonic piece of $C_8(O_8)_V$, as calculated in [24], can be casted as follows

$$C_8(O_8)_V|_{\text{gluons}} = \frac{1}{\pi^2} [ -4 \langle \hat{\Theta}_1 \rangle - 196 \langle \hat{\Theta}_2 \rangle - 292 \langle \hat{\Theta}_3 \rangle + 540 \langle \hat{\Theta}_4 \rangle ]$$

with

$$\hat{G}_{\mu\nu} = G_{\mu\nu} \frac{q^2}{2}.$$  

A reference estimate of (62) may be obtained using the vacuum saturation approximation [1, 36] in which case

$$C_8(O_8)_V|_{\text{gluons}} \approx - \mathcal{O}(10^{-3} - 10^{-2}) \text{GeV}^8$$

where (53), (57) were used. However, it should be kept in mind that there are no theoretical arguments to support this approximation [37–38] and, in fact, some authors have claimed deviations of up to factors of ten [37, 39]. In any case, notice that even the sign of (68) disagrees with the FESR result (60). The huge factor needed to reconcile (68) with (60), though, does not necessarily translate into huge deviations from vacuum saturation for the individual matrix elements $\langle O_{1-4} \rangle$ as these appear in the combination (62). As easily checked, it would be enough to admit e.g. individual mild deviations, but with appropriate signs, to recover (60).

The most serious problem lies in the relative sign difference between (60) and (61), as the pure gluonic contributions to $C_8(O_8)_V$ should be the same in the vector and axial-vector channels. This may be an indication that the contribution from the other operators which appear in the complete expression for $C_8(O_8)_V$, usually assumed to be non-leading, might be important. There are, in fact, three additional operators involving gluon fields together with the source current $j^a_{\mu} = D_{\mu} G_{ab}$, as well as ten quark operators. Although the Wilson coefficients for the vector case have been computed in [24] this is not enough as there are no theoretical estimates for many of these matrix elements.

The values of the $d = 8$ vacuum condensates (60)–(61) are, of course, meant to be the complete result, i.e. they include the contribution from all possible 17 operators. In view of their stability in $s_0$ and of the improvement achieved in the Laplace sum rules as well as in the heat evolution tests, both for the vector and axial-vector channels, we feel that these results should be reliable to some reasonable extent.

9 Conclusions

Available experimental data on semileptonic decays of the $\tau$-lepton, $\tau \rightarrow v_\tau + n \pi$, allows for a determination of the vector and axial-vector spectral functions (cf. Sect. 4). This information has been used here in order to extract the values of the leading vacuum condensates appearing in the Operator Product Expansion of current correlators. Our technical procedure was based on a combined use of FESR, Laplace and Gaussian transform QCD sum rules. In this fashion we have been able to perform valuable consistency checks and verify explicitly the duality interplay
between the hadronic data and the QCD parameters so determined. In Table 1 we collect our results and show the standard values (3), (14) and (21) for comparison. It is rewarding to find that the use of two independent sets of experimental data lead to essentially the same result for the gluon condensate. The latter is consistent with the standard value (3) as well as with higher values derived from $e^+e^-$ data [11–12] and other sources [2, 6, 13–14], but it is barely consistent with a recent s-wave charmonium determination [7]. Our value for the dimension $d = 6$ four-quark condensate $C_6\langle O_6 \rangle_v$, though, indicates a clear deviation from the standard vacuum saturation approximation, in agreement with earlier claims [9, 11–12, 15–17]. On the other hand, our result for $C_6\langle O_6 \rangle_A$ has the same sign as expected from theoretical considerations (V.S.), cf. (21), but its value shows also a sizable deviation from vacuum saturation (for a lucid and detailed presentation of the vacuum saturation approximation see [40]).

We have also determined $C_8\langle O_8 \rangle_{v,A}$ using FESR and found that its inclusion in the Laplace and Gaussian transforms leads to an improved duality between QCD and the data. However, in view of the rather high weight of the spectral function one should not expect the same level of accuracy as for lower dimensional condensates. In any case, we find a clear sign difference between $C_8\langle O_8 \rangle_v$ and $C_8\langle O_8 \rangle_A$, which is also supported by the stability of the results. This could be an indication that the pure four-gluon operator contributions to $C_8\langle O_8 \rangle$ may not be leading. In view of its potential phenomenological consequences, this possibility deserves a separate theoretical study which is, however, beyond the scope of this paper.

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