Nuclear Physics B302 (1988) 645-667
North-Holland, Amsterdam

COSMOLOGIES WITH VARIABLE NEWTON'S "CONSTANT"

C. WETTERICH

Deutsches Elektronen Synchrotron, DESY Hamburg, Fed. Rep. Germany

Received 24 September 1987

We discuss cosmologies where the Planck length is not a fundamental constant but rather evolves with time. The dynamics which should be responsible for today's tiny value of this length scale are governed by the effective potential of a Brans-Dicke type theory. Qualitative properties of this potential depend on the short distance behaviour of the unifying fundamental theory. We discuss criteria for the asymptotic behaviour of realistic cosmologies and show that the role of a possible cosmological constant is quite different from the case of standard cosmology.

1. Introduction

In our present understanding of fundamental laws of nature we observe mass scales of two different orders of magnitude. On the one side there are the mass scales characteristic for the standard $SU(3) \times SU(2) \times U(1)$ model which we may identify with the QCD scale Λ_{QCD} and the W-boson mass M_W. But we also know that much higher mass scales appear in our world: The Planck mass M_P, characteristic for gravity, is 17 orders of magnitude bigger than M_W. The observed baryon asymmetry in the universe together with the observed stability of the proton suggests a scale M_X characteristic for baryon number violation which should not be too much below M_P. Also the smallness of neutrino masses (if they are not zero) could be explained by a high scale characteristic for lepton number (or $B-L$) violation. The approach to this problem adopted most frequently takes M_P as a fundamental (intrinsic) mass scale of the theory. The question then arises why other scales like M_W are so much smaller than M_P. In the gravitational sector one needs to solve the cosmological constant problem, i.e. to explain why today's value of the Hubble parameter H_0 is much smaller than M_W^2/M_P (or even M_P).

In this paper we investigate the alternative approach where M_P is not a fundamental mass parameter but rather a property of today's state of the world, typically

0550-3213/88/$03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

given by a vacuum expectation value (vev) of some scalar field. Its present value may then be a consequence of the evolution of the universe, a hypothesis proposed long ago by Dirac [1]. Such a theory may have some other fundamental mass scale m which we suppose to be much smaller than M_P. As a typical example m could be given by Λ_{QCD} or M_W, but it could even be much smaller. Alternatively the theory may have no intrinsic mass scale. In the latter case dilatation (scale) symmetry can only be broken spontaneously. For both alternatives M_P is given by the present value of a scalar field χ which is much bigger than m and therefore not directly related to an intrinsic scale. Typical "initial" conditions for cosmology do not require anymore that all scales are of order M_P. The scales for initial conditions would rather be given by m or even be completely random. A central question in this scenario is the following: How does χ evolve with time so that its present value is M_P?

The most popular proposal in this direction is the model of Brans and Dicke [2]. (See also the description in ref. [3].) This model assumes a massless scalar χ which has no self-interactions or interactions with other fields except for the graviton. The Einstein-Hilbert term in the action $(M_P^2/16\pi)R$ is replaced by $\chi^2 R$ and a kinetic term for χ is supplemented. In the light of modern particle theory and its unification with gravity, however, there seems to be no good reason why χ should not be interacting. In general, we expect some potential, $V(\tilde{\varphi}_i, \chi)$, where $\tilde{\varphi}_i$ denotes the degrees of freedom of the low-energy model like the Higgs doublet. It is not surprising that the combined cosmology for gravity and χ will crucially depend on the form of this potential. The purpose of this paper is a study of cosmologies with field dependent Newton's constant in presence of a potential V.

From the point of view of the low-energy standard SU(3) × SU(2) × U(1) model the field χ plays the role of a physical cutoff scale. For energies beyond this scale physics is expected to change dramatically. For example, a much higher symmetry may become visible. Actually, the standard model very probably needs some physical cutoff as an implication of the triviality of φ^4-theory (which very likely applies to the weak Higgs sector and perhaps to the abelian U(1) theory*). In absence of any experimental results indicating new (intermediate) scales, it seems not unnatural to identify χ with this physical cutoff. What we have in mind is a model with a *variable* cutoff length l. For example, in the lattice regularization of the standard model this would mean that the lattice distance l becomes a dynamical degree of freedom. For most purposes the field $\chi \sim l^{-1}$ may be thought of as a background field. Its evolution must be such that l becomes much smaller than M_W^{-1}.

What are the dynamics of the cutoff length l resulting from its interaction with the standard model degrees of freedom $\tilde{\varphi}_i$? First we expect nonrenormalizable interactions of the type $l^2\tilde{\varphi}^6$ or $l^2(\bar{\psi}\psi)^2$ with $\tilde{\varphi}$ the Higgs doublet and ψ a fermion.

* For a thorough discussion of triviality and its implications see ref. [4].

They correspond to the "finite volume effects" for the lattice or the baryon number violating four fermion interactions in grand unified models. More general, such interactions are proportional l^P, $P>0$, and vanish for $l \to 0$. Next there can be logarithmic contributions like $\tilde{\varphi}^4 \ln(l\tilde{\varphi})$ from the running of dimensionless coupling constants. Finally, the symmetry also allows a mass type term $l^{-2}\tilde{\varphi}^2$ for the Higgs doublet. Terms proportional l^{-P} are dangerous. They blow up for small l and tend to make any small value $(l\tilde{\varphi}) \ll 1$ impossible. Such terms must be absent for a realistic model (or at least their coefficient must be very small). To a good approximation $V(\tilde{\varphi}_i, l)$ should be finite or at most logarithmically divergent for $l \to 0$.

We are interested in a model where for small enough l (so that l^P corrections are negligible) the observed $SU(3) \times SU(2) \times U(1)$ model becomes a renormalizable theory without any mass like terms $\sim l^{-P}$. In contrast to the usual renormalization procedure where l is only a technical device, this becomes in our context a strong physical assumption on the relative decoupling of the short distance degrees of freedom. It is closely related to the existence of a gauge hierarchy. We will adopt this assumption throughout most of this paper. We also note another difference from usual renormalization, namely that the renormalized dimensionless couplings at some fixed low-energy scale may now depend on l.

As an example of a physical cutoff length l one may consider higher dimensional theories. Here the degree of freedom l is identified with some characteristic length scale of the internal space. Our assumption implies a relative decoupling of the low energy degrees of freedom from the short distance degrees of freedom which are connected with the dynamics of the internal space. In this sense the low-energy world should "lose its memory" about the scale of the internal space (up to possible logarithmic corrections). One should note that l is not necessarily simply the volume degree of freedom of the internal space. The decoupling of the low-energy degrees of freedom may obtain as a consequence of an internal space changing also its shape as l varies. Rough properties of the low-energy world like gauge symmetries and quantum numbers of chiral fermions do not depend on detailed properties of the internal space but follow from isometries and index considerations [5]. The dimensionless couplings, however, will depend on the shape. This may contribute to the dependence of these couplings on l.

In a more general context we may consider some fundamental theory which is a system of infinitely many degrees of freedom (as, for example, strings). It may nevertheless be possible to describe the physics of our world by much less properly chosen degrees of freedom, in a way that all other additional degrees of freedom only describe small corrections, negligible for most purposes. This is related to the concept of universality classes in statistical mechanics. Universality classes are usually characterized by symmetries and a few other basic properties. We know – even if we don't understand why – that our world belongs to a universality class characterized by the symmetry $\text{gen}_4 \times SU(3) \times SU(2) \times U(1)$ in a phase where

SU(3) is confined and SU(2) × U(1) is spontaneously broken to $U(1)_{em}$. (Here gen_4 denotes general coordinate and Lorentz transformations in four dimensions.) As a consequence of the symmetries, such a universality class should have a description in terms of four-dimensional space time with a finite number of fields which include the gauge bosons and the Higgs doublet. In addition the fermionic content of this universality class must be specified – the quantum numbers of chiral fermions. The interactions among these fields should be renormalizable in the limit where the other "irrelevant" degrees of freedom can be neglected.

If we study the dynamical evolution of possible states the "ground state" associated with a certain universality class may be viewed as a fixed point in the space of states which is approached asymptotically by many trajectories. (As an example from statistical mechanics the ground state could correspond to the thermodynamic equilibrium state at the critical temperature of a second order phase transition.) For a description of the *approach* to the ground state, however, one also needs to consider the "irrelevant" degrees of freedom which only decouple in the asymptotic ground state. (For the statistical analogue this would be the transition from non-equilibrium to equilibrium.) The decoupling of degrees of freedom in a general field theory is most easily described by some mass scale (l^{-1}) growing huge compared to the mass scales characteristic for the ground state (Λ_{QCD}, M_W). This rises the mass of the irrelevant degrees of freedom compared to the modes of the standard model. In this sense our study of the evolution of l is a study on how the irrelevant degrees of freedom decouple dynamically during the approach to the ground state characteristic for the gen_4 × SU(3) × SU(2) × U(1) universality class.

Actually, there are two ways how degrees of freedom can decouple for $l \to 0$. For most irrelevant modes the mass will grow $\sim l^{-1}$. This results in *local* non-renormalizable effective interactions between the modes of the standard model. As a second possibility the mass of such a particle may be zero or remain small for $l \to 0$, whereas its couplings to the modes of the standard model vanish for $l \to 0$. This happens if no renormalizable couplings are permitted by the symmetry. In particular, this is the case for the graviton which decouples for $M_P \to \infty$. Massless modes cannot be eliminated in favour of local nonrenormalizable interactions between the standard model modes. They must often be kept for the description of low-energy phenomena. Despite their small coupling strength they can play a role for physics at very long distances.

In our approach the length l itself is one of the irrelevant degrees of freedom. It is not clear a priori if it belongs to the first category (with mass $\sim l^{-1}$) or to the second one. Since l is a scalar and a singlet under SU(3) × SU(2) × U(1), only one renormalizable coupling with the modes of the standard model is allowed by the symmetry, namely the coupling with the Higgs doublet of the form $l^{-2}\tilde{\varphi}^2$. This coupling (unless it is tiny) is in contradiction with the existence of our world being described by the gen_4 × SU(3) × SU(2) × U(1) universality class and should therefore be discarded. If the mode l has mass $\sim l^{-1}$ it will only play a role during very early cosmology, for example in higher dimensional models of inflation [6]. In contrast, if this mode remains light it possibly can also influence late cosmology and may be related to the fate of the cosmological constant. In this paper we investigate this latter scenario.

As we have already mentioned the very assumption of our world being described by a $\mathrm{gen}_4 \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ universality class implies that the product lM_W must be very small. There are in principle two possibilities for the asymptotic behaviour: either $lM_W \to 0$ or lM_W approaches a very small constant. In both cases the small value lM_W implies the existence of a gauge hierarchy! In a sense, the gauge hierarchy problem is now turned upside down: We do not start from a short distance theory and try to explain the occurrence of small mass scales. We rather start from the observed standard model with scales Λ_{QCD} and M_W and try to describe how the other degrees of freedom of a more fundamental theory decouple as a result of dynamics. One may wonder if the assumption of a $\mathrm{gen}_4 \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ universality class also implies a solution to the cosmological constant problem since for $l \to 0$ gravity should decouple and the ground state should be flat Minkowski space. There is unfortunately no such automatic solution of the cosmological constant problem. Since $M_P \sim l^{-1}$ the curvature scalar $\tilde{R}$ will indeed be proportional $l^2 m^4$ with m some suitable scale of the low energy model. It vanishes for $l \to 0$ as it should. But a value $m \sim M_W$ gives a value for $\tilde{R}$ more than 50 orders of magnitude bigger than observed. Additional physics is therefore needed for an understanding of the smallness of the cosmological constant. We will study this question in detail.

To summarize this discussion we hope that we have convinced the reader that the potential $V(\tilde{\varphi}_i, l)$ can never be derived from the standard model alone. Its properties reflect the way how our observed world emerges from some more fundamental theory. Our study therefore needs a guess how the decoupling of the irrelevant degrees of freedom might be realized. This situation has an important consequence for our understanding of cosmology: The potential $V(\tilde{\varphi}_i, l)$ can crucially influence even late cosmology, but it cannot be determined from covariance arguments plus standard model physics alone. Additional unknown physics is needed and we should not be too surprised if the way how the standard model arises from a fundamental theory leads to observable deviations from the standard hot big bang cosmology even at a late stage of its evolution!

In sect. 2 we give the action for the coupled system of a low energy degree of freedom $\tilde{\varphi}$ (which we take the scalar doublet), the physical cutoff scale $\chi = l^{-1}$ and gravity. We derive the coupled system of field equations in presence of matter or radiation. We concentrate on the special case of a Roberson-Walker metric with $k = 0$. In sect. 3 we discuss as a first most simple example for a nonvanishing potential the asymptotic solutions of the Brans-Dicke theory with cosmological constant. This corresponds to a potential $V(\tilde{\varphi}_0, l) = V_0$ for $\partial V/\partial\tilde{\varphi}(\tilde{\varphi}_0, l) = 0$. In contrast to the unacceptable exponential behaviour of the scale factor $a(t)$ for standard cosmology with cosmological constant, we find now a power law expansion for $a(t)$ even for $V_0 > 0$! The Hubble parameter decreases like ηt^{-1}. This demonstrates the possible drastic differences between cosmologies with fixed M_P or with $M_P \sim \chi$ in case of a nonvanishing potential. A nonzero V_0 is much less harmful if the Planck mass is given by a dynamical degree of freedom.

In sect. 4 we develop an important alternative language for the description of our models which is related to Weyl scaling. This version is formulated in terms of ratios like $\tilde{\varphi}/\chi$, $\tilde{g}^{\mu\nu}/\chi^2$, $\tilde{R}/\chi^2$. The Planck mass appears now as a constant. In this version the short distance degree of freedom with properly normalized kinetic term is $\sigma \sim \ln \chi$. Also the effective potential is modified. It has a typical exponential dependence on σ. The description of cosmology is closer to a standard gravity theory in this language. The new physics associated with the dynamical Planck mass appears now in the specific form of the effective potential. Of course, the Weyl scaled formulation is strictly equivalent to the formulation of sect. 2. In sect. 5 we formulate criteria for a realistic overall behaviour of cosmology based on the observed properties of late cosmology and on the successful description of nucleosynthesis and background radiation by the standard hot big bang model. We discuss the solutions of sect. 3 with respect to these criteria. More general potentials and the corresponding cosmologies are discussed in a subsequent paper [7] with emphasis on the fate of dilatation symmetry [8].

2. Variable short distance scale

As a convenient formalism for the discussion of this paper we will use the effective action for the coupled system of the Higgs doublet ϕ, the short distance length l and the graviton $\tilde{g}_{\mu\nu}$. These fields may be considered as background fields. The effective action is thought to be obtained in the usual way by integrating the quantum fluctuations in presence of sources and performing a Legendre transformation. (Of course, for the gravity sector this assumes the existence of a consistent quantum theory of gravity.) We neither know the fundamental theory nor are we able to solve a complex quantum theory. Both would be necessary to compute the effective action. For our discussion we mainly use the symmetry properties of the effective action, scale arguments and a few qualitative assumptions on the l dependence of the effective potential. This will be sufficient to establish the asymptotic behavior of cosmology. The cosmological equations are the classical field equations obtained from a variation of the effective action. Our treatment can easily be generalized to include other degrees of freedom of the standard model as for example the effective σ-model for QCD at small momentum. Alternatively, one can imagine that the QCD degrees of freedom are integrated out and their effects included in the effective action. We will adopt the latter approach in order to keep the discussion simple.

We write the effective action in the form

$$S = S^0 + S^M,$$

$$S^0 = -\int d^4x\, \tilde{g}^{1/2}\left\{ l^{-2}\tilde{R} - 4\omega l^{-4}\,\partial_\mu l\,\partial^\mu l - \partial_\mu\phi^\dagger\,\partial^\mu\phi + V(\phi, l)\right\}. \tag{2.1}$$

By our assumption we should obtain a standard renormalizable SU(3) × SU(2) × U(1) field theory including the weak doublet for $l \ll M_W^{-1}$. In this limit S should become the corresponding effective action for ϕ on flat space with S^0 containing the standard kinetic term and the effective potential V. In particular, V should remain finite (or diverge at most logarithmically for $l \to 0$). It should therefore not contain terms $\sim l^{-P}$, $P > 0$. Gravity is a nonrenormalizable interaction, and its coupling, Newton's "constant", should vanish for $l \to 0$. We normalize l so that the coefficient of the curvature scalar in S^0 (which is the inverse of Newtons "constant" divided by 16π) is l^{-2}. We want tc consider the physical cutoff length l as a dynamical degree of freedom and there should be a kinetic term for l. The most general term not involving more than two derivatives would be $f(l,\phi)\partial_\mu l\,\partial^\mu l$. Neglecting the dependence on ϕ the only possible function not involving mass scales is $f \sim l^{-4}$. (If l is interpreted as an internal length scale of a higher dimensional theory, kinetic terms $\sim l^{-4}\partial_\mu l\,\partial^\mu l$ are indeed obtained from dimensional reduction of invariants with dimensionless couplings.) In order to ensure stability of Minkowski space for $l \to 0$ we require

$$\omega > -\tfrac{3}{2}. \tag{2.2}$$

(This condition will become more apparent later.) The effective potential V may in general depend on l and we will discuss various possibilities. V is the only piece in S_0 which could contain mass scales. According to our assumptions V either contains no mass scale at all or its mass scales are much smaller than the Planck mass. In general, the effective action will only be approximated by S^0 and we collect all other pieces in S^M. S^M contains all higher derivative terms (including non-local terms) in the effective action as well as (ϕ dependent) deviations of $f(l,\phi)$ from $4\omega l^{-4}$. We also note that for cosmology the effective action has to be evaluated on a background with nonvanishing entropy, for example in thermo-dynamical equilibrium at temperature T. We formally include all such effects either in V or in S^M. Both V and S^M may therefore in general depend on other (time-dependent) functions characterizing the state of the universe, like temperature T, energy density $\tilde{\rho}$, pressure $\tilde{p}$ or classical expectation values of other fields.

We may bring S^0 into a more standard form by using

$$\chi = l^{-1}. \tag{2.3}$$

For the doublet field ϕ we write

$$\phi(x)=\exp\left(i\,\boldsymbol{\alpha}(x)\frac{\boldsymbol{\tau}}{2}\right)\tilde{\varphi}(x) \tag{2.4}$$

with $\tilde{\varphi}(x)$ the (real) modulus of ϕ. We will neglect the degrees of freedom $\boldsymbol{\alpha}(x)$. The effective action S^0 for χ and $\tilde{\varphi}$ reads

$$S^0=-\int \mathrm{d}^4x\,\tilde{g}^{1/2}\left\{\chi^2\tilde{R}-4\omega\,\partial_\mu\chi\,\partial^\mu\chi-\partial_\mu\tilde{\varphi}\,\partial^\mu\tilde{\varphi}+V(\tilde{\varphi},\chi)\right\}. \tag{2.5}$$

The field equations are

$$\tilde{\varphi}_{;\ \mu}^{\ \mu}+\frac{1}{2}\frac{\partial V}{\partial\tilde{\varphi}}=\tilde{q}^{\varphi}, \tag{2.6}$$

$$\chi_{;\ \mu}^{\ \mu}+\frac{1}{8\omega}\frac{\partial V}{\partial\chi}+\frac{1}{4\omega}\tilde{R}\chi=\frac{1}{4\omega}\tilde{q}^{\chi}, \tag{2.7}$$

$$\tilde{R}_{\mu\nu}-\tfrac{1}{2}\tilde{R}\tilde{g}_{\mu\nu}=\frac{1}{2\chi^2}\left(\tilde{T}^{\chi}_{\mu\nu}+\tilde{T}^{\varphi}_{\mu\nu}+V\tilde{g}_{\mu\nu}+\tilde{T}^{M}_{\mu\nu}\right), \tag{2.8}$$

$$\tilde{T}^{\chi}_{\mu\nu}=-4\omega\chi_{;\rho}\chi_{;}^{\ \rho}\tilde{g}_{\mu\nu}+8\omega\chi_{;\mu}\chi_{;\nu}+2(\chi^2)_{;\mu\nu}-2(\chi^2)_{;\ \rho}^{\ \rho}\tilde{g}_{\mu\nu}, \tag{2.9}$$

$$\tilde{T}^{\varphi}_{\mu\nu}=-\tilde{\varphi}_{;\rho}\tilde{\varphi}_{;}^{\ \rho}\tilde{g}_{\mu\nu}+2\tilde{\varphi}_{;\mu}\tilde{\varphi}_{;\nu}. \tag{2.10}$$

Here we define

$$\tilde{T}^{M}_{\mu\nu}=2\tilde{g}^{-1/2}\frac{\delta S^M}{\delta\tilde{g}^{\mu\nu}},$$

$$\tilde{q}^{\varphi}=\tfrac{1}{2}\tilde{g}^{-1/2}\frac{\delta S^M}{\delta\tilde{\varphi}},$$

$$\tilde{q}^{\chi}=\tfrac{1}{2}\tilde{g}^{-1/2}\frac{\delta S^M}{\delta\chi}. \tag{2.11}$$

These quantities contain in particular the effects of incoherent excitations (entropy). They fulfil the identity

$$\tilde{T}^{M\mu\nu}{}_{;\nu}+2\chi_{;}^{\ \mu}\tilde{q}^{\chi}+2\tilde{\varphi}_{;}^{\ \mu}\tilde{q}^{\varphi}=0. \tag{2.12}$$

We may call $\tilde{T}^{M}_{\mu\nu}$ the energy-momentum tensor of matter and note that it is covariantly conserved only if the fields $\tilde{\varphi}$, χ are static or their associated $\tilde{q}$ are zero. Contraction of (2.8) gives the useful identities (with $\tilde{T}^M = \tilde{T}^{M\mu}{}_{\mu}$)

$$\tilde{R} = -\frac{1}{2\chi^2}\left\{\tilde{T}^M + 4V - 2\tilde{\varphi}_{;\rho}\tilde{\varphi}_{;}^{\rho} - 8\omega\chi_{;\rho}\chi_{;}^{\rho} - 6(\chi^2)_{;}^{\rho}{}_{\rho}\right\}, \tag{2.13}$$

$$(6+4\omega)(\chi^2)_{;}^{\rho}{}_{\rho} = \tilde{T}^M + 4V - 2\tilde{\varphi}_{;\rho}\tilde{\varphi}_{;}^{\rho} - \chi\frac{\partial V}{\partial\chi}. \tag{2.14}$$

For $\omega \to -\frac{3}{2}$ the kinetic term for χ^2 vanishes and χ ceases to be an independent degree of freedom – it would then be determined as a function of $\tilde{\varphi}$ by the vanishing of the right-hand side of (2.14). We note that for V independent of χ the source term for $\Box\chi^2$ is the trace of the energy momentum tensor $\tilde{T}^{\varphi}_{\mu\nu} + \tilde{T}^M_{\mu\nu} + V\tilde{g}_{\mu\nu}$.

Let us now consider an isotropic and homogenous universe with the usual Robertson-Walker scale factor $\tilde{a}(t)$:

$$\tilde{g}_{00} = 1, \qquad \tilde{g}_{ij} = -\tilde{a}^2(t)\mathring{g}_{ij}, \qquad \tilde{\varphi} = \tilde{\varphi}(t), \qquad \chi = \chi(t),$$

$$\tilde{T}^M_{00} = \tilde{\rho}\tilde{g}_{00}, \qquad \tilde{T}^M_{ij} = -\tilde{p}\tilde{g}_{ij}, \qquad \tilde{q}^{\varphi} = \tilde{q}^{\varphi}(t), \qquad \tilde{q}^{\chi} = \tilde{q}^{\chi}(t). \tag{2.15}$$

Motivated by the success of inflationary cosmology we will further assume that $\tilde{a}^{-2}$ can be neglected compared to $\tilde{H}^2 = (\dot{\tilde{a}}/\tilde{a})^2$ up to the present time ($k = 0$ universe). In addition we suppose (for late cosmology) that $\tilde{T}^M_{\mu\nu}$ is dominated by incoherent matter fluctuations with

$$\tilde{p} = \left(\tfrac{1}{3}n - 1\right)\tilde{\rho} \tag{2.16}$$

and $n = 3$ ($n = 4$) for the matter dominated (radiation dominated) period. We introduce

$$\psi = \chi^2,$$
$$q^{\psi} = \frac{\chi}{2\omega}\tilde{q}^{\chi}. \tag{2.17}$$

This leads to the field equations

$$\tilde{H}^2 = \frac{1}{6\psi}\left(V + \dot{\tilde{\varphi}}^2 + \omega\frac{\dot{\psi}^2}{\psi} - 6\tilde{H}\dot{\psi} + \tilde{\rho}\right),$$

$$\dot{\tilde{\rho}} + n\tilde{H}\tilde{\rho} = -2\dot{\tilde{\varphi}}\tilde{q}^{\varphi} - 2\omega\frac{\dot{\psi}}{\psi}q^{\psi},$$

$$\ddot{\tilde{\varphi}} + 3\tilde{H}\dot{\tilde{\varphi}} + \frac{1}{2}\frac{\partial V}{\partial\tilde{\varphi}} = \tilde{q}^{\varphi},$$

$$\ddot{\psi} + 3\tilde{H}\dot{\psi} - \frac{1}{6+4\omega}\left\{4V + (4-n)\tilde{\rho} - 2\dot{\tilde{\varphi}}^2 - 2\psi\frac{\partial V}{\partial\psi}\right\} = q^{\psi}. \tag{2.18}$$

3. Brans-Dicke theory with cosmological constant

As a first simple example for cosmological solutions we consider the case where the potential V is independent of χ. In addition we assume that $\tilde{\varphi}$ is settled at a corresponding minimum of V so that $\dot{\tilde{\varphi}}=0$. The value of the potential at its minimum, V_0, plays the role of a cosmological constant. We also take $\tilde{q}^{\chi}=\tilde{q}^{\varphi}=0$ and the energy-momentum tensor for matter is therefore conserved. For a radiation dominated ($n=4$) or matter dominated ($n=3$) period the field equations read

$$\tilde{H}^2=\frac{V^0}{6\psi}+\frac{\tilde{\rho}}{6\psi}+\frac{\omega}{6}\frac{\dot{\psi}^2}{\psi^2}-\tilde{H}\frac{\dot{\psi}}{\psi}, \tag{3.1}$$

$$\dot{\tilde{\rho}}+n\tilde{H}\tilde{\rho}=0, \tag{3.2}$$

$$\ddot{\psi}+3\tilde{H}\dot{\psi}=\frac{1}{6+4\omega}\{4V_0+(4-n)\tilde{\rho}\}. \tag{3.3}$$

For $V_0=0$ these are the field equations of the Brans-Dicke theory [2, 3].

Here we are interested in solutions which approach asymptotically for large t the evolution

$$\tilde{H}=\tilde{\eta}t^{-1}, \qquad \tilde{a}=a_0 t^{\tilde{\eta}}. \tag{3.4}$$

Conservation of the energy momentum tensor (3.2) immediately implies

$$\tilde{\rho}=\rho_0 t^{-n\tilde{\eta}}. \tag{3.5}$$

For positive (negative) $\tilde{\eta}$ the scale factor $\tilde{a}(t)$ expands (shrinks) and $\tilde{\rho}$ decreases (increases) whereas for $\tilde{\eta}=0$ both $\tilde{a}$ and $\tilde{\rho}$ are constant. For negative $\tilde{\eta}$ the universe is asymptotically $\tilde{\rho}$ dominated (V_0 can be neglected) whereas $\tilde{\eta}>0$ leads to a V_0 dominated universe (for $V_0\neq 0$). For solutions with $\tilde{\eta}=0$ the ratio of matter density and cosmological constant ρ_0/V_0 goes to a constant.

The evolution of ψ depends critically on $n, \tilde{\eta}$ and the sign of V_0. For $n=4$ or $\tilde{\rho}=0$ (V_0 dominated universe) eq. (3.3) has the solution (for $\tilde{\eta}\neq\pm\frac{1}{3}$)

$$\psi=\psi_0+\psi_1 t^{1-3\tilde{\eta}}+\frac{1}{(1+3\tilde{\eta})(3+2\omega)}V_0 t^2. \tag{3.6}$$

One finds the following asymptotic behaviour for $n=4$

$$\lim_{t\to\infty}\psi(t)=\begin{cases}\dfrac{V_0t^2}{(1+3\tilde{\eta})(3+2\omega)} & \text{for } \tilde{\eta}>-\frac{1}{3}, \quad V_0>0\\ \dfrac{V_0}{3+2\omega}t^2\ln(t/t_0) & \text{for } \tilde{\eta}=-\frac{1}{3}, \quad V_0>0\\ \psi_1 t^{1-3\tilde{\eta}} & \text{for } \tilde{\eta}<-\frac{1}{3} \quad \text{or for } \tilde{\eta}<\frac{1}{3}, \quad V_0=0\\ \psi_1\ln(t/t_0) & \text{for } \tilde{\eta}=\frac{1}{3}, \quad V_0=0\\ \psi_0 & \text{for } \tilde{\eta}>\frac{1}{3}, \quad V_0=0.\end{cases} \tag{3.7}$$

For negative V_0 and $\tilde{\eta}\geqslant-\frac{1}{3}$ there is no asymptotic solution of the form (3.4) since ψ is by definition positive. For the matter dominated case $n=3$ one finds similarly

$$\lim_{t\to\infty}\psi(t)=\begin{cases}\dfrac{V_0t^2}{(1+3\tilde{\eta})(3+2\omega)} & \text{for } \tilde{\eta}>0, \quad V_0>0\\ \dfrac{(V_0+\frac{1}{4}\rho_0)t^2}{(3+2\omega)} & \text{for } \tilde{\eta}=0, \quad V_0>-\frac{1}{4}\rho_0\\ \dfrac{\rho_0t^{2-3\tilde{\eta}}}{(2-3\tilde{\eta})(6+4\omega)} & \text{for } \tilde{\eta}<0 \quad \text{or for } \tilde{\eta}<\frac{2}{3}, \quad V_0=0\\ \psi_1\ln(t/t_0) & \text{for } \tilde{\eta}=\frac{2}{3}, \quad V_0=0\\ \psi_0 & \text{for } \tilde{\eta}>\frac{2}{3}, \quad V_0=0.\end{cases} \tag{3.8}$$

There is no $\tilde{H}=\tilde{\eta}t^{-1}$ asymptotic behaviour for positive $\tilde{\eta}$ and negative V_0.

Finally, possible solutions must fulfil (3.1). Depending on ω and V_0, we find the following types of solutions:

(i) $V_0>0, \quad \omega>-\frac{1}{2}, \quad n=3$ or 4:

$$\tilde{\eta}=\omega+\frac{1}{2}, \quad V_0 \text{ dominated}, \quad \psi\sim t^2.$$

(ii) $V_0>0, \quad -\frac{3}{2}<\omega\leqslant-\frac{1}{2}, \quad n=4$:

$$\tilde{\eta}=0, \quad \frac{\rho_0}{V_0}=-\frac{3(1+2\omega)}{3+2\omega}, \quad \psi\sim t^2.$$

(iii) $V_0>0, \quad -1<\omega\leqslant-\frac{1}{2}, \quad n=3$

or $V_0<0, \quad -\frac{3}{2}<\omega<-1, \quad n=3$:

$$\tilde{\eta}=0, \quad \frac{V_0+\rho_0}{4V_0+\rho_0}=-\frac{\omega}{3+2\omega}, \quad \psi\sim t^2.$$

$$\text{(iv)} \qquad V_0 \text{ arbitrary}, \quad -\tfrac{4}{3} < \omega < -1, \quad n = 3$$

$$\text{or} \qquad V_0 = 0, \quad \omega > -\tfrac{4}{3}, \quad n = 3:$$

$$\tilde{\eta} = \frac{2(1+\omega)}{3\omega+4}, \quad \tilde{\rho}\text{-dominated}, \quad \psi \sim t^{2-3\tilde{\eta}}.$$

$$\text{(v)} \qquad V_0 = 0, \quad -\tfrac{3}{2} < \omega, \quad n = 4:$$

$$\tilde{\eta} = \tfrac{1}{2}, \quad \psi = \tfrac{2}{3}\rho_0. \tag{3.9}$$

There is a solution similar to (v) for $V_0 = 0$, $n = 3$, $\omega > -\frac{3}{2}$ with $\tilde{\eta} = \frac{2}{3}$ provided the asymptotic behaviour of ψ is not the logarithmic behaviour of (3.8) but rather ψ goes to a constant $\psi_0 = \frac{3}{8}\rho_0$. No solutions of the type (3.4) exist for $n = 4$, $V_0 < 0$ or for $n = 3$, V_0 negative and $\omega \geqslant -1$ or V_0 positive and $-\frac{3}{2} < \omega \leqslant -\frac{4}{3}$ or $\omega = -1$.

In order to derive these solutions, we have assumed that $\tilde{\varphi}$ is time independent. We show in the appendix that more general cosmologies with

$$\tilde{\varphi} \sim t^{\tilde{\alpha}}, \qquad \chi \sim t^{\tilde{\beta}} \tag{3.10}$$

can be mapped by field rescalings and corresponding rescalings of ω and V onto cosmologies where $\tilde{\varphi}$ is asymptotically constant. (This is not possible for $\tilde{\alpha} = \tilde{\beta}$ and we will discuss this case in detail in ref. [7]). Using this freedom, our assumptions for the above Brans-Dicke solutions consist in the χ independence of V_0 in the appropriately scaled version.

4. Weyl scaling

Before discussing these solutions (and other more realistic scenarios) let us note that there is another useful picture of our coupled system where Newtons constant is kept fixed. We may introduce an arbitrary mass scale M and scale all fields by appropriate powers of χ/M:

$$\begin{aligned} g_{\mu\nu} &= w^{-2}\tilde{g}_{\mu\nu}, \\ g^{1/2} &= w^{-4}\tilde{g}^{1/2}, \\ \tilde{R} &= w^{-2}\left\{R - 6(\ln w)_{;\ \mu}^{\ \mu} - 6(\ln w)_{;}^{\ \mu}(\ln w)_{;\mu}\right\}, \\ w &= M/\chi. \end{aligned} \tag{4.1}$$

Choosing M proportional to todays observed Planck mass M_P

$$M^2 = \frac{M_P^2}{16\pi} \tag{4.2}$$

one obtains the standard form for the gravitational interactions with fixed Newton's constant:

$$S^0 = -\int d^4x\, g^{1/2}\left\{ M^2R - (6+4\omega)M^2\chi^{-2}\chi_{;}{}^{\mu}\chi_{;\mu} - \frac{M^2}{\chi^2}\tilde{\varphi}_{;}{}^{\mu}\tilde{\varphi}_{;\mu} + \frac{M^4}{\chi^4}V \right\}. \quad (4.3)$$

Similarly we rescale the scalar doublet $\tilde{\varphi}$

$$\varphi = \frac{\tilde{\varphi}}{\chi}M = w\tilde{\varphi}. \quad (4.4)$$

(Note that φ is now the ratio between $\tilde{\varphi}$ and χ times M and the expectation value of the Higgs doublet is therefore measured in units of the Planck mass normalized to today's value.) The kinetic term of χ involves derivatives of $\ln\chi$ and we define

$$\sigma = M\ln(\chi/M) = -M\ln(lM). \quad (4.5)$$

In this picture, the action reads

$$S^0 = -\int d^4x\, g^{1/2}\left\{ M^2R - (6+4\omega)\sigma_{;}{}^{\mu}\sigma_{;\mu} - \frac{\varphi^2}{M^2}\sigma_{;}{}^{\mu}\sigma_{;\mu} - \varphi_{;}{}^{\mu}\varphi_{;\mu} - 2\frac{\varphi}{M}\varphi_{;\mu}\sigma_{;}{}^{\mu} + W \right\}, \quad (4.6)$$

$$W = \exp\left(-4\frac{\sigma}{M}\right)V = \frac{M^4}{\chi^4}V. \quad (4.7)$$

For $\omega > -\frac{3}{2}$ and neglecting terms $\sim \varphi/M$ this is the standard action for gravity coupled to two scalar fields φ and σ with potential $W(\varphi, \sigma)$. (The factor $6+4\omega$ can be absorbed by a trivial rescaling of σ.)

At this point it may seem that our discussion will not lead to anything new. However, the possible new physics is hidden in the specific form of the scalar potential W. We choose a definition of M so that today's value of σ is zero. The ratio φ/M should be small today. The potential W is today equal to V. Assume now that the cutoff length l was larger in the past than today. In this case σ was negative and W was larger than V by a factor $(l_{\text{past}}/l_{\text{today}})^4$. Similarly, if V has a minimum for a given value of $\tilde{\varphi}$ we find from (4.4) that φ was in the past much larger than $\tilde{\varphi} = \varphi_{\text{today}}$. If at some moment χ was of the order $\tilde{\varphi}$, our rescaled version leads to φ of the order of M_{P} for this time. Cosmologies with l decreasing from $\tilde{\varphi}^{-1}$ to M^{-1} correspond in the rescaled version to an expectation value of φ decreasing from M

to $\tilde{\varphi}$! It may be instructive to see how given terms in V appear in W:

$$\begin{aligned} \lambda\tilde{\varphi}^4 &\to \lambda\varphi^4 , \\ \mu^2\tilde{\varphi}^2 &\to \mu^2\exp(-2\sigma/M)\varphi^2 , \\ \varepsilon &\to \varepsilon\exp(-4\sigma/M) . \end{aligned} \tag{4.8}$$

A cosmological constant today of the order $\alpha\tilde{\varphi}^4$ would appear to be of the order αM^4 for $\chi = \tilde{\varphi}$ and similarly a mass term today $\mu^2 \approx \tilde{\varphi}^2$ appears of the order M^2 for $\chi = \tilde{\varphi}$. If we come back to the example of the last section with V depending only on $\tilde{\varphi}$ and assume that $\tilde{\varphi}$ has settled at $\partial V/\partial\tilde{\varphi} = 0$ we find

$$\begin{aligned} \frac{\partial W}{\partial \varphi} &= 0 , \\ \frac{\partial W}{\partial \sigma} &= -\frac{4V_0}{M}\exp\left(-\frac{4\sigma}{M}\right) . \end{aligned} \tag{4.9}$$

For positive (negative) V_0 there is a driving force which tends to increase (decrease) σ. The mass matrix (with $\frac{1}{2}\partial^2 V/\partial\tilde{\varphi}^2 = \mu^2$) is (up to normalization of σ)

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 W}{\partial \varphi^2} &= \exp\left(-\frac{2\sigma}{M}\right)\mu^2 , \\ \frac{1}{2}\frac{\partial^2 W}{\partial \sigma^2} &= \exp\left(-\frac{2\sigma}{M}\right)\frac{\varphi^2}{M^2}\mu^2 + 8\exp\left(-\frac{4\sigma}{M}\right)\frac{V_0}{M^2} , \\ \frac{1}{2}\frac{\partial^2 W}{\partial \varphi\,\partial \sigma} &= \exp\left(-\frac{2\sigma}{M}\right)\frac{\varphi}{M}\mu^2 . \end{aligned} \tag{4.10}$$

For $\sigma = 0$, ($\varphi^2 \ll M^2$) one finds one eigenvalue about μ^2 and the other $8V_0/M^2$. Excitations above the cosmological background comprise the standard Higgs doublet φ plus an additional long range scalar field σ. For $V_0 = 0$ this is the massless Brans-Dicke scalar. For $V_0 \approx \Lambda^4_{\mathrm{QCD}}$ the interaction mediated by excitations of σ would have a typical range

$$l_\sigma = \left(\frac{8V_0}{M^2}\right)^{-1/2} \approx (2\cdot 10^{-11}\ \mathrm{eV})^{-1} \approx 10\ \mathrm{km} . \tag{4.11}$$

We will come back to this interaction in more detail in ref. [7].

The field equations from (4.6), neglecting the φ/M corrections to the kinetic terms*, are

$$\varphi_{;\ \mu}^{;\mu} + \frac{1}{2}\frac{\partial W}{\partial \varphi} = q^{\varphi},$$

$$(6+4\omega)\sigma_{;\ \mu}^{;\mu} + \frac{1}{2}\frac{\partial W}{\partial \sigma} = q^{\sigma},$$

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \frac{1}{2M^2}\left(T^{\sigma}_{\mu\nu} + T^{\varphi}_{\mu\nu} + Wg_{\mu\nu} + T^{M}_{\mu\nu}\right) = \frac{1}{2M^2}T_{\mu\nu},$$

$$T^{\sigma}_{\mu\nu} = (6+4\omega)\left(2\sigma_{;\mu}\sigma_{;\nu} - \sigma_{;}^{\ \rho}\sigma_{;\rho}g_{\mu\nu}\right),$$

$$T^{\varphi}_{\mu\nu} = 2\varphi_{;\mu}\varphi_{;\nu} - \varphi_{;}^{\ \rho}\varphi_{;\rho}g_{\mu\nu}. \tag{4.12}$$

The quantities q^{φ}, q^{σ} and $T^{M}_{\mu\nu}$ are defined similar as in (2.11)

$$T^{M}_{\mu\nu} = 2g^{-1/2}\frac{\delta S^M}{\delta g^{\mu\nu}}, \qquad q^{\varphi} = \tfrac{1}{2}g^{-1/2}\frac{\delta S^M}{\delta\varphi}, \qquad q^{\sigma} = \tfrac{1}{2}g^{-1/2}\frac{\delta S^M}{\delta\sigma} \tag{4.13}$$

and one has the relations

$$T^{M}_{\mu\nu} = \frac{M^2}{\chi^2}\tilde{T}^{M}_{\mu\nu} = w^2\tilde{T}^{M}_{\mu\nu},$$

$$q^{\varphi} = w^3\tilde{q}^{\varphi},$$

$$q^{\sigma} = w^3\tilde{q}^{\chi} + \frac{1}{2M}T^{M}_{\mu\nu}g^{\mu\nu}. \tag{4.14}$$

If we use again the standard form of the Robertson-Walker metric for a homogeneous and isotropic universe ($g_{00}=1$, $g_{ij} = -a^2(t)\mathring{g}_{ij}$) the rescaled formulation of this section is related to sects. 2, 3 by a coordinate transformation of the time variable (with $\tilde{t}$ the time coordinate in sects. 2, 3)

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = \frac{M}{\chi} = w. \tag{4.15}$$

The space-like comoving coordinates x^i need not to be rescaled and one has for the

* This approximation is valid for $(\varphi/M) \ll \omega + \frac{3}{2}$. For $\omega = -\frac{3}{2}$ one may use the field equation from (4.3) to express χ as a functional of $\tilde{\varphi}$ before choosing an appropriately scaled field φ.

scale factor and the Hubble parameter

$$\tilde{a} = wa\,,$$
$$\tilde{H} = w^{-1}H + w^{-2}\dot{w}\,. \tag{4.16}$$

Similarly, one finds for the energy-momentum tensor

$$\tilde{\rho} = w^{-4}\rho, \qquad \tilde{p} = w^{-4}p\,. \tag{4.17}$$

With these scalings (and noting $(\chi^2)_{;\mu\nu} \to (\chi^2)_{;\mu\nu} + 4\chi_{;\mu}\chi_{;\nu} - 2\chi^{;\rho}_{\;}\chi_{;\rho}g_{\mu\nu}$ etc.) one verifies that the field equations (2.18) are equivalent (up to φ/M corrections) to the rescaled version

$$H^2 = \frac{1}{6M^2}(E + \rho)\,,$$
$$E = W + \dot{\varphi}^2 + (6 + 4\omega)\dot{\sigma}^2\,,$$
$$\dot{E} = -6H(E - W) - (\dot{\rho} + nH\rho)\,,$$
$$\dot{\rho} + nH\rho = -2\dot{\varphi}q^{\varphi} - 2\dot{\sigma}q^{\sigma}\,,$$
$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{1}{2}\frac{\partial W}{\partial\varphi} = q^{\varphi}\,,$$
$$(6 + 4\omega)(\ddot{\sigma} + 3H\dot{\sigma}) + \frac{1}{2}\frac{\partial W}{\partial\sigma} = q^{\sigma}\,. \tag{4.18}$$

Here E is the total energy of the scalar fields φ and σ and $E - W$ is the kinetic energy part.

We can reexpress the equation for energy momentum conservation by the original $\tilde{q}^{\varphi}, \tilde{q}^{\chi}$ (2.11):

$$\dot{\rho} + nH\rho - (n-4)\frac{\dot{\sigma}}{M}\rho = -2\exp\left(-\frac{3\sigma}{M}\right)(\dot{\varphi}\tilde{q}^{\varphi} + \dot{\sigma}\tilde{q}^{\chi})\,. \tag{4.19}$$

For $\tilde{q}^{\varphi} = \tilde{q}^{\chi} = 0$ we note for the radiation dominated epoch ($n = 4$) that the energy momentum tensor for matter is conserved in both formulations, whereas for $n = 3$ this is true only for $\dot{\sigma} = 0$. In addition to the possibly unusual form of the potential W this is a second important difference in comparison to standard cosmology. This deviation from energy momentum conservation is related to the time dependence of particle masses which dominate in the matter dominated epoch. In fact, if the particle masses depend on σ, $m = m(\sigma)$, there will be an additional contribution to

$\dot{\rho}$ from the change in m

$$\Delta\dot{\rho} = \frac{\partial m}{\partial \sigma}\dot{\sigma}\left(\frac{\rho}{m}\right). \tag{4.20}$$

If for the unscaled version the particle masses would be independent of χ and $\tilde{\varphi}$ these masses would read after Weyl scaling

$$m = \tilde{m}w = \tilde{m}\exp(-\sigma/M). \tag{4.21}$$

This explains the additional term on the left-hand side of eq. (4.19). In a realistic theory particle masses will depend on $\tilde{\varphi}$ and possibly also on χ so that we expect nonzero $\tilde{q}^{\varphi}$ and $\tilde{q}^{\chi}$*.

We finally indicate the Weyl scaling for additional scalars τ, fermions ψ and gauge fields A_μ:

$$\begin{aligned} \tau &= w\tilde{\tau}, \\ \psi &= w^{3/2}\tilde{\psi}, \\ A_\mu &= \tilde{A}_\mu. \end{aligned} \tag{4.22}$$

It is easy to check (with (4.1) and (4.4)) that all (gauge + gravity) covariant kinetic terms are transformed into themselves plus additional terms involving derivatives of σ. Dimensionless couplings like the gauge couplings, Yukawa couplings and quartic scalar couplings remain unchanged under Weyl scaling, whereas couplings with dimension of mass are scaled with an appropriate power of w (compare (4.8) and (4.21)).

5. Conditions for realistic cosmologies

Explicit solutions of the cosmological equations (4.18) or (2.18) depend on the form of $W(V)$ and may in general be quite complicated. Rough asymptotic features are often more easily obtained and we will concentrate on "realistic" cosmologies fulfilling a few criteria for their asymptotic behaviour. The first three criteria concern the overall behaviour. We formulate them in a language with Newtons constant held fixed.

(i) We want to describe an expanding universe and require the asymptotic behaviour (remember $k = 0$ by assumption)

$$H = \eta t^{-1}, \qquad \eta > 0. \tag{5.1}$$

* Even in the Weinberg-Salam model with standard cosmology there is a nonzero contribution $q^{\varphi} = \alpha\rho/\varphi$ from the φ dependence of particle masses. If the universe is dominated by massive neutrinos α is of the order one whereas for a baryon dominated universe $\alpha \approx$ quark mass/nucleon mass. The presence of q^{φ} leads to a ρ dependent shift in φ with $\dot{\varphi} \sim \alpha\mu_{\varphi}^{-2}\varphi^{-1}\dot{\rho}$. The correction to energy conservation is tiny today, $\Delta\dot{\rho} \sim \dot{\rho}\rho/\varphi^4$.

(Constraints on η may be derived from the deceleration parameter q_0, or from the age of the universe, $t_0 = \eta/H_0$.)

(ii) The observed luminous matter density is today of the order H^2M^2 (about one percent of the critical density). This should be no accident and we demand an asymptotic evolution

$$\rho \sim H^2M^2 \sim t^{-2}. \tag{5.2}$$

(iii) There should have been a transition when atoms formed and the photon gas decoupled. This happened when the temperature fell below a typical ionization energy of hydrogen which is proportional $he^4\varphi$, $T_D \approx 10^{-10}\varphi$. (We assume here that dimensionless couplings like the electromagnetic coupling constant e and Yukawa coupling h did not change much during the history of the universe.) If the universe was once very hot with T of the order of φ we conclude that ρ must decrease faster than φ^4. The constraint for a powerlaw behaviour is

$$\varphi \sim t^{\alpha}, \qquad \alpha > -\tfrac{1}{2}. \tag{5.3}$$

From (5.1) and (5.3) one obtains the relation

$$\frac{H}{M} = \left(C\frac{\varphi}{M}\right)^{-1/\alpha}. \tag{5.4}$$

This tells us immediately that no cosmology can determine both $H(\varphi)$ and φ simultaneously independent of time. If the theory has no small quantity at all, the constant C must be of the order one. A value of α near $-\frac{1}{4}$ would give today's Hubble parameter near the observed value $H_0 \approx 2 \cdot 10^{-33}$ eV. (If α changes during the evolution of the universe, for example at the transition between radiation and matter dominated period, one should use an appropriate mean value of α to predict H_0 as function of φ/M.) For α different from zero there is a time independent relation between the Hubble parameter and typical particle masses given by φ. On the other hand, todays observed value $\varphi/M \approx 10^{-16}$ has no fundamental significance, but is rather due to the oldness of the universe. If in addition the ratio Λ_{QCD}/φ is time independent, all small dimensionless quantities appear only as a consequence of the age of the universe and an old hypothesis of Dirac [1] would be realized. In contrast, for $\alpha = 0$ the ratio φ/M must be a small quantity characteristic for the theory and is in principle calculable from its fundamental parameters. There is no time independent relation between H and particle masses. This is the case of standard cosmology.

(iv) There are several observations [9] from which upper bounds on the time variation of Newton's constant are derived.

$$|\kappa| = \left|\frac{\dot{G}}{G}\right|_{t_0} \leqslant 10^{-11}/\mathrm{yr}. \tag{5.5}$$

Actually, the above limits apply in cosmologies where typical particle masses are time independent. Assuming that Λ_{QCD}/φ is constant, all particle masses are proportional to φ. They depend on time for $\alpha \neq 0$. One therefore should replace

$$\kappa = \frac{d}{dt}(\varphi^2 G)/\varphi^2 G\bigg|_{t_0} = \frac{2\alpha}{\eta} H_0 . \tag{5.6}$$

(A calculation in the picture with variable Newton's "constant" leads of course to the same result. For $\alpha \approx -\frac{1}{4}$ one derives from (5.5) the bound $\eta \gtrsim 2-5$.)

Other constraints are based on the successful explanation of helium abundance in standard cosmology and the related prediction of the temperature of the background radiation [3]. We assume that the observed helium abundance by weight (near 24%) is produced cosmologically when the temperature dropped below a typical nuclear dissociation temperature T_N characteristic for nucleosynthesis. The conditions that an appreciable amount of helium is produced cosmologically are best formulated in the picture where nuclear reaction rates are time independent. These reaction rates depend on two scales: The Fermi constant $\tilde{\varphi}^{-2}$ sets the scale for weak interaction rates, the neutron lifetime and also for the electron mass m_e and the quark masses appearing in the neutron proton mass difference. On the other hand Λ_{QCD} determines nucleon masses, binding energies and strong interaction rates. We concentrate on a time independent ratio $\Lambda_{QCD}/\tilde{\varphi}$ so that the only difference from the standard picture is the possible variation of the Planck mass compared to $\tilde{\varphi}$ for $\alpha \neq 0$*.

In the picture with constant $\tilde{\varphi}$ one has to compare typical weak or strong nuclear reaction rates with the relative change of temperature with time.

$$\frac{\dot{\tilde{T}}}{\tilde{T}} = \frac{1}{4}\frac{\dot{\tilde{\rho}}}{\tilde{\rho}} = -\tfrac{1}{4} n\tilde{H} = -\tilde{\eta}\tilde{t}^{-1} . \tag{5.7}$$

For standard cosmology ($\alpha = 0$) the temperature range relevant for nuclear synthesis (10^{10}–10^9 °K) corresponds to time scales between 1 and 10^2 sec. If $\tilde{\varphi}^2 G_N$ varies with time, the time scale for nucleosynthesis is multiplied (for $\tilde{\eta} \neq 0$, and $\tilde{H}^2$ of the order $\tilde{\rho}/\chi^2$) by a factor

$$\left(\frac{G_N}{G_0}\right)^{-1/2} = \frac{\chi_N}{M} , \tag{5.8}$$

where G_N and G_0 are Newton's "constant" at the time of nucleosynthesis and today.

* For $\tilde{\varphi}/\chi \to$ const the standard picture is valid and we can replace the scale $\tilde{\varphi}$ by φ.

The scale factor χ_N/M can be evaluated in either one of the pictures with constant $\tilde{\varphi}$ or constant M_P

$$\frac{\chi_N}{M} = \frac{\varphi_0}{\varphi_N} = \left(\frac{t_0}{t_N}\right)^{\alpha} = \left(\frac{\tilde{t}_N}{\tilde{t}_0}\right)^{\tilde{\beta}}. \tag{5.9}$$

(This assumes that $\tilde{\varphi}$ was constant before Weyl scaling. Again, α stands for an appropriate mean value between nucleosynthesis and today if the time evolution of φ has changed during that period and similar for $\tilde{\beta}$.) The ratio $\tilde{t}_N/\tilde{t}_0$ is of the order 10^{-15}. Even small values of $|\tilde{\beta}|$ give huge changes in the time scale relevant for nucleosynthesis.

The cosmologically produced He abundance and associated abundances of deuterium and ^{3}He depend very critically on the time scale of the evolution of the universe [3]. If the time scale is moderately shortened the He abundance increases. For χ_N/M smaller than 10^{-1} to 10^{-2} it starts decreasing again and finally becomes tiny since there would not be enough time to produce helium before the temperature falls too low. For $\chi_N/M > 1$ the time of nucleosynthesis would be postponed and more neutrons could decay before. Already by an increase of the time scale by a factor 100 almost all neutrons would have decayed before He could be formed. Taking together the cosmologically observed abundances of ^{4}He, ^{3}He and d, possible deviations from the standard time scale must be small, typically in the range of the effect of one neutrino species more or less contributing to the density of the universe. Nucleosynthesis can be regarded as an extremely good test on the time evolution of $\varphi^2 G_N$. Any cosmological explanation of nucleosynthesis gives stringent conditions:

(v) The time scale during nucleosynthesis should be modified by less than 10% compared to standard cosmology with three neutrino species. This leads to

$$|\tilde{\beta}| \lesssim 3 \cdot 10^{-3}. \tag{5.10}$$

The same condition holds for α. A value $\alpha \approx -\frac{1}{4}$ required for Dirac's small number hypothesis is clearly inconsistent with cosmological nucleosynthesis. Cosmological nucleosynthesis suggests that the ratio φ/M ($\tilde{\varphi}/\chi$) should become almost constant asymptotically, with $\alpha = \tilde{\beta} = 0$ between the time of nucleosynthesis and today.

(vi) The observed abundances of ^{4}He and ^{3}He + d give information on the baryon density at the time of nucleosynthesis. This can be used to predict today's background temperature T_0 of the photon gas. For $T \sim a^{-1}$ during the radiation dominated period one finds the correct order of magnitude for T_0. For $n = 4$ the law $aT =$ const follows from the conservation of $T^M_{\mu\nu}$. One concludes that the right-hand side of (4.19) must be very small during the radiation dominated period. We require (with indices N and C for nucleosynthesis and combination of atoms)

$$\tfrac{1}{10} \leqslant \frac{T_N a_N}{T_C a_C} \leqslant 10. \tag{5.11}$$

Having established criteria for realistic cosmologies we can apply them to the solutions of sect. 3. It is convenient to translate these solutions to the picture with constant M_P. For $\tilde{H}=\tilde{\eta}\tilde{t}^{-1}$ and $\psi=\psi_0\tilde{t}^{2\tilde{\beta}}$ one obtains ($\tilde{\beta}>-1$)

$$H=\eta t^{-1},$$

$$\eta=\frac{\tilde{\eta}+\tilde{\beta}}{1+\tilde{\beta}},$$

$$t\sim\tilde{t}^{\tilde{\beta}+1},$$

$$\chi\sim t^{\beta},$$

$$\beta=\frac{\tilde{\beta}}{1+\tilde{\beta}}. \tag{5.12}$$

We note that solutions $\tilde{H}\sim\tilde{t}^{-1}$ are transformed into $H\sim t^{-1}$ if χ is proportional to $\tilde{t}^{\tilde{\beta}}$, $\tilde{\beta}>-1$. Even a contracting universe in the unscaled version ($\tilde{\eta}<0$) will appear as expanding after Weyl scaling provided $\tilde{\beta}>-\tilde{\eta}$! For realistic cosmologies one should have $\rho\sim t^{-2}$. From (4.17) and (5.12) one finds, with $\tilde{\rho}\sim\tilde{t}^{-\tilde{\gamma}}$

$$\rho\sim t^{-(4\tilde{\beta}+\tilde{\gamma})/(1+\tilde{\beta})}. \tag{5.13}$$

For the generalized Brans-Dicke solutions of sect. 3 one has $\tilde{\gamma}=n\tilde{\eta}$. One finds $\rho\sim t^{-2}$ for cases (ii), (iii), (iv) and (v) of eq. (3.9), whereas for case (i) ρ decreases faster than t^{-2}. It is interesting that even models with nonvanishing cosmological constant ($V_0=0$) lead to cosmologies with $H\sim t^{-1}$, $\rho\sim t^{-2}$, in sharp contrast to standard gravity where a nonvanishing cosmological constant implies an exponential behaviour for H and ρ.

A solution with $\tilde{\varphi}\sim\tilde{t}^{\tilde{\alpha}}$ reads after Weyl scaling

$$\varphi\sim t^{\alpha},$$

$$\alpha=\frac{\tilde{\alpha}-\tilde{\beta}}{1+\tilde{\beta}}. \tag{5.14}$$

Condition (5.3) implies (for $\tilde{\alpha}=0$) that $\tilde{\beta}$ must be smaller than 1. This condition is violated for all solutions except those with $V_0=0$ and $\tilde{\eta}>0$ (cases (iv) and (v)). This is the standard Brans-Dicke theory. We could think about a solution starting with (3.9)v) in the radiation dominated period which makes a transition to the asymptotic solution (3.9)iv) once the temperature has decreased sufficiently to enter the matter dominated epoch. This would be consistent with the three conditions

(5.1)–(5.3) for the overall behaviour of realistic cosmologies. During the radiation dominated period one has $\tilde{\beta}=0$ and energy momentum is conserved. Conditions (v) and (vi) are fulfilled. However, for the matter dominated period the condition of (almost) vanishing $\tilde{\beta}$ requires $\tilde{\eta}$ to be very near $\frac{2}{3}$. This is the asymptotic value for $\omega \to \infty$. As is well known, the Brans-Dicke theory converges to standard cosmology in the limit $\omega \to \infty$. It is compatible with observation for $\omega \gtrsim 500$.

For the Brans-Dicke type solution (3.9)ii) with $V_0>0$, $n=4$ the coupled system of gravitation, scalar singlet χ and radiation would lead to realistic cosmology. The only problems come from the variation of the ratio $\tilde{\varphi}/\chi$. It is not difficult to construct potentials where $\tilde{\varphi}/\chi$ instead of $\tilde{\varphi}$ reaches asymptotically a constant value. Then a positive cosmological constant $V_0>0$ is compatible with the power law behaviour of standard cosmology. We describe such a model, with adequate generalizations and an extension for the matter dominated epoch, in a subsequent paper [7]. There we discuss the fate of dilatation symmetry. We will see how this symmetry is intimately related to the behaviour of the theory under variation of a physical cutoff length l.

Appendix

The action (2.5) remains form invariant under the following rescaling of fields (with arbitrary scale M and $\delta > -1$)

$$\chi=(\bar{\chi}/M)^{\delta}\bar{\chi},$$

$$\tilde{\varphi}=(\bar{\chi}/M)^{\delta}\bar{\varphi},$$

$$\tilde{g}_{\mu\nu}=(\bar{\chi}/M)^{-2\delta}\bar{g}_{\mu\nu}. \tag{A.1}$$

This corresponds to taking instead of l some power of l as basic short distance length scale. Expressing the action in the new fields $\bar{\chi}$, $\bar{\varphi}$, $\bar{g}_{\mu\nu}$ leads to a rescaling $\omega \to \bar{\omega}$, $V \to \bar{V}$ according to

$$\left(\frac{3+2\bar{\omega}}{3+2\omega}\right)^{1/2}=1+\delta, \tag{A.2}$$

$$\bar{V}=\left(\frac{\bar{\chi}}{M}\right)^{-4\delta}V. \tag{A.3}$$

If some cosmology leads to an asymptotic behaviour $\chi \sim t^{\tilde{\beta}}$, $\tilde{\varphi} \sim t^{\tilde{\alpha}}$ with $\tilde{\alpha} \neq \tilde{\beta}$ one can choose a scaling with

$$\delta=\frac{\tilde{\alpha}}{\tilde{\beta}-\tilde{\alpha}} \tag{A.4}$$

in order to obtain $\overline{\varphi}$ asymptotically constant. If $\tilde{\alpha} \neq \tilde{\beta}$ we can always use this scaling, so that the asymptotic behaviour is either $\tilde{\varphi}/\chi \to$ const or $\tilde{\varphi} \to$ const. This defines ω and V unambiguously except for $\tilde{\varphi}/\chi \to$ const. We note that the condition $\omega > -\frac{3}{2}$ remains conserved under rescaling.

References

[1] P.A.M. Dirac, Nature 139 (1937) 323; Proc. Roy. Soc. A165 (1938) 199
[2] C. Brans and R.H. Dicke, Phys. Rev. 124 (1961) 925
[3] S. Weinberg, Gravitation and cosmology (Wiley, 1972)
[4] M. Lüscher and P. Weisz, Nucl. Phys. B290 [FS20] (1987) 25
[5] E. Witten, Nucl. Phys. B186 (1981) 412;
C. Wetterich, Nucl. Phys. B223 (1983) 109;
E. Witten, Proc. Shelter Island II Conf. (MIT press, 1984)
[6] Q. Shafi and C. Wetterich, Phys. Lett. 129B (1983) 387; 152B (1985) 51
[7] C. Wetterich, Nucl. Phys. B302 (1988) 668
[8] R.D. Peccei, J. Solà and C. Wetterich, Phys. Lett. B195 (1987) 183
[9] R. Hellings et al., Phys. Rev. Lett. 18 (1983) 1609;
W.J. Marciano, Phys. Rev. Lett. 52 (1984) 489