

Path integral on a group manifold and the lattice gauge theory Hamiltonian

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Received 7 October 1987

Abstract. For a quantum mechanical system living on the manifold of a compact simple Lie group we present explicit formulae for the quantum corrections, both in the Hamiltonian and, for the most common time discretization, in the path integral. As a special application of this rather general procedure, we compare, for lattice gauge theories, the path integral corresponding to the Kogut–Susskind Hamiltonian and the Wilson action. The latter is shown to correspond to a very special but elegant way of discretizing the time variable.

I Introduction

Recently there has been some renewed interest in the path integral formulation of quantum mechanical systems and quantum field theories on curved spaces, or, more general, on topologically nontrivial manifolds. In quantum field theory Christ and Lee [1] several years ago pointed out that, when formulating Yang–Mills theories in the Coulomb gauge, new terms appear in the action integral. They result from a nontrivial metric in the space of the gauge-fixed field variables [2–4], and they lead, in perturbation theory, to new interaction vertices. These terms had been overlooked before [5]. In the context of quantum mechanical problems there has been some recent progress in calculating path integrals, which now allows to handle quite a few problems in the path integral formulation which had been untractable before ([6, 7] and references therein). In most of these cases symmetries are playing an essential rôle. This motivates a strong interest in formulating path integrals on group manifolds [8, 9]. In string theory one faces the task of doing quantum mechanics on topologically nontrivial manifolds (e.g. Riemann mani-

folds with nonzero genus). Although it may be too much to expect that one might be able to write down a closed expression for the action integral on the whole manifold, one should be able to formulate the theory, at least, on coordinate patches. The basic task then is the same as in the other examples: to handle the path integral of a quantum mechanical system on a manifold with a nonflat metric, e.g. on group manifolds.

There exists a well-established procedure for both canonical quantization [10–12] and writing down the path integral for quantum mechanics on curved spaces [13–21]. In [1–4] this formalism has partially been used to determine the correct Yang–Mills Hamiltonian in the Coulomb gauge. In the context of quantum mechanical problems, however, not much use has been made yet of this procedure. In [6, 7] special cases of group manifolds have been studied, such as $SU(2)$, $SO(n)$, $SU(1, 1)$ and $SO(n, m)$. All these group manifolds have in common that they can easily be embedded in euclidian (or pseudo-euclidian) flat space, and this special property has been made use of for deriving the path integral. It is clear that this method does not work for other groups of interest [8]. There is also a potential danger in this way of deriving the path integral [3]. It therefore seems very much preferable to directly use the standard procedure, which is always applicable [9]. In the first part of this paper we perform, in a rather explicit manner, both canonical quantization and the derivation of the path integral for a general compact simple Lie group, following the standard routine of [13–21]. In particular, we explicitly calculate the quantum corrections which are necessary for the correct formulation of the quantum theory.

There is an interesting application in lattice gauge theories, namely the interrelation between the Wilson action and the corresponding lattice Hamiltonian. Usually a lattice gauge theory is defined through a partition function on a 4-dimensional euclidian lattice, using the Wilson action in the Boltzmann-factor. In

* Supported by BMFT, 05 4HH 92P/3, Bonn, FRG

order to derive the corresponding Hamiltonian, one singles out the time direction of the lattice, defines the transfer matrix and finally takes the lattice spacing in time direction to zero [22, 23]. For the simplest nonabelian case of $SU(2)$ the Hamiltonian has been found by Kogut and Susskind [24]

$$H = -\frac{g_t^2}{2a_s} \sum_{\langle i,j \rangle} \Delta_{LB}\{i,j\} + \frac{2}{g_s^2 a_s} V. \tag{1.1}$$

Here the kinetic term (electric energy) comes as a sum over all links, and for each link we have the Laplace–Beltrami operator on the $SU(2)$ group manifold S_3 . The second term (magnetic energy) plays the rôle of the potential which depends on all the link variables. g_t is the coupling constant in time direction and a_s is the special lattice spacing. The (asymmetric) Wilson action in the temporal gauge (a_t is the lattice spacing in time direction)

$$S = \sum_k \left\{ \frac{2a_s}{g_t^2 a_t} \sum_{\langle i,j \rangle} \text{Re Tr} [1 - U_{ij}^{-1}(k+1)U_{ij}(k)] - \frac{2a_t}{g_s^2 a_s} V \right\} \tag{1.2}$$

can be viewed as the action of the quantum mechanical path integral of (1.1): each link variable U_{ij} is like a particle with mass $m = a_s/g_t^2$ living on the $SU(2)$ manifold. If we now apply to (1.1) the standard procedure for deriving the path integral, the resulting quantum action will, in general, look much more complicated than the Wilson action (1.2). We shall show, however, that the path integral based upon (1.2) matches the Hamiltonian (1.1): the Wilson action “has chosen” a very clever way of discretizing time. From the point of view of the standard routine this discretization scheme may look peculiar, but the simplicity of (1.2) is certainly striking. We shall show that the equivalence of (1.1) and (1.2) generalizes to any compact simple Lie group. We thus end with the conclusion that, with a time discretization scheme which at first sight looks complicated, the path integral can always be cast into the elegant “Wilson form”. As a by-product, we present a device for finding rather easily the lattice Hamiltonian for Lie groups other than $SU(2)$.

This paper is organized as follows. We first (Sect. II) review, for a general compact simple Lie group, how canonical quantization is done and how the path integral is derived. In Sect. III we then turn to lattice gauge theories and study the interrelation between the Wilson action and the lattice Hamiltonian. In an appendix we briefly outline how the (mostly well-known) results for $SU(2)$ are reproduced.

II Quantum theory on a group manifold

a) Metrical quantities of the group

In the following we consider a compact simple Lie

group G . It may be construed as a differentiable manifold M furnished with a group structure. Elements of G correspond to points on M and may be parametrized in terms of the real coordinates ω^l . As usual $\omega = 0$ determines the unity element of the group. The dimension n of the manifold is identical to that of the group (i.e. the dimension of the associated Lie algebra as a vector space). The group structure is fixed if we know the composition function Φ which determines the group multiplication. Let ω_1^l and ω_2^l be the parameters of two elements g_1 and g_2 of G . Then

$$\omega^l = \Phi^l(\omega_1, \omega_2), \quad l = 1, \dots, n \tag{2.1}$$

corresponds to the product $g = g_1 g_2$ of these two elements. The left auxiliary functions are defined as [25],

$$\eta^{ll_1}(\omega) = \left. \frac{\partial \Phi^l(\omega, \omega_1)}{\partial \omega_1^{l_1}} \right|_{\omega_1=0} \tag{2.2}$$

The inverse of η are the components of the Maurer–Cartan form σ :

$$\eta^{l_1 l_1} \sigma^{l_1 l_2} = \delta^{l_1 l_2}. \tag{2.3}$$

The associativity of the group multiplication

$$\Phi(\omega_3, \Phi(\omega_2, \omega_1)) = \Phi(\Phi(\omega_3, \omega_2), \omega_1) \tag{2.4}$$

leads via differentiation to Lie’s second theorem

$$\eta^{l_1 l_1} \partial_{l_1} \eta^{l_2 l_3} - \eta^{l_2 l_2} \partial_{l_2} \eta^{l_1 l_3} = f^{l_1 l_2 l_3} \eta^{l_3 l_4} \tag{2.5}$$

where ∂_l denotes $\partial/\partial \omega^l$. In terms of the σ -fields this relation is also called the Maurer–Cartan equation:

$$\partial_{l_3} \sigma^{l_1 l_2} - \partial_{l_2} \sigma^{l_1 l_3} = f^{l_1 l_2 l_3} \sigma^{l_4 l_5} \sigma^{l_5 l_3}. \tag{2.6}$$

In (2.5) and (2.6), $f^{l_1 l_2 l_3}$ are the structure constants of the group. Since we are considering compact simple groups, the structure constants can be chosen to be totally antisymmetric.

Now we consider a matrix representation $U(\omega)$ of G , satisfying (without loss of generality)

$$\partial_l U|_{\omega=0} = -iT^l \tag{2.7}$$

where T^l are the generators of the group in the chosen representation. The T^l satisfy the commutation relation

$$[T^{l_1}, T^{l_2}] = i f^{l_1 l_2 l_3} T^{l_3}, \tag{2.8}$$

and they are normalized to

$$\text{Tr } T^{l_1} T^{l_2} = \frac{1}{2} \delta^{l_1 l_2}. \tag{2.9}$$

The left auxiliary functions are related to derivatives of U . This can be seen by differentiating

$$U(\Phi(\omega_1, \omega_2)) = U(\omega_1)U(\omega_2) \tag{2.10}$$

with respect to ω_2 at $\omega_2 = 0$. This yields

$$\eta^{l_2 l_1} \partial_{l_2} U = -i U T^{l_1}, \tag{2.11}$$

or, in terms of σ^*

$$\sigma^{i1} T^{i1} = iU^{-1} \partial_i U. \tag{2.12}$$

Equations (2.11) and (2.12) imply that

$$L^i = \eta^{i1} \partial_{i1} \tag{2.13}$$

is the infinitesimal generator of transformations via group multiplication from the right: $U \rightarrow UU'$. Note that σ (and therefore also η and L) are invariant under global transformations from the left: $U \rightarrow U'U$. The role of the left and right is reversed, when in (2.2) the derivative of Φ is taken with respect to the first argument. This would lead to right invariant $\bar{\sigma}$'s and $\bar{\eta}$'s and to the right invariant generators of left multiplications. In this sense left and right multiplications are completely on an equal footing.

The "natural" metric on M is expected to be invariant under global both left and right multiplication. If G is a simple compact group, this bi-invariant metric is unique up to multiplication by a positive constant [26]:

$$g_{i_1 i_2} = \sigma^{i_1 i_2} \sigma^{i_2 i_1} = \bar{\sigma}^{i_1 i_2} \bar{\sigma}^{i_2 i_1}. \tag{2.14}$$

Our choice of the multiplicative constant is motivated by the desire that σ (and not a multiple of σ) can be interpreted as a vielbein field. Then the inverse of the metric reads:

$$g^{i_1 i_2} = \eta^{i_1 i_2} \eta^{i_2 i_1}. \tag{2.15}$$

Note that this definition (2.14) of the metric is independent of any representation. For a given representation U , satisfying (2.7)–(2.9), the metric can also be written as:

$$g_{i_1 i_2} = -2 \text{Tr} [U^{-1} (\partial_{i_1} U) U^{-1} (\partial_{i_2} U)]. \tag{2.16}$$

For the purpose of quantization we also need the Christoffel symbols and the scalar curvature. We use the conventions

$$\Gamma_{i_1 i_2}^{i_3} = \frac{1}{2} g^{i_3 i_4} (\partial_{i_1} g_{i_2 i_4} + \partial_{i_2} g_{i_1 i_4} - \partial_{i_4} g_{i_1 i_2}), \tag{2.17}$$

$$R = g^{i_1 i_2} (\partial_{i_3} \Gamma_{i_1 i_2}^{i_3} - \partial_{i_1} \Gamma_{i_2 i_3}^{i_3} + \Gamma_{i_1 i_2}^{i_3} \Gamma_{i_3 i_4}^{i_4} - \Gamma_{i_1 i_4}^{i_3} \Gamma_{i_2 i_3}^{i_4}). \tag{2.18}$$

Making use of (2.5) and (2.6), these quantities can be calculated to be

$$\begin{aligned} \Gamma_{i_1 i_2}^{i_3} &= \frac{1}{2} \eta^{i_3 i_4} (\partial_{i_1} \sigma^{i_4 i_2} + \partial_{i_2} \sigma^{i_4 i_1}) \\ &= \eta^{i_3 i_4} (\partial_{i_1} \sigma^{i_4 i_2} + \frac{1}{2} f^{i_4 i_3 i_6} \sigma^{i_5 i_1} \sigma^{i_6 i_2}) \end{aligned} \tag{2.19}$$

$$R = \frac{1}{4} f^{i_1 i_2 i_3} f^{i_1 i_2 i_3}. \tag{2.20}$$

Now we have all the tools at our disposal which are needed for the quantization procedure. We finally note that many of these calculations greatly simplify if one makes use of the calculus of differential forms [25, 27]. For our purposes, however, it will be more appropriate to stay in the component formulation: this applies, in

particular, to the calculation of the quantum corrections in the path integral.

Since the left auxiliary functions η and correspondingly all metrical quantities are not explicitly known except for $U(1)$ and $SU(2)$, we now want to give a Taylor expansion of η . It can be obtained from the Campbell–Baker–Hausdorff formula (for a recent account, see [28]):

$$\begin{aligned} \eta^{i1}(\omega) &= \delta^{i1} + \frac{1}{2} f^{i1i2i} \omega^{i2} + \frac{1}{12} f^{i1i2i3} f^{i3i4i} \omega^{i2} \omega^{i4} \\ &\quad - \frac{1}{720} f^{i1i2i3} f^{i3i4i5} f^{i5i6i7} f^{i7i8i} \omega^{i2} \omega^{i4} \omega^{i6} \omega^{i8} \\ &\quad + O(\omega^6). \end{aligned} \tag{2.21}$$

This leads to the approximate expression for the metric:

$$\begin{aligned} g^{i1i} &= \delta^{i1i} - \frac{1}{12} f^{i1i2i3} f^{i3i4i} \omega^{i2} \omega^{i4} \\ &\quad + \frac{1}{240} f^{i1i2i3} f^{i3i4i5} f^{i5i6i7} f^{i7i8i} \omega^{i2} \omega^{i4} \omega^{i6} \omega^{i8} \\ &\quad + O(\omega^6), \end{aligned} \tag{2.22}$$

which may be used in order to calculate approximations to all metrical quantities. Clearly, this form of η^{i1i} and g^{i1i} is only valid for the parametrization $U = e^{-i\omega^i T^i}$. Any other parametrization would imply different composition functions and therefore different metrical quantities.

b) Canonical quantization

Since the correct quantization procedure, in spite of being well established for a long time [10, 12], is not yet sufficiently well-known, it may be useful to recapitulate the basic ideas. First let us consider canonical quantization. The classical kinetic Hamiltonian is of the form $\frac{1}{2} \tilde{g}^{im} p_i p_m$ where p_i are the canonical momenta and \tilde{g}^{im} is the metric (rescaled by the mass), which in general depends on the coordinates q^i . Upon quantization, q and p become operators obeying the canonical commutation relations. This leads to an operator ordering ambiguity in the kinetic Hamiltonian. It can be resolved by requiring that the Hamilton operator has the correct classical limit, that it is invariant with respect to arbitrary coordinate transformations and, finally, that it is hermitean with respect to the canonical integration measure $d^n q \sqrt{g}$. The unique result is*

$$H = \frac{1}{2} g^{-1/4} p_i \tilde{g}^{i1i2} g^{1/2} p_{i2} g^{-1/4} + V \tag{2.23}$$

where

$$\tilde{g}_{i1i2} = m g_{i1i2}, \quad g = \det(g_{i1i2}) \tag{2.24}$$

and V is assumed to depend only on q . In the coordinate representation the momenta are

$$p_i = -i g^{-1/4} \partial_i g^{1/4} = -i \partial_i - \frac{i}{4} g^{i1i2} (\partial_i g_{i2i1}). \tag{2.25}$$

* (2.12) shows that $\sigma^{i1} d\omega^i$ indeed is the Maurer Cartan 1-form: $(\sigma^{i1} d\omega^i) T^{i1} = iU^{-1} dU$ where d is the exterior derivative on M [25]

* We ignore a possible curvature term, which is present in [10], but forbidden in [11], since, for our case, the curvature is a constant anyway

This again follows from the hermiticity requirement with respect to the canonical integration measure. In this representation the kinetic Hamiltonian is the Laplace–Beltrami operator

$$H_{\text{kin}} = -\frac{1}{2}g^{-1/2}\partial_{l_1}\tilde{g}^{l_1l_2}g^{1/2}\partial_{l_2} \\ = -\frac{1}{2}\tilde{g}^{l_1l_2}\partial_{l_1}\partial_{l_2} + \frac{1}{2}\tilde{g}^{l_1l_2}\Gamma_{l_1}{}^{l_3}{}_{l_2}\partial_{l_3}. \quad (2.26)$$

The form of H given in (2.23) is not suitable for practical purposes. Hence it must be reordered according to an appropriate scheme [19]. We shall make use of the standard and the Weyl ordered forms of H_{kin} : standard:

$$H_{\text{kin}} = \frac{1}{2}\tilde{g}^{l_1l_2}p_{l_1}p_{l_2} - \frac{i}{2}(\partial_{l_1}\tilde{g}^{l_1l_2})p_{l_2} \\ - \frac{1}{8}(\partial_{l_1}\partial_{l_2}\tilde{g}^{l_1l_2}) + \Delta V_1, \quad (2.27)$$

Weyl:

$$H_{\text{kin}} = \frac{1}{8}(\tilde{g}^{l_1l_2}p_{l_1}p_{l_2} + 2p_{l_1}\tilde{g}^{l_1l_2}p_{l_2} + p_{l_1}p_{l_2}\tilde{g}^{l_1l_2}) + \Delta V_1. \quad (2.28)$$

In both cases:

$$\Delta V_1 = \frac{1}{8}(\tilde{g}^{l_1l_2}\Gamma_{l_1}{}^{l_3}{}_{l_4}\Gamma_{l_2}{}^{l_4}{}_{l_3} - \tilde{R}). \quad (2.29)$$

On the group manifold, g, Γ and R are given by (2.15), (2.19) and (2.20), respectively. Now ΔV_1 can be calculated to be:

$$\Delta V_1 = \frac{1}{8m}(\partial_{l_1}\eta^{l_2l_1})(\partial_{l_2}\eta^{l_1l_2}). \quad (2.30)$$

Our final item in the canonical context are the generators of group transformations. The hermitian left-invariant generators of right multiplications are

$$\hat{L}^{l_1} = -\frac{1}{2}(\eta^{l_2l_1}p_{l_2} + p_{l_2}\eta^{l_2l_1}) \\ = -\eta^{l_2l_1}p_{l_2} + \frac{i}{2}(\partial_{l_2}\eta^{l_2l_1}). \quad (2.31)$$

In the coordinate representation \hat{L} agrees, up to a factor of i , with (2.13). The generators satisfy the algebra

$$[\hat{L}^{l_1}, \hat{L}^{l_2}] = i f^{l_1l_2l_3}\hat{L}^{l_3}, \\ [\hat{L}^l, \hat{L}^2] = 0. \quad (2.32)$$

For a representation U with (2.7)–(2.9) application of the generators yields:

$$[\hat{L}^l, U] = U T^l. \quad (2.33)$$

The quadratic Casimir operator of L coincides with (2.27) and (2.30):

$$H_{\text{kin}} = \frac{1}{2m}\hat{L}^2, \quad (2.34)$$

i.e. the kinetic part of the Hamilton operator is just the quadratic Casimir operator of the group [25]. Note that working with the right invariant generators of left multiplications \hat{R}^l would have led to (2.34) with \hat{L}^2 replaced by \hat{R}^2 .

c) Path integral quantization

Let us now turn to the path integral description of the quantum system. The path integral approach to the quantization on curved spaces (or on flat spaces in non cartesian coordinates) has also been known for many years [13–21]. Although it is well understood by now, it may—especially for the community of particle physicists—be helpful to present the procedure in some detail.*

The path integral is a device for the calculation of the probability that a state $|q'\rangle$ at a time t' evolves into a state $|q''\rangle$ at a later time t'' :

$$\text{PI} = \langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t''-t')} | q' \rangle. \quad (2.35)$$

Usually the first step for the evaluation of PI is the insertion of intermediate states. For thus purpose we need the completeness relations and normalizations of the p - and q -eigenstates. They are fixed by the definition of the scalar product. In accordance with (2.25) we choose to use the coordinate invariant (geometric) integration measure:

$$\langle \psi | \chi \rangle = \int d^n q g^{1/2}(q) \psi^*(q) \chi(q). \quad (2.36)$$

Dividing the time integral $t'' - t'$ into N equal parts of size ε , the path integral may be written as ($e^a = \lim_{N \rightarrow \infty} (1 + a/N)^N$)

$$\text{PI} = \lim_{N \rightarrow \infty} g^{-1/4}(q'') g^{-1/4}(q') \int \left(\prod_{k=1}^{N-1} d^n q_k \right) \\ \cdot \prod_{k=0}^{N-1} K(k+1, k), \quad (2.37)$$

where $q_0 = q', q_N = q''$, and the short time kernel is defined by

$$K(k+1, k) := g^{1/4}(q_{k+1}) g^{1/4}(q_k) \langle q_{k+1} | 1 - i\varepsilon H | q_k \rangle. \quad (2.38)$$

The potential contained in H does not provide any difficulties, since it depends only on q . The kinetic part, however, cannot be used in the form of (2.23).** Instead, it has to be reordered into a form belonging to a “ λ -ordering” [19]. The most convenient (and most commonly used) one is the Weyl-ordering scheme,

* Most textbooks describe the path integral approach only in flat systems using cartesian coordinates. One of the exceptions, the textbook of T.D. Lee [29], uses a normalization which we consider to be somewhat unnatural: instead of our eq. (2.36) he uses

$$\langle \psi | \chi \rangle = \int d^n q \psi^*(q) \chi(q),$$

i.e. the integration measure is not invariant under general coordinate transformations

** In special cases, it is possible to proceed from now on in a somewhat different manner: rather than inserting p -eigenstates, as we shall do below, one can evaluate the matrix element of H directly [30]. The resulting path integral corresponds to a special discretization scheme, which may deviate from the midpoint rule

defined by:

$$\{p^r q^m\}_W = \frac{1}{2^m} \sum_{l=0}^m \frac{m!}{l!(m-l)!} q^{m-l} p^r q^l. \quad (2.39)$$

For $r=2$ this reduces to (2.28) (this is posed as an exercise in Lee's book [29]). Therefore we have:

$$\langle q_{k+1} | H_{\text{kin}} | q_k \rangle = \langle q_{k+1} | \frac{1}{2} \{ \tilde{g}^{l_1 l_2} p_{l_1} p_{l_2} \}_W + \Delta V_1(q) | q_k \rangle \quad (2.40)$$

$\tilde{g}^{l_1 l_2}$ is expanded in a power series in q , and we use

$$\langle q_{k+1} | \{ q^m p^r \}_W | q_k \rangle = \bar{q}_k^m \cdot g^{-1/4}(q_{k+1}) g^{-1/4}(q_k) \int \frac{d^n p}{(2\pi)^n} p^r e^{ip \cdot \Delta_k} \quad (2.41)$$

where $\bar{q}_k := \frac{1}{2}(q_{k+1} + q_k)$ and $\Delta_k := q_{k+1} - q_k$. This gives for the short time kernel:

$$K(k+1, k) = \int \frac{d^n p}{(2\pi)^n} e^{ip \cdot \Delta_k} \cdot [1 - i\varepsilon (\frac{1}{2} \tilde{g}^{l_1 l_2}(\bar{q}_k) p_{l_1} p_{l_2} + \Delta V_1 + V)], \quad (2.42)$$

where ΔV_1 and the potential V may be taken at arbitrary q , for instance at \bar{q} . From here it is obvious that Weyl-ordering of the Hamiltonian corresponds to a "midpoint discretization" of the metric. Hence there is a one-to-one correspondence between discretization and the quantum correction ΔV_1 . As an example for another discretization scheme, standard ordering would have given

$$K(k+1, k) = \int \frac{d^n p}{(2\pi)^n} e^{ip \cdot \Delta_k} \left[1 - i\varepsilon (\frac{1}{2} \tilde{g}^{l_1 l_2}(q_{k+1}) p_{l_1} p_{l_2} - \frac{i}{2} (\partial_{l_1} \tilde{g}^{l_1 l_2})(q_{k+1}) p_{l_2} - \frac{1}{8} \partial_{l_1} \partial_{l_2} \tilde{g}^{l_1 l_2} + \Delta V_1 + V) \right] \quad (2.43)$$

Since in the path integral we only need to be precise up to $O(\varepsilon)$, the integrand of the kernel (2.42) is exponentiated. Then the p -integration is gaussian and can be performed

$$\text{PI} = g^{-1/4}(q'') g^{-1/4}(q') \lim_{N \rightarrow \infty} \int \left(\prod_{k=1}^{N-1} d^n q_k \right) \cdot \prod_{k=0}^{N-1} \left\{ g^{1/2}(\bar{q}_k) \cdot \exp i \left[\frac{1}{2\varepsilon} \tilde{g}_{l_1 l_2}(\bar{q}_k) \Delta_k^{l_1} \Delta_k^{l_2} - \varepsilon \Delta V_1 - \varepsilon V \right] \right\}. \quad (2.44)$$

where a constant normalization factor has been omitted. Equation (2.44) shows that Δ is of order $\sqrt{\varepsilon}$, since

$$\Delta^{l_1} \Delta^{l_2} \doteq i\varepsilon \tilde{g}^{l_1 l_2}, \quad (2.45)$$

where the inverse metric is taken at \bar{q} of the corresponding time slice. Here the symbol \doteq denotes equi-

valence as far as use in the path integral is concerned [13].

In order to compare (2.44) to the usual covariant (configuration space) path integral, which has the integration measure $\prod_k d^n q_k \sqrt{g(q_k)}$, we still have to manipulate the measure. By Taylor expansion of $g(q_k)$ and $g(q_{k+1})$ around \bar{q}_k up to order ε (i.e. Δ^2), we obtain:

$$g^{1/2}(\bar{q}_k) = g^{1/4}(q_{k+1}) g^{1/4}(q_k) \left(1 - \frac{1}{8} \Delta_k^{l_1} \Delta_k^{l_2} \partial_{l_1} \partial_{l_2} \Gamma_{l_2}^{l_3} \right) \doteq g^{1/4}(q_{k+1}) g^{1/4}(q_k) \left(1 - \frac{i\varepsilon}{8} \tilde{g}^{l_1 l_2} \partial_{l_1} \Gamma_{l_2}^{l_3} \right). \quad (2.46)$$

This is exponentiated and inserted into (2.44):

$$\text{PI} = \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^n q_k g^{1/2}(q_k)) \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{1}{2\varepsilon} \tilde{g}_{l_1 l_2}(\bar{q}_k) \Delta_k^{l_1} \Delta_k^{l_2} - \varepsilon \Delta V - \varepsilon V \right] \right\}, \quad (2.47)$$

where

$$\Delta V = \frac{1}{8} \tilde{g}^{l_1 l_2} \partial_{l_1} \Gamma_{l_2}^{l_3} + \Delta V_1 \quad (2.48)$$

and again we have omitted the constant normalization factor.

Equation (2.47) requires some discussion. Most important, the path integral is *not* simply the integral over the field variables of the exponential of the classical action. An additional term ΔV has appeared which has to be interpreted as a quantum correction, since it has its origin in the noncommutativity of the operators p and q . Our discussion shows that this new term strongly depends upon the way in which time is discretized. We have started from the Weyl-ordered Hamiltonian, and in our result g is taken at the midpoint value \bar{q} . Another ordering could have led, for example, to $g(q_{k+1})$ ("standard ordering") or $g(q_k)$ ("antistandard ordering"), and in each case we would have found a different result for ΔV . Therefore, writing the exponent of (2.47) as an integral (without further specification)

$$S = \int dt \left(\frac{1}{2} \tilde{g}_{l_1 l_2} \dot{q}^{l_1} \cdot \dot{q}^{l_2} - V - \Delta V \right) \quad (2.49)$$

is extremely misleading: the kinetic term seems to be the same in any scheme of time discretization. But ΔV will differ from scheme to scheme, and (2.49) then suggests different answers for different schemes. This cannot be correct, since our starting point, the matrix element (2.35) does not depend upon our choice of discretization of time. The resolution to this lies in the fact that the kinetic term in (2.47) *does* depend upon the scheme. In order to be unambiguous one should therefore either avoid taking in (2.47) the limit $\varepsilon \rightarrow 0$ or supply the expression (2.49) with the additional specification of the discretization scheme.

In the literature (2.47) is often written in a different form:

$$\begin{aligned}
 \text{PI} = & \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^n q_k g^{1/2}(q_k)) \\
 & \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{1}{2\varepsilon} \tilde{g}_{l_1 l_2}(\bar{q}_k) \Delta_k^{l_1} \Delta_k^{l_2} + \frac{1}{\varepsilon} \varepsilon \tilde{R} \right. \right. \\
 & + \frac{1}{48\varepsilon} (\partial_{l_1} \partial_{l_2} \tilde{g}_{l_3 l_4} - 2\tilde{g}_{l_5 l_6} \Gamma_{l_1}^{l_5} \Gamma_{l_2}^{l_6} \Gamma_{l_3}^{l_6} \Gamma_{l_4}^{l_6}) \\
 & \left. \left. \cdot \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \Delta_k^{l_4} \right] \right\}. \tag{2.50}
 \end{aligned}$$

The equivalence can be checked using a generalization of (2.45):

$$\begin{aligned}
 \Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4} \\
 \doteq -\varepsilon^2 (\tilde{g}^{l_1 l_2} \tilde{g}^{l_3 l_4} + \tilde{g}^{l_1 l_3} \tilde{g}^{l_2 l_4} + \tilde{g}^{l_1 l_4} \tilde{g}^{l_2 l_3}). \tag{2.51}
 \end{aligned}$$

Finally we want to write down the path integral for our quantum theory on a group manifold. ΔV_1 has already been calculated in (2.30). With a similar calculation we obtain:

$$\begin{aligned}
 \text{PI} = & \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^n \omega_k \det(\sigma(\omega_k))) \\
 & \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} (\sigma^{l_1} \sigma^{l_2})(\bar{\omega}_k) \Delta_k^{l_1} \Delta_k^{l_2} - \varepsilon V \right. \right. \\
 & - \frac{\varepsilon}{8m} ((\partial_{l_1} \eta^{l_2 l}) (\partial_{l_2} \eta^{l_1 l}) - \eta^{l_1 l} \partial_{l_1} \partial_{l_2} \eta^{l_2 l}) \\
 & \left. \left. - \eta^{l_1 l} (\partial_{l_1} \eta^{l_2 l}) \eta^{l_3 l_4} \partial_{l_2} \sigma^{l_4 l_3} \right] \right\}. \tag{2.52}
 \end{aligned}$$

In order to cast this equation into the form of (2.50) we have to evaluate the last term in the exponent of (2.50), which leads to:

$$\begin{aligned}
 \text{PI} = & \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^n \omega_k \det(\sigma(\omega_k))) \\
 & \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} (\sigma^{l_1} \sigma^{l_2})(\bar{\omega}_k) \Delta_k^{l_1} \Delta_k^{l_2} \right. \right. \\
 & + \frac{\varepsilon}{24m} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} \\
 & \left. \left. + \frac{m}{24\varepsilon} \sigma^{l_1} \partial_{l_2} \partial_{l_3} \sigma^{l_4} \Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4} - \varepsilon V \right] \right\}. \tag{2.53}
 \end{aligned}$$

This form of the path integral and, more general, (2.50) is particularly well-suited for comparing (2.53) with any other path integral formulation which uses a different discretization. One expands the exponent of the latter path integral around the midpoint $\bar{\omega}$. If the resulting power series in Δ (up to order ε) agrees with (2.53), both path integrals describe the same physics, namely a quantum mechanical system on the group manifold. In the following section such a comparison

will be carried out for the Wilson action of a lattice gauge theory.

III Lattice gauge theory

Lattice gauge theories serve as an interesting application of the results of the preceding section. As we shall see, they provide an instructive example, how a very peculiar way of discretizing time may lead to a particularly appealing form of the path integral. For simplicity, we limit ourselves to unitary representations of the gauge group, which is thought to be a compact and simple Lie group.

Usually, a lattice gauge theory is defined through the partition function on the 4-dimensional euclidian lattice, using the Wilson action in the Boltzman factor. In order to derive the lattice Hamiltonian, one goes into the temporal gauge $A_0^l = 0$ (l is an algebra-index) and singles out the time direction:

$$\begin{aligned}
 Z = & \int \prod_k \left(\prod_{\{i,j\}} dU_{ij}(k) \right) \\
 & \cdot \exp \left\{ - \sum_k \left[\frac{2a_s}{g_s^2 a_t} \sum_{\{i,j\}} \text{Tr}(1 - \text{Re } U_{ij}^\dagger(k+1) U_{ij}(k)) \right. \right. \\
 & \left. \left. - \frac{2a_t}{g_s^2 a_s} V_k \right] \right\} \tag{3.1}
 \end{aligned}$$

where

$$V_k = \sum_{\substack{\text{plaquettes} \\ P}} \text{Tr}(1 - \text{Re } U(\partial P)). \tag{3.2}$$

Here a_s and a_t are the lattice spacings in spacelike and timelike directions, respectively. g_s and g_t are the two lattice coupling constants which in this asymmetric lattice have to be distinguished from each other. In (3.1) and (3.2) the lattice has been sliced: k refers to the time slice, and $\{i, j\}$ labels spacelike links. The sum in (3.2) then extends over all spacelike plaquettes belonging to the time-slice k . Finally the Hamiltonian is derived by writing (3.1) in terms of the transfer-matrix: in the limit $a_t \rightarrow 0$ one then obtains the lattice Hamiltonian (Kogut–Susskind–Hamiltonian):

$$H = \frac{g_t^2}{2a_s} \sum_{\{i,j\}} L_{ij}^2 + \frac{2}{g_s^2 a_s} V. \tag{3.3}$$

The hermitian generators L_{ij}^l satisfy the algebra

$$[L_{ij}^{l_1}, L_{ij}^{l_2}] = i f^{l_1 l_2 l_3} L_{ij}^{l_3}, \tag{3.4}$$

$$[L_{ij}^l, U_{ji}] = U_{ji} T^l. \tag{3.5}$$

All these operators act onto states which are normalized according to

$$\int (dU) |U\rangle \langle U| = 1, \tag{3.6}$$

where (dU) is the Haar-measure of the gauge group.

Let us return for a moment to the integral of (3.1). In the limit $a_t \rightarrow 0$ (with fixed length of the lattice in the time direction) it can be viewed as a path integral,

where each link is like a quantum mechanical “particle” which lives on the group manifold. It is then clear that the derivation of the Hamiltonian from (3.1), which we have just reviewed is the “inverse” of the procedure described in the previous section*. There one starts from the Hamiltonian and then derives the path integral. Since the “path integral” (3.1) does not have quite the form that we would expect from the considerations of the previous section, we shall apply the standard procedure to the Hamiltonian (3.3) and then compare the resulting path integral with (3.1).

Let us first show that, for each link, the kinetic part of (3.3) agrees with (2.34). To this end we observe that the algebra of (3.4) and (3.5) coincides with (2.32) and (2.33), if we identify $\hat{L} = L_{ij}$, $U = U_{ji}$ and $m = a_s/g_t^2$. Furthermore the integration measures of the normalization conditions of the states of (3.6) and (2.36), when applied to the group manifold, agree. Hence, for each link, the kinetic part of the Kogut–Susskind–Hamiltonian describes a quantum theory of a particle with mass a_s/g_t^2 constrained to move on the gauge group manifold. Therefore we can write the Hamiltonian as

$$H = \frac{g_t^2}{2a_s} \sum_{\langle i,j \rangle} [\{g^{l_1 l_2}(\omega) p_{l_1} p_{l_2}\}_w(i,j) + \Delta V_1(i,j)] + \frac{2}{g_s^2 a_s} V \quad (3.7)$$

where ω_{ij}^l are the parameters of the group at the link $\{i,j\}$, and $g^{l_1 l_2}(\omega)$ and ΔV_1 are given in (2.16) and (2.30), respectively. In terms of the canonical operators the hermitian generators of right group transformations at each link are given by: (cf. (2.31))

$$L_{ij}^l = -\frac{1}{2}(\eta^{l_1 l} p_{l_1} + p_{l_1} \eta^{l_1 l})(i,j). \quad (3.8)$$

As a by-product, this comparison provides an easy method for finding, for a general group, the kinetic part of the lattice Hamiltonian: rather than going through the transfer matrix formalism, it can be obtained by canonical quantization directly.

If we would now apply the standard method of the previous section to the kinetic part of the Kogut–Susskind–Hamiltonian, (3.3), we would, of course, end up with (2.53) for each link. It therefore remains to be shown that Wilson’s form of the path integral, (3.1), is identical to (2.53): the only reason why, at least at first sight, (3.1) looks quite different from (2.53), lies in the use of a very special scheme of time discretization. In order to see this we shall rewrite (3.1) into the discretization scheme of (2.53). Then the only difference between (3.1) and (2.53) will be recognized to be an overall normalization constant. Let us begin with the

exponent of (3.1) (in the following we shall disregard the potential which is irrelevant for our discussion; we also suppress the summation over the links, and we write ε instead of a_t):

$$S_{\text{wilson}}^{\text{kin}} = \sum_k \frac{2a_s}{g_t^2 \varepsilon} \text{Tr}[1 - \text{Re } U^+(k+1)U(k)] = \sum_k \frac{a_s}{g_t^2 \varepsilon} \text{Re Tr}[(U(k+1) - U(k))^+ \cdot (U(k+1) - U(k))]. \quad (3.9)$$

Obviously, the time discretization in (3.9) does not correspond to the midpoint rule used in (2.53). For comparison, we have to expand the kinetic part of the Wilson action around the “midpoint” of all time intervals, keeping all terms up to order ε (terms of higher order than ε are irrelevant in the path integral). Since the kinetic term has a factor ε^{-1} and $\Delta_k = \omega_{k+1} - \omega_k$ is of order $\sqrt{\varepsilon}$, we have to expand up to fourth order in Δ^* . This gives:

$$U(k+1) - U(k) = \Delta_k^l \partial_{l_1} U + \frac{1}{24} \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \partial_{l_1} \partial_{l_2} \partial_{l_3} U, \quad (3.10)$$

where the derivatives of U have to be taken at the midpoint $\bar{\omega}_k = \frac{1}{2}(\omega_{k+1} + \omega_k)$. The quadratic term gives the classical kinetic term:

$$\frac{a_s}{g_t^2 \varepsilon} \Delta^{l_1} \Delta^{l_2} \text{Re Tr}[(\partial_{l_1} U)^+ (\partial_{l_2} U)] = \frac{a_s}{2g_t^2 \varepsilon} \sigma^{l_1 l_2} \sigma^{l_1 l_2} \Delta^{l_1} \Delta^{l_2} = \frac{m}{2\varepsilon} g_{l_1 l_2} \Delta^{l_1} \Delta^{l_2}, \quad (3.11)$$

where we have defined the mass parameter precisely as in the Hamiltonian approach. The quartic term of $S_{\text{Wilson}}^{\text{kin}}$ reads:

$$\frac{m}{12\varepsilon} \Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4} \text{Re Tr}[(\partial_{l_1} U^+) (\partial_{l_2} \partial_{l_3} \partial_{l_4} U)]. \quad (3.12)$$

Using (2.9) and (2.12) and the identity

$$X^{l_1} X^{l_2} \text{Re Tr } i T^{l_1} T^{l_2} T^l = 0, \quad (3.13)$$

valid for any X because of the hermiticity of the generators, the Wilson action can be simplified to

$$S_{\text{wilson}}^{\text{kin}} = \sum_k \left\{ \frac{m}{2\varepsilon} g_{l_1 l_2}(\bar{\omega}_k) \Delta_k^{l_1} \Delta_k^{l_2} + \left[\frac{m}{24\varepsilon} \sigma^{l_1 l_2} \partial_{l_2} \partial_{l_3} \sigma^{l_1 l_4} - \frac{m}{12\varepsilon} \sigma^{l_5 l_1} \sigma^{l_6 l_2} \sigma^{l_7 l_3} \sigma^{l_8 l_4} \text{Re Tr } T^{l_5} T^{l_6} T^{l_7} T^{l_8} \right] \cdot \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \Delta_k^{l_4} \right\}. \quad (3.14)$$

* Throughout this section we shall use euclidian time. Contact with the previous section has therefore to be made through the usual Wick rotation

* It has already been shown in [31], that an expansion only up to second order leads to erroneous results, as it had to be expected

If we compare this result with the exponent of the path integral expression of the gauge group of (2.53), there is a difference:

$$\Delta S = \sum_k \left(\frac{\varepsilon}{24m} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} + \frac{m}{12\varepsilon} \sigma^{l_3 l_1} \sigma^{l_6 l_2} \sigma^{l_7 l_3} \sigma^{l_8 l_4} \operatorname{Re} \operatorname{Tr} T^{l_5} T^{l_6} T^{l_7} T^{l_8} \cdot \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \Delta_k^{l_4} \right). \quad (3.15)$$

In the path integral, however, ΔS is equivalent to a constant (use (2.51)):

$$\begin{aligned} \Delta S &= \sum_k \left[\frac{\varepsilon}{24m} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} - \frac{\varepsilon}{12m} \operatorname{Re} \operatorname{Tr} (2 T^{l_1} T^{l_1} T^{l_2} T^{l_2} + T^{l_1} T^{l_2} T^{l_1} T^{l_2}) \right] \\ &= \sum_k \frac{\varepsilon}{8m} (\frac{1}{2} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} - 2 \operatorname{Re} \operatorname{Tr} T^{l_1} T^{l_1} T^{l_2} T^{l_2}). \end{aligned} \quad (3.16)$$

$$(3.17)$$

Hence the only difference between the path integrals of (3.1) and (2.53) may safely be absorbed into the normalization of the path integral. In this way it is explicitly verified that the standard procedure for the derivation of a path integral and the transfer matrix formalism are indeed inverse operations.

Our result also implies that the path integral on a group manifold can be cast into an especially simple form, namely (we now include the correct normalization factor):

$$\begin{aligned} &\langle q'', t'' | q', t' \rangle \\ &= \exp \left\{ i \frac{t'' - t'}{16m} [f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} - 4 \operatorname{Re} \operatorname{Tr} ((T^l T^l)^2)] \right\} \\ &\cdot \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \varepsilon} \right)^{N \cdot n/2} \int \prod_{k=1}^{N-1} (d^n \omega_k g^{1/2}(\omega_k)) \\ &\cdot \exp \left\{ i \sum_{k=0}^{N-1} \frac{2m}{\varepsilon} \operatorname{Tr} [1 - \operatorname{Re} U^+(\omega_{k+1}) U(\omega_k)] \right\}. \end{aligned} \quad (3.18)$$

Here the U 's form a unitary representation of the group, parametrized by ω and satisfying (2.7)*.

Finally, we want to state the result for ΔS for the case of $SU(N)$. Using general properties of $SU(N)$ in the fundamental representation [32], one finds:

$$\Delta S = \sum_k \frac{\varepsilon}{16m} \frac{N^2 - 1}{N}. \quad (3.19)$$

* Note that all expressions for the quantum mechanics on the group manifold, (2.34), (2.52), (2.53), and (3.18), also apply to the case of nonlinear sigma models, which are the field theoretic extensions of our quantum mechanical system

IV Conclusion

In this paper we first have reviewed the basic formulae for quantum mechanics on the manifold of a compact simple Lie group, both for canonical quantization and for the derivation of the path integral. Particular attention has been given to the non-cartesian nature of any parametrization of the group, and explicit expressions have been presented for the quantum corrections which are associated with this feature.

In the second part we have used formulae in order to gain, in lattice gauge theories, further insight into the relationship between the Wilson action and the lattice Hamiltonian (Kogut–Susskind–Hamiltonian). The latter is shown to be the canonical Hamiltonian on the group manifold (up to the potential part). This identification is in agreement with the picture that attached to each (spacial) link there is a group manifold, and the link variables U_{ij} behave like “quantum mechanical particles” living on these group spaces. This may be considered as the lattice counterpart of the continuum fibre bundle picture. The identification of the Hamiltonians also allows to write the lattice Hamiltonian in terms of canonical coordinate and momentum operators and of the left-auxiliary functions η of the Lie group. In the parametrization $U = \exp(-i\omega^l T^l)$ the Campbell–Baker–Hausdorff formula provides a tool to calculate η (and hence the Hamiltonian) to any given order of accuracy.

The usual four dimensional euclidian partition function with the Wilson action, on the other hand, becomes a path integral when (in temporal gauge) the timelike lattice spacing is taken to zero (keeping the spacelike lattice spacing fixed). The action integral of this path integral (“Wilson form”), however, does not look at all like the one that follows from applying the standard rules (with a simple time discretization) to the lattice Hamiltonian (“standard form”). We have explicitly demonstrated that both forms are equivalent (up to an irrelevant normalization constant): the (superficial) difference lies in the way in which time is discretized. The “Wilson form” corresponds to a very special discretization scheme, whereas the “standard form” e.g. uses the midpoint rule.

We finally like to stress that this equivalence of two seemingly different forms of the path integral not only applies to the context of lattice gauge theories. For any quantum system on a compact simple Lie group the path integral can be written in the elegant “Wilson form”.

Acknowledgement. We wish to thank F. Steiner for useful remarks.

Appendix A

Path integral on the group manifold of $SU(2)$

In $SU(2)$ the generators are $T^l = \frac{1}{2}\tau^l$ with τ^l being the

Pauli-matrices. They satisfy

$$T^{l_1} T^{l_2} = \frac{1}{2} \delta^{l_1 l_2} + \frac{i}{2} \varepsilon^{l_1 l_2 l_3} T^{l_3}, \quad (\text{A.1})$$

where $\varepsilon^{l_1 l_2 l_3}$ is the totally antisymmetric Levi-Civita tensor in three dimensions. We want to present the path integral for $SU(2)$ in two different wide-spread representations, namely $U = \exp(-iB^l T^l)$ [33] and $U = x^0 + ix^l \tau^l$ [34].

a) First we want to use the familiar parametrization: [33]

$$U = e^{-iT^l B^l} = c - 2i \frac{B^l T^l}{B} s, \quad (\text{A.2})$$

where

$$B = \sqrt{B^l B^l}, \quad c = \cos \frac{B}{2}, \quad s = \sin \frac{B}{2}. \quad (\text{A.3})$$

In this case, the composition functions are rather complicated, but we do not need to know them. The σ 's can be calculated using (2.12), which is applicable since (A.2) satisfies (2.7):

$$\sigma^{l_1 l_2} = \frac{2c s}{B} P^{l_1 l_2} + \frac{B^{l_1} B^{l_2}}{B^2} + 2 \frac{s^2}{B^2} \varepsilon^{l_1 l_2 l_3} B^{l_3}, \quad (\text{A.4})$$

where the projector P is defined by

$$P^{l_1 l_2} = \delta^{l_1 l_2} - \frac{B^{l_1} B^{l_2}}{B^2}. \quad (\text{A.5})$$

This implies the left-auxiliary functions

$$\eta^{l_1 l_2} = \frac{1}{2} \frac{c}{s} B P^{l_1 l_2} + \frac{B^{l_1} B^{l_2}}{B^2} - \frac{1}{2} \varepsilon^{l_1 l_2 l_3} B^{l_3}. \quad (\text{A.6})$$

(2.14) and (2.15) can be used to calculate the metric and its inverse: Equations

$$g_{l_1 l_2} = \frac{4s^2}{B^2} P^{l_1 l_2} + \frac{B^{l_1} B^{l_2}}{B^2}, \quad (\text{A.7})$$

$$g^{l_1 l_2} = \frac{B^2}{4s^2} P^{l_1 l_2} + \frac{B^{l_1} B^{l_2}}{B^2}. \quad (\text{A.8})$$

Note that the Taylor expansion of (A.8) coincides with (2.22), as it should be. The Christoffel symbols can be calculated by using (2.17) or (2.19):

$$\Gamma_{l_1 l_2}^{l_3} = \left(\frac{1}{B} - \frac{2cs}{B^2} \right) P^{l_1 l_2} \frac{B^{l_3}}{B} + \left(\frac{c}{2s} - \frac{1}{B} \right) \left(P^{l_1 l_3} \frac{B^{l_2}}{B} + P^{l_2 l_3} \frac{B^{l_1}}{B} \right), \quad (\text{A.9})$$

and the curvature is given by (2.20):

$$R = \frac{1}{4} \varepsilon^{l_1 l_2 l_3} \varepsilon^{l_1 l_2 l_3} = \frac{3}{2}. \quad (\text{A.10})$$

The quantum correction ΔV_1 in the Weyl-ordered Hamiltonian is (cf. (2.29) and (2.30))

$$\Delta V_1 = \frac{1}{8m} \left(-\frac{3}{2} - \frac{1}{2} \frac{c^2}{s^2} - \frac{1}{s^2} + \frac{c}{2s^3} B + \frac{2}{B^2} \right). \quad (\text{A.11})$$

The quantum corrections in the path integral, which uses the midpoint discretization, may be written in two alternative forms, either as a q -dependent potential or as a power series in Δ . In the form of a potential it reads (cf. (2.47) and (2.48)):

$$\Delta V = \frac{1}{8m} \left(-\frac{3}{2} - \frac{1}{2} \frac{c^2}{s^2} - \frac{5}{2s^2} + \frac{c}{s^3} B + \frac{4}{B^2} \right). \quad (\text{A.12})$$

For the power series in Δ we have, for each time slice (cf. (2.50) and (2.53)):

$$\begin{aligned} \Delta L = & \frac{\varepsilon}{4m} + \frac{m}{6\varepsilon} \left\{ \left[\frac{3}{2} \frac{cs}{B^3} - \frac{s^2}{B^4} (2 + c^2) \right] P^{l_3 l_4} \right. \\ & \left. + \left[-\frac{1}{B^2} - \frac{c^2}{2B^2} + 6 \frac{s^2}{B^4} \right] \frac{B^{l_3} B^{l_4}}{B^2} \right\} \\ & \cdot P^{l_1 l_2} \Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4}, \end{aligned} \quad (\text{A.13})$$

which is, of course, equivalent to $-\varepsilon \cdot \Delta V$. Hence the correct path integral on the $SU(2)$ group manifold takes the form:

$$\begin{aligned} \text{PI} = & \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^3 B_k g^{1/2}(B_k)) \\ & \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} g_{l_1 l_2}(\bar{B}_k) \Delta_k^{l_1} \Delta_k^{l_2} - \varepsilon \cdot V(\bar{B}_k) \right. \right. \\ & \left. \left. + \frac{\varepsilon}{8m} \left(\frac{3}{2} + \frac{1}{2} \frac{c^2}{s^2} + \frac{5}{2s^2} - \frac{c}{s^3} B - \frac{4}{B^2} \right) \right] \right\} \end{aligned} \quad (\text{A.14})$$

or

$$\begin{aligned} \text{PI} = & \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^n B_k g^{1/2}(B_k)) \\ & \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} g_{l_1 l_2}(\bar{B}_k) \Delta_k^{l_1} \Delta_k^{l_2} - \varepsilon \cdot V(\bar{B}_k) \right. \right. \\ & \left. \left. + \frac{\varepsilon}{4m} + \frac{m}{6\varepsilon} \left(\left[\frac{3}{2} \frac{cs}{B^3} - \frac{s^2}{B^4} (2 + c^2) \right] P^{l_3 l_4} \right. \right. \right. \\ & \left. \left. + \left(-\frac{1}{B^2} - \frac{c^2}{2B^2} + 6 \frac{s^2}{B^4} \right) \frac{B^{l_3} B^{l_4}}{B^2} \right) \right. \\ & \left. \left. \cdot P^{l_1 l_2} \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \Delta_k^{l_4} \right] \right\}, \end{aligned} \quad (\text{A.15})$$

where we have omitted the common normalization factor.

b) In this part of the appendix we consider the parametrization [34]:

$$U = x^0 + ix^l \tau^l, \quad x^0 = \sqrt{1 - x^l x^l}. \quad (\text{A.16})$$

Note that this parametrization does not fulfil (2.7), hence the results of Sect. II should be applied with some care. Therefore we start from the very beginning. The composition function reads:

$$\Phi^l(x_1, x_2) = x_1^0 x_2^l + x_1^l x_2^0 - \varepsilon^{ll_1 l_2} x_1^{l_1} x_2^{l_2}, \tag{A.17}$$

which implies for the left auxiliary functions

$$\eta^{ll_1} = x^0 \delta^{ll_1} + \varepsilon^{ll_1 l_2} x^{l_2}. \tag{A.18}$$

Let us mention that (A.18) cannot be obtained from (A.6) by simply applying the coordinate transformation $B^l \rightarrow x^l(B) = -s \cdot B^l/B$ and using the proper transformation behaviour of the vielbein η . This would yield

$$\tilde{\eta}^{ll_1}(x) = \eta^{ll_2}(B(x)) \frac{\partial x^{l_1}}{\partial B^{l_2}} = -\frac{1}{2} \eta^{ll_1}(x). \tag{A.19}$$

The reason for this lies in the fact that when changing the parameterization from (A.2) to (A.16) we also make a change of the orthogonal basis of the Lie algebra according to $T \rightarrow -\tau$. As a result, under this combined transformation, the vielbein receives an additional factor of -2 , compared to the simple coordinate transformation. The redefinition of the basis also changes the structure constants: (2.5) and (A.18) lead to

$$f^{l_1 l_2 l_3} = -2 \varepsilon^{l_1 l_2 l_3}. \tag{A.20}$$

For the σ 's we find

$$\sigma^{ll_1} = x^0 \delta^{ll_1} + \frac{x^l x^{l_1}}{x^0} - \varepsilon^{ll_1 l_2} x^{l_2}. \tag{A.21}$$

Instead of (2.12) they satisfy

$$U^{-1} \partial_l U = i \tau^l \sigma^{ll_1}, \tag{A.22}$$

which again indicates the change in the basis of the algebra. σ and η give the metric and its inverse:

$$g_{l_1 l_2} = \delta^{l_1 l_2} + \frac{x^{l_1} x^{l_2}}{x^{0^2}}, \tag{A.23}$$

$$g^{l_1 l_2} = \delta^{l_1 l_2} - x^{l_1} x^{l_2}. \tag{A.24}$$

The Christoffel symbols and the scalar curvature are calculated to be:

$$\Gamma_{l_1 l_2}^{l_3} = x^{l_3} g_{l_1 l_2}, \tag{A.25}$$

$$R = \frac{1}{4} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} = \varepsilon^{l_1 l_2 l_3} \varepsilon^{l_1 l_2 l_3} = 6. \tag{A.26}$$

The difference between (A.26) and (A.10) can be understood by the following transformation: first perform the coordinate transformation $B \rightarrow x$, which leaves R invariant and then rescale the metric by a factor of 4 (due to the change of the basis in the algebra), which enlarges R by the same factor.

The quantum correction ΔV_1 in the Weyl-ordered Hamiltonian is given by (see (2.29) and (2.30))

$$\Delta V_1 = \frac{1}{8m} \left(\frac{x^l x^l}{x^{0^2}} - 6 \right) \tag{A.27}$$

The quantum corrections in the midpoint rule discretized path integral can be stated as (cf. (2.47), (2.48), (2.50) and (2.53)):

$$\Delta V = \frac{1}{8m} \left(5 \frac{x^l x^l}{x^{0^2}} - 3 \right), \tag{A.28}$$

$$\Delta L = \frac{\varepsilon}{m} + \frac{m}{24\varepsilon} g_{l_1 l_2} (g_{l_3 l_4} + 3g_{l_3 l_5} g_{l_4 l_6} x^{l_5} x^{l_6}) \cdot \Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4}. \tag{A.29}$$

Therefore the correct path integral reads:

$$\begin{aligned} \text{PI} = \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^3 x_k g^{1/2}(x_k)) \\ \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} g_{l_1 l_2}(\bar{x}_k) \Delta_k^{l_1} \Delta_k^{l_2} \right. \right. \\ \left. \left. - \frac{\varepsilon}{8m} \left(5 \frac{x^l x^l}{x^{0^2}} - 3 \right) - \varepsilon V(\bar{x}_k) \right] \right\}, \end{aligned} \tag{A.30}$$

or

$$\begin{aligned} \text{PI} = \lim_{N \rightarrow \infty} \int \prod_{k=1}^{N-1} (d^3 x_k g^{1/2}(x_k)) \\ \cdot \exp \left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} g_{l_1 l_2}(\bar{x}_k) \Delta_k^{l_1} \Delta_k^{l_2} - \varepsilon V(\bar{x}_k) \right. \right. \\ \left. \left. + \frac{\varepsilon}{m} + \frac{m}{24\varepsilon} g_{l_1 l_2} (g_{l_3 l_4} + 3g_{l_3 l_5} g_{l_4 l_6} x^{l_5} x^{l_6}) \right. \right. \\ \left. \left. \cdot \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \Delta_k^{l_4} \right] \right\}, \end{aligned} \tag{A.31}$$

where, again, we have suppressed the common normalization factor.

References

1. N.H. Christ, T.D. Lee: Phys. Rev. D22 (1980) 939
2. N.K. Falck, A.C. Hirshfeld: Ann. Phys. (N.Y.) 144 (1982) 34
3. K. Gawedzki: Phys. Rev. D26 (1982) 3593
4. B.E. Baaquie: Phys. Rev. D32 (1985) 2774
5. see, e.g., E.S. Abers, B.W. Lee: Phys. Rep. 9 (1973) 1
6. C. Grosche, F. Steiner: Z. Phys. C—Particles and Fields 36 (1978) 699; Phys. Lett. A123 (1987) 319
7. M. Böhm, G. Junker: J. Math. Phys. 28 (1987) 1978
8. M.S. Marinov, M.V. Terentyev: Sov. J. Nucl. Phys. 28 (1978) 729
9. M.S. Marinov, M.V. Terentyev: Fortschr. Phys. 27 (1979) 511
10. B.S. DeWitt: Phys. Rev. 85 (1952) 653
11. M. Omote, H. Sato: Prog. Theor. Phys. 47 (1972) 1367
12. W. Pauli: Handbuch der Physik V/1: Prinzipien der Quantentheorie I. Berlin, Göttingen, Heidelberg: Springer 1958 (first edition 1933), p.39
13. B.S. DeWitt: Rev. Mod. Phys. 29 (1957) 377
14. D.W. McLaughlin, L.S. Schulman: J. Math. Phys. 12 (1971) 2520
15. F.A. Berezin: Theor. Math. Phys. 6 (1971) 194
16. I.W. Mayes, J.S. Dowker: J. Math. Phys. 14 (1973) 434
17. M. Mizrahi: J. Math. Phys. 16 (1975) 2201
18. J.L. Gervais, A. Jevicki: Nucl. Phys. B110 (1976) 93
19. H. Leschke, M. Schmutz: Z. Phys. B—Condensed Matter 27 (1977) 85
20. M. Omote: Nucl. Phys. B120 (1977) 325
21. A.C. Hirshfeld: Phys. Lett. A67 (1978) 5
22. M. Lüscher: Commun. Math. Phys. 54 (1977) 283
23. M. Creutz: Phys. Rev. D15 (1977) 1128
24. J. Kogut, L. Susskind: Phys. Rev. D11 (1975) 395

25. Y. Choquet–Bruhat et al.: Analysis, manifolds and physics
Amsterdam: North-Holland 1982
26. J. Milnor: Adv. Math. 21 (1976) 293
27. The $SO(3)$ case is treated in C.W. Misner, K.S. Thorne, J.A.
Wheeler: Gravitation. New York: Freeman 1970
28. D. Gensing, G. Gensing: Z. Phys. C—Particles and Fields 33
(1986) 307
29. T.D. Lee: Particle physics and introduction to field theory. Chur:
Harwood 1981
30. C. Borgs: ETH-preprint, 1987
31. A. Patrasciou: Phys. Rev. Lett. 54 (1985) 1102
32. A.J. Macfarlane et al.: Commun. Math. Phys. 11 (1968) 77
33. J.B. Bronzan: Phys. Rev. D31 (1985) 2020
34. V.F. Müller, W. Rühl: Nucl. Phys. B230 FS10 (1984) 49