Z. Phys. C – Particles and Fields 37, 143–158 (1987)

Zeitschrift für Physik C Particles and Fields

Scaling behaviour of Creutz ratios in $SU(2)$ lattice gauge theory

F. Gutbrod

Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg 52, Notkestrasse 85, Federal Republic of Germany

Received 24 September 1987

Abstract. An analysis of the scaling behaviour of Creutz ratios on large lattices is given for $SU(2)$ gauge theory. The β-interval is $2.5 \leqq \beta \leqq 2.8$. Under a factor 2 scaling test, after multiplicative corrections for lattice artifacts, the Monte Carlo data show deviations from scaling, which are similar for all values of β. The ratios can be fitted successfully by a sum of three perturbative terms and an exponentially decreasing nonperturbative term. For many ratios the latter turns out to be very small, and its size dependence at fixed β is consistent with that of an area term in the Wilson loops. The deviation of the corresponding exponents from the ones expected for an area term gives a coherent explanation of the observed departures from scaling. It is well possible that for fixed spatial extension (in lattice units) nonperturbative contributions vanish so fast that they cannot be interpreted as physical effects.

1 Introduction

In two previous publications [1, 2] the scaling properties of $SU(2)$ lattice gauge theory (LGT) have been studied at $\beta=2.6$ and $\beta=2.7$ with Monte Carlo methods. The lattice size of 24^4 gave reason to expect rather small finite lattice size effects. Statistically significant deviations from perturbative scaling behaviour have been observed, the observables being Creutz ratios [3] formed out of planar Wilson loops. In this paper I will present additional Monte Carlo (MC) data at $\beta=2.5$ and $\beta=2.8$, and a thorough discussion of the trends observed in the whole set of data will be given.

The goal of these calculations is to find out to what extent LGT allows for scale transformations, here by a factor 2, sticking to the standard one plaquette action. That such a transformation is possible is generally regarded as a necessity, if continuum physics is to be associated with observables measured on the lattice. In many previous studies in $SU(3)$ LGT [4–7] and $SU(2)$ LGT [8–10], it was either assumed or made likely that, within the accessible region of β and lengths, this possibility was given. At the same time clear evidence was found that the Callan-Symanzik $\bar{\beta}$-function differs appreciably from the well-known perturbative two-loop expression. A return to the perturbative $\bar{\beta}$-function at the upper end of the β-intervals under study was indicated with limited significance. The behaviour for $SU(2)$ is broadly the same as for $SU(3)$, and information from other observables, notably the critical temperature in $SU(3)$ [11, 12], is in accord with this picture*.

From the present data one now can draw more detailed conclusions. The scaling properties depend, over the whole range of β, strongly on the geometry of the Creutz ratios. Let us, for the sake of the argument, simplify the situation and make a distinction** between those ratios containing at least one side of length $1a$ (a=lattice unit) and between those where the minimal length $l_{\min}$ is $\geqq 2a$. Considering first Creutz ratios with $l_{\min} \geqq 2a$, the data show that a possible return to perturbative 2-loop scaling (commonly called asymptotic scaling, AS) is not very fast. One finds that from $\beta=2.5$ to $\beta=2.8$ the deviations from AS have been reduced at most by 30%, if measured in the variable $\Delta\beta$. This variable is the shift in β necessary to change the scale by a factor 2. Among the ratios in this group there is still a statistically significant spread in the scaling behaviour, i.e. in $\Delta\beta$, but an average value of $\Delta\beta=0.225$ at $\beta=2.8$ is reasonably

* In later publications it was found out that systematic errors were larger than thought originally [13, 14]

** It will turn out in the later analysis that this distinction is not sufficient representative, and it has to be compared with the prediction from the two-loop $\bar{\beta}$-function $\Delta\beta_{AS}=0.274$. At $\beta=2.5$, the lowest values of $\Delta\beta$ observed are around $\Delta\beta\approx 0.20$. This explains the above statement.

On the other hand, many of the ratios with $l_{min}=1a$ have approached AS rather closely at $\beta=2.8$. The spread in $\Delta\beta$ between the two classes is thus quite large, and it does not seem to decrease rapidely with β. This can be interpreted as evidence for scale breaking. The most straightforward conclusion from these Monte Carlo data therefore is that the region in β, where there is one universal $\bar{\beta}$-function, has not been reached yet. Before this has to be accepted, however, a thorough discussion of possible systematic errors is necessary, and this is attempted here.

As has been found out long ago [15, 16], Creutz ratios do not show perfect scaling in perturbation theory, although the scale breaking effects are relatively small even at length $1a$. These lattice artifacts are supposed to vanish as $1/l_{min}^2$ for $l_{min}\to\infty$, and over the range of β of this work it was possible – judged optimistically – to increase l_{min} from $3a$ to $5a$ only. This was achieved by spending a factor 30 more computer power at $\beta=2.8$ than at $\beta=2.5$. For statements about scaling it is necessary to use ratios with smaller l_{min}, and there the dangerous influence of lattice artifacts is still large. This is due to a rapid flattening of the β-dependence of Creutz ratios of fixed geometrical size. The resulting sensitivity of $\Delta\beta$ with respect to lattice artifacts can be expressed by stating that an ad hoc increase of the ratios with $l_{min}=1a$ by about 5% will make the scaling violations vanish. Such a shift is comparable to the size of lattice artifacts, which is about 10% for ratios with $l_{min}=3a$, according to one loop perturbation theory. Of course, it is possible to correct for the calculated finite a effects, but statements on scaling will still be influenced by assumptions on the amount of scale breaking in the uncalculated two-loop, in higher order and in nonperturbative contributions.

On the other hand, the flat β-dependence may be a clue to the interpretation of the data and to an error estimate. It can be understood as a smallness of nonperturbative contributions which from dimensional reasons are expected to have a steep exponential decrease in β. This is in contrast to perturbative contributions, which vanish with powers of $1/\beta$. The different β-dependence may allow to separate the nonperturbative contributions numerically, and one may go further and determine also higher order terms in the perturbative expansion from the β-dependence of ratios. They will come out reasonably small. If this picture of low order perturbative dominance is correct – and at the moment the data are fully consistent in this respect –, the amount of lattice artifacts in the unknown terms is not so important. The situation is then as follows:

- For many ratios with $l_{min}=1$, the third and higher order perturbative terms are of the order of 10% of the first and second perturbative contribution, whereas the nonperturbative contributions are of the order of 1–5%. It is then a mild assumption that the amount of scale breaking in the third order perturbative contribution does not differ from that of the leading ones by more than 10% (which is the total amount of scale breaking anyhow). This allows to obtain a systematic error in $\Delta\beta$ smaller than 0.01 and this is sufficiently small to conclude that, as far as scaling is concerned, the ratio has approached the regime of AS closely.
- For $l_{min}>2$, both the higher order perturbative contributions and the nonperturbative part are of the order of 10–30%. Due to the influence of the latter the β-dependence is much steeper than in the previous case, and also for many of these ratios the lattice artifacts are somewhat smaller ($\leqq 5\%$). Thus a reasonable error in the amount of finite a effects of higher order terms again will lead to an error in $\Delta\beta<0.01$.

The above statements come from a fit to the β-dependence of the ratios in terms of a power series in the renormalized coupling constant (including the third order as one free parameter), and an exponentially decreasing nonperturbative contribution χ_{np}. This indeed leads to a very small χ_{np} for all those ratios, for which the scaling test gives a $\Delta\beta$ close to $\Delta\beta_{AS}$. Where the statistical errors of the ratios are small enough, the fit is good within $2\cdot 10^{-4}$ relative deviation, and the x^2 is normal. In some cases the behaviour of the third order term under scaling can be tested, and it turns out to be in agreement with that of the leading terms.

An independent evidence for the perturbative scale breaking being under control is the following: After application of a correction for lattice artifacts, the ratios with $l_{min}=1$ have $\Delta\beta$'s, which fall on a smooth curve as function of a variable R_p. This is the fraction of the perturbative contribution (to one loop accuracy) within the Creutz ratio as determined by Monte Carlo methods. Even within this group the calculated scale breaking varies strongly, and the removal of these variations is a nontrivial test. This has been observed in [2] already.

The above determination of χ_{np} opens the possibility to look into its scaling behaviour separately. Now finite a effects cannot be defined in a way suitable to correct for them at finite l_{min} and β. Since, however, for $l_{min}=1$ the nonperturbative contributions are rather small, their detailed scaling properties are probably unimportant, and the relevant question is whether this smallness itself is a lattice artifact. By this it is meant that after scaling by factors 2 and more and an appropriate shift in β the contribution would be much larger. Now for the ratio with the smallest loops there is the possibility to scale by factors 2, 3 and 4. It turns out that χ_{np} increases under scaling at fixed β by factors 4.7, 8.6 and 12.2 resp., which is compatible with what is expected for an area term. But for all ratios the exponential decrease is steeper than for an area term. Thus there is no indication that χ_{np} is particularly small for $l_{min}=1$, and at least in this case one can conclude that the scaling test is not strongly influenced by uncontrollable lattice artifacts.

The discrepancy between size dependence and magnitude of the exponential slope can be observed for all ratios where the β-dependence can be studied also for the scaled ratios. This seems to be the basic reason both for the deviation from AS and from scaling.

In Sect. 2 the relevant material from the MC-simulations is listed, and some indication for the smallness of finite lattice size effects is given. In Sect. 3 the perturbative expansion is defined, and in Sect. 4 the numerical analysis for Creutz ratios will be given in detail. Section 5 contains results for the static $q\bar{q}$-potential, and in Sect. 6 conclusions are drawn.

2 Monte Carlo data

The analysis uses the material listed in Table 1. Everything is based on the standard Wilson action

$$S=-\beta\sum_{p} W_p(1,1) \tag{1}$$

with $W_p(1,1)$ being half the trace of the plaquette operator, and $\beta=4/g_0^2$. The lattices were updated with the heatbath method in vectorized code, and planar Wilson loops $W(l_1,l_2)$ have been measured after 10 or 25 sweeps with the multihit method [17]. For all lattices the sequence of link updating has been changed in short intervals by rotating the lattices and by varying the starting points of the vectors randomly. Of course, the regularity, inherent in vectorized code, namely proceeding in steps of $2a$ through large parts of the lattice, could not be avoided. However, the very precise data at $\beta=2.8$ show no irregularity in Wilson loops, if there one side is fixed and the other is varied up to length $12a$ by steps of $1a$.

The boundary condition "twi/hel" for the lattice of size "12×16" means the following: The lattice is of the asymmetric form $12\times12\times16\times16$ with different boundary conditions in the two planes. In the 16×16-plane helical boundary conditions with shift $s=1$ were applied as to speed up the vectorized code [22] (see [2] for definitions). In the 12×12-plane twisted periodic boundary conditions [23–25] where applied in order to reduce finite lattice size effects. At $\beta=2.6$ and $\beta=2.8$ there is sufficiently good statistics to discuss finite size effects in detail.

This is done directly for Creutz ratios [3] defined by

$$\chi(l)=-\ln(W(l_1,l_2)\,W(l_3,l_4)/W(l_5,l_6)\,W(l_7,l_8)) \equiv\frac{l_1l_2|l_3l_4}{l_5l_6|l_7l_8} \tag{2}$$

with

$$l_1+l_2+l_3+l_4=l_5+l_6+l_7+l_8. \tag{3}$$

Table 1. Survey of statistics collected on various lattices

β	L	No. of sweeps	No. of sweeps discarded	Group	Boundary conditions	Maximal size of measured loops
2.35	12	14000	1000	full $SU(2)$	helical	6×6
2.4	12	27000	1000	icosaheder	periodic	6×6
2.45	12	34000	4000	icosaheder	helical, $s=1$	6×6
	12	41000	4000	full $SU(2)$	helical, $s=1$	6×6
2.5	12	22000	4000	icosaheder	helical	6×6
	12	42000	2000	full $SU(2)$	helical, $s=1$	6×6
	24	6400	1000	full $SU(2)$	periodic	8×8
2.55	12×16	85000	1000	full $SU(2)$	twi/hel, $s=1$	8×8
2.6	12	14400	2000	full $SU(2)$	periodic	6×6
	12×16	63000	1000	full $SU(2)$	twi/hel, $s=1$	8×8
	24	10000	1000	full $SU(2)$	periodic	8×8
2.7	24	50000	9000	full $SU(2)$	helical, $s=2$	8×10
2.8	24	8600	2000	full $SU(2)$	periodic	10×10
	24×32	106000	20000	full $SU(2)$	twi/hel, $s=2$	12×12

In Table 2 a small sample of ratios is given for $\beta=2.6$ and $\beta=2.8$ and for various lattice sizes. The type of ratios selected is a good approximation to the static $q\bar{q}$-potential differences at distance $(R-1)a$. The errors are estimated by the standard binning method where the bin length was increased until about 20 bins remained. Especially at $\beta=2.8$ no clear saturation of the errors with increasing bin length was observed, which indicates that the correlation time may be of the order of a few thousand (heatbath) sweeps. The observed increase in the error, however, is not very large, e.g. for $W(1,1)$ at $\beta=2.8$ it moves from $7\cdot10^{-7}$ to 1.2×10^{-6}.

For the asymmetric lattices the contributions from the 16×16-plane (case a) and from the 12×16-planes (case b) are listed separately. We see that for the ratios with $R\leqq6$ the finite size effects are of the order of 2% in the case of the 12×16-lattice as compared to the 24-lattice, whereas the 12-lattice has already 3% deviation in the smallest ratio listed. There is a barely significant spread between the two planes on the 12×16-lattice, which is absent in the data at $\beta=2.55$ and opposite in sign at $\beta=2.8$. Most probably it is a statistical fluctuation. Although at $\beta=2.8$ the differences between the 24-lattice and the 24×32-lattice are statistically not fully significant, they are compatible with the trend observed at $\beta=2.6$ between the 12-lattice and the larger lattices. When comparing these one has to be aware that the two β-values are connected by a scale transformation of almost a factor 2, and thus one has to compare small ratios at $\beta=2.6$ with larger ones at $\beta=2.8$. Incidentally it should be noted that the finite size differences and agreements are more significant on the basis of Wilson loops directly.

Preliminary calculations on a lattice of size 12×16 without twist show that the finite size effects are roughly half of those for the 12^4-lattice. Also freezing of Polyakov lines was observed over periods of 5000 sweeps or more. The spatial correlation of these lines is rather different beyond distances of $3a$ as compared to the case of twisted boundary conditions. Thus the introduction of twist seems to be quite effective in reducing finite size effects, i.e. saving computer storage and allowing more sweeps at a given β. Assuming approximate scaling behaviour of finite size effects when keeping l/L fixed, we have good reason to believe that on the large lattices the effects for Creutz ratios with $l\leqq12$ are in the order of 2–3% at most. This is much less than the statistical errors. Of course, in the factor 2 scaling test these finite size effects are reduced further. In [26] troubles were reported with respect to twisted boundary conditions. Specifically the plaquette value required 50000 iterations to converge on a 6^4-lattice. None such irregularities could be observed in the present runs.

A selection of Creutz ratios for the largest lattices at $\beta=2.5$, 2.7 and 2.8 is listed in the Appendix. More material can be found in [1].

3 Perturbative expansion of Creutz ratios

The material summarized above will be analyzed under the hypothesis that the Creutz ratios are, to a large extent, dominated by perturbative contributions. This will be tested with respect to the β-dependence in the interval $2.5\leqq\beta\leqq2.8$. It also has to be assumed with respect to the factor 2 scaling test in order to remove lattice artifacts. The definition of the residual nonperturbative contribution is accomplished by a fit to the MC-data being a sum of a polynomial in the renormalized coupling constant and of a term with an exponential β-dependence appropriate for an operator with a higher dimension. The definition of the renormalized perturbative expansion will follow the arguments of [18].

The bare coupling constant g_0^2 is a poor expansion parameter, since in the present region $g_0^2\geqq1.4$ large coupling constant renormalizations occur even at distances $1a$ and $2a$. If, however, one renormalizes g^2

Table 2. Evidence for finite size effects for Creutz ratios. These are of the form $\frac{R,R-1|R-1,R-2}{R-1,R-1|R,R-2}$

β	L	$R=4$	$R=5$	$R=6$	$R=7$
2.6	12	0.07085 (49)	0.04274 (112)	0.0307 (27)	–
	$12\times16, a$	0.07263 (17)	0.04613 (31)	0.03439 (54)	0.02840 (82)
	$12\times16, b$	0.07284 (22)	0.04628 (39)	0.03512 (77)	0.02771 (104)
	24	0.07301 (14)	0.04636 (17)	0.03514 (35)	0.02866 (61)
β	L	$R=5$	$R=6$	$R=7$	$R=8$
2.8	24	0.03105 (9)	0.02144 (24)	0.01614 (34)	0.01231 (55)
	$24\times32, a$	0.03140 (8)	0.02167 (14)	0.01665 (19)	0.01322 (29)
	$24\times32, b$	0.03136 (5)	0.02174 (7)	0.01680 (9)	0.01388 (15)

at a length l_0 characteristic for a Creutz ratio*, the second term in the expansion with respect to the renormalized coupling constant $g_r^2(l_0)$ will be in the order of 10% only [18]. The relation between g_0^2 and $g_r^2(l_0)$ is, up to 2-loop accuracy,

$$1/g_r^2(l_0) - 1/g_0^2 + \beta_0 \ln\left(\frac{\Lambda^2}{\Lambda_{\text{latt}}^2} \frac{l_0^2}{a^2}\right) + \frac{\beta_1}{\beta_0} \ln(g_r^2(l_0)/g_0^2) = 0 \tag{4}$$

where

$$\beta_0 = 22/3(4\pi)^2, \qquad \beta_1 = 136/3(4\pi)^4. \tag{5}$$

The Λ-parameter on the lattice is

$$\Lambda_{\text{latt}} = a^{-1}(\beta_0 g_0^2)^{-\beta_1/2\beta_0^2} \exp(-1/2\beta_0 g_0^2) \tag{6}$$

and the ratio of the Λ-parameters is given by [15, 18]

$$\Lambda = 17.6 \Lambda_{\text{latt}}. \tag{7}$$

Now it will be assumed that the expansion with respect to $g_r^2(l_0)$ can be truncated after terms of $O(g_r^6(l_0))$. In order to fit the β-dependence of the full MC-Creutz ratios, $\chi_{\text{MC}}(l, \beta)$, I thus make the ansatz

$$\chi_{\text{MC}}(l, \beta) = \sum_{i=1,n} g_r^{2i}(l_0)\, \chi_r^{(i)}(l) + \chi_{\text{np}}(l) \exp(-\gamma_l(\beta - \beta_s)) \tag{8}$$

with $n=3$ and $\beta_s = 2.5$.

The first and second order terms in this expansion are related to the tree and one loop expansion terms on the lattice [16], $\chi_a^{(1)}(l)$ and $\chi_a^{(2)}(l)$, by

$$\chi_r^{(1)}(l) = \chi_a^{(1)}(l) \tag{9}$$

$$\chi_r^{(2)}(l) = \chi_a^{(2)}(l) - \beta_0\, \chi_a^{(1)}(l) \ln\left(\frac{\Lambda^2}{\Lambda_{\text{latt}}^2} \frac{l_0^2}{a^2}\right). \tag{10}$$

There are 3 free parameters, $\chi_r^{(3)}(l)$, $\chi_{\text{np}}(l)$ and γ_l to fit 4 accurate MC-data.

The expansion (8) with $n=2$ and $\chi_{\text{np}}=0$ will also be used to define perturbative corrections to the factor 2 scaling test. In this case the last term in (4) has to be dropped for consistency, since it comes from a two-loop calculation, and no 2-loop terms are available for ratios analytically. As in [1, 2] I shall express the perturbative scale breaking of ratios by a correction factor $c_p(l)$, which on the tree level is given by

$$c_{p,\text{tree}}(l)\, \chi_r^{(1)}(l, L) = \chi_r^{(1)}(2l, 2L). \tag{11}$$

The dependence of the ratios on the lattice size L has been made explicit. On the one-loop level the analogous definition of the correction factor is

$$c_{p,l_0}(l)\, \chi_p(\beta, l, l_0, L) = \chi_p(\beta + \Delta\beta_{\text{AS, 1 loop}}, 2l, 2l_0, 2L) \tag{12}$$

* Such a renormalization is necessary in the continuum anyhow

with

$$\chi_p(\beta, l, l_0, L) = \sum_{i=1,2} g_{r,\,1\,\text{loop}}^{2i}(l_0)\, \chi_r^{(i)}(l, L). \tag{13}$$

The quantity $\Delta\beta_{\text{AS, 1 loop}}$ follows from (6), if there β_1 is set to zero. These perturbative correction factors, of course, depend on l_0, but the dependence is weak within a reasonable variation of l_0. If the ratio is close to a potential difference between R_1 and R_2, a natural choice of l_0 is $l_0 = (R_2 - R_1))/2$. I shall assume that also for other ratios a similar choice is adequate, namely that l_0 is given by the average of the smallest nonidentical lengths of the ratio. The dependence of the fit to the β-dependence on the choice of l_0 will be discussed in Sect. 4.2.

In (12) it has assumed implicitely that all higher order corrections can be absorbed into $g_r^2(l_0)$. If, however, the higher order terms in χ_p are chosen as a geometrical series, the perturbative expression becomes independent of l_0 and takes the form

$$\chi_p(\beta, l, L) = g_0^2\, \chi_a^{(1)}(l, L)/(1 - g_0^2\, \chi_a^{(2)}(l, L)/\chi_a^{(1)}(l, L)). \tag{14}$$

This "Padéized" version has been used in [1] and [2] to define $c_{p,\,1\,\text{loop}}(l)$ analogously to (12):

$$c_{p,\,1\,\text{loop}}(l)\, \chi_p(\beta, l, L) = \chi_p(\beta + \Delta\beta_{\text{AS, 1 loop}}, 2l, 2L) \tag{15}$$

The agreement of $c_{p,\,1\,\text{loop}}(l)$ with the previous choice of the correction factor, $c_{p,l_0}(l)$, is good but not perfect, if l_0 is chosen as above. Differences are in the order of 1–2%. It is reassuring to observe that the inclusion of the third order term, determined numerically from the fit to the β-dependence, improves the agreement for the above choice of l_0. Some l_0-dependence of the correction factors remains and probably can only be removed when accurate MC-data at slightly higher β become available, since then the perturbative expansion is defined much more reliably.

Now we are ready to define $\Delta\beta$ under "multiplicative improvement" by solving

$$c_p(l)\, \chi_{\text{MC}}(\beta - \Delta\beta, l, L) = \chi_{\text{MC}}(\beta, 2l, 2L) \tag{16}$$

for $\Delta\beta$. The β-dependence on the lhs is defined by quadratic interpolation among data from fixed β-values, and $c_p(l)$ is taken as one of the above definitions. This method of improvement leads to sensible results if the higher order perturbative contributions either have the same relative amount of scale breaking as the combination of the first two terms, or if they are quite small altogether. The fit to the β-dependence suggests that for many ratios both is true: The relative contribution of the higher order terms to the full ratio is in the order of 10%, and the ratio of the appropriately scaled third order terms agrees within 5% with the Padéized $c_{p,\,1\,\text{loop}}(l)$.

Also the nonperturbative terms, as defined via (8), may suffer from lattice artifacts, but I am not aware of a useful definition of these. For the perturbative expansion scale breaking by lattice effects can be defined in the limit $\beta \to \infty$ term by term. But the nonperturbative terms vanish exponentially fast in this limit, so numerically it is not possible to determine scale breaking at large β, and furthermore an extrapolation down to smaller β would be undefined in view of the unknown $\bar{\beta}$-function. A qualitative basis for a discussion of finite a-effects is given by the empirical observation that the nonperturbative contributions roughly have dimension a^2, i.e. they contribute to Creutz ratios proportional to the differences of the areas entering the ratio. There are exceptions to this observations just among the ratios which closely follow AS in the sense that χ_{np} is smaller than for the bigger ratios. One may ask whether this smallness of χ_{np} is a consequence either of the fact that these ratios have $l_{\min}=1$, or of their geometrical shape. Under the first possibility χ_{np} of size $2l$ should scale by much more than by a factor 4, which is not the case (see Table 5 below). Of course, minor deviations ($<50\%$) are unimportant if the nonperturbative contribution is of the order of 1%. The case of the smallest ratio where multiple scaling is possible was already mentioned. It is treated in Subsect. 4.4 as well as the application of additive improvement to be discussed now.

Multiplicative improvement has the advantage to allow the correction of ratios individually, but it for instance overestimates the correction if the higher order perturbative contributions have no scaling violations. In this case the "additive improvement" [5, 16, 20, 21] leads to better results, and the method should be studied as an alternative. There linear superpositions of ratios are formed such that the lattice artifacts are cancelled to an accuracy including terms of $O(g_0^4)$. It is reasonable to require that ratios with different signs of lattice artifacts are combined linearly in order to avoid superpositions with negative coefficients, which could lead to unwanted large cancellations. Such ratios are certainly available, but in most cases they have rather different geometrical shape and/or size. Because of the suspicion that the scaling properties of ratios depend on the size, one should restrict additive improvement to the cases where superposition of ratios with similar geometry is possible.

4 Numerical results

4.1 Factor 2 scaling test

Also the new data at $\beta=2.5$ and $\beta=2.8$ show the phenomenon that the values of $\Delta\beta$, evaluated according to (16), scatter over a large interval. In [2] it has been observed that these values can be ordered by the introduction of the variable

$$R_p(l)=\chi_p(l)/\chi_{\mathrm{MC}}(l) \tag{17}$$

where the perturbative ratio $\chi_p(l)$ is defined in (14). All quantities are evaluated at $\beta-\Delta\beta_{\mathrm{AS}}$. This variable probably underestimates the full perturbative contribution to the ratio $\chi_{\mathrm{MC}}(l)$. The reason is that the fit to the β-dependence leads to a positive third order perturbative contribution, and it might be better to use the result of this fit to define R_p. For simplicity I shall stick to the definition (17) here. The close correlation between $\Delta\beta$ and R_p is shown for all β in Figs. 1–3. Only a subset of ratios has been included into the plots according to the following selection criteria: a) The correction factor $c_{p,\,1\,\mathrm{loop}}$ should not deviate from 1 by more than 0.12. b) The difference between $c_{p,\mathrm{tree}}$ and $c_{p,\,1\,\mathrm{loop}}$ should not exceed 0.05. c) The sum of areas in the numerator of the ratio minus the sum of areas in the denominator is restricted to 1 or 2.

Several comments are necessary here:

- The similarity between the curves excludes the

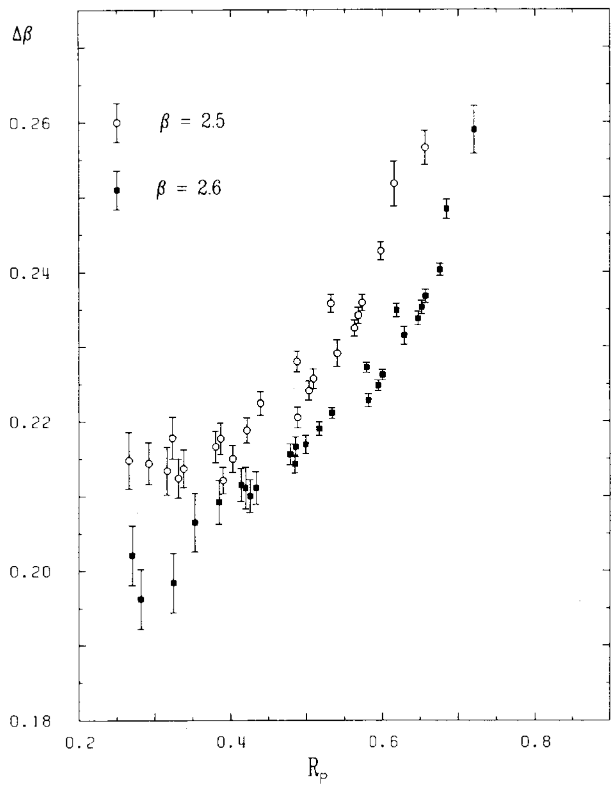

Fig. 1. Values of $\Delta\beta$ for improved Creutz ratios at $\beta=2.5$ (circles) and $\beta=2.6$ (squares). The abszissa is defined in (17). Asymptotic scaling approximately corresponds to the upper border line

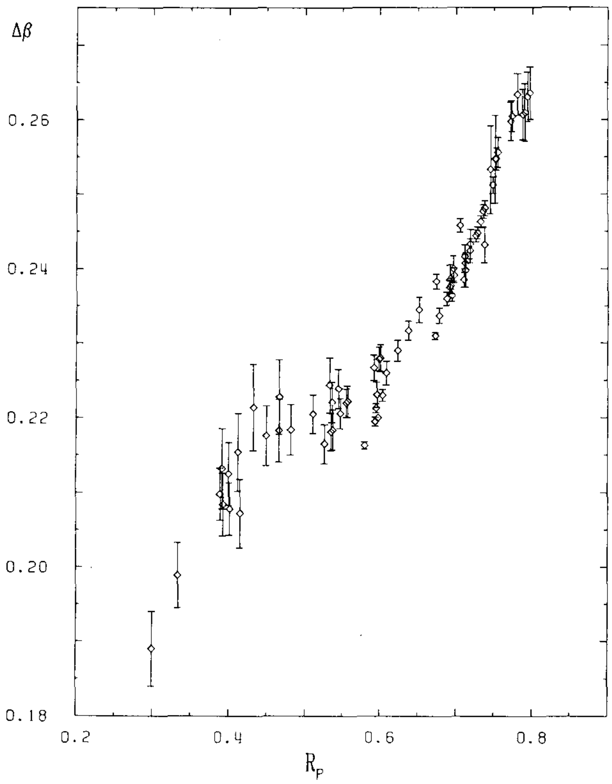

Fig. 2. Values of $\Delta\beta$ for improved Creutz ratios at $\beta=2.7$. The abszissa is defined in (17). Asymptotic scaling approximately corresponds to the upper border line

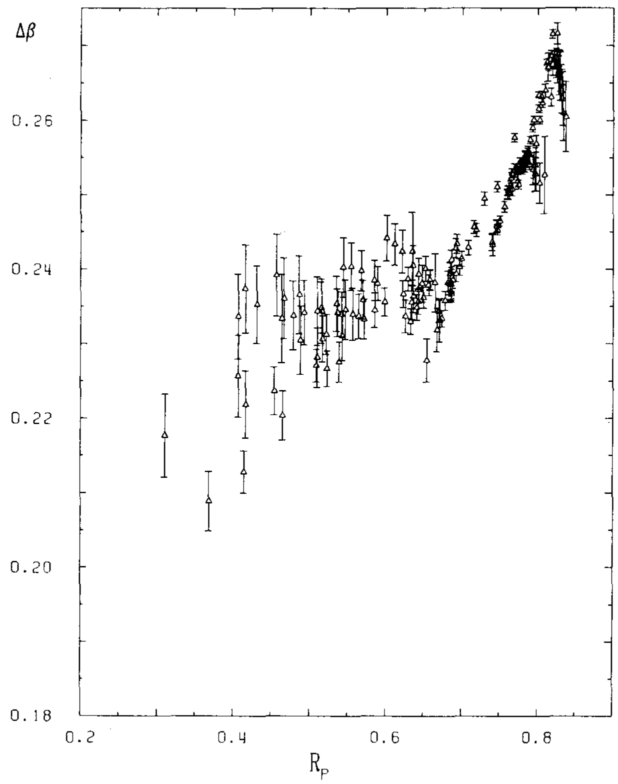

Fig. 3. Values of $\Delta\beta$ for improved Creutz ratios at $\beta=2.8$. The abszissa is defined in (17). Asymptotic scaling approximately corresponds to the upper border line

presence of uncontrolled statistical fluctuations. The errors of the differences between closeby ratios are probably smaller than the statistical errors quoted, since there are fine structures, i.e. small deviations in $\Delta\beta$ for neighbouring ratios which show up at all β in identical magnitude.

- The importance of improvement is immense. In Fig. 4 the results for $\Delta\beta$ at $\beta=2.8$ are shown with and without multiplicative improvement. The unimproved ratios (bars only) form various clusters which come from ratios with similar shape. The difference between improved and unimproved ratios shows a discontinuity around $R_p=0.66$, where ratios with $l_{\min}>1$ start to show up. Another irregularity is at $R_p=0.69$. None of these breaks is reflected by a significant discontinuity in the improved ratios. This gives much support to the belief that multiplicative improvement is sensible.

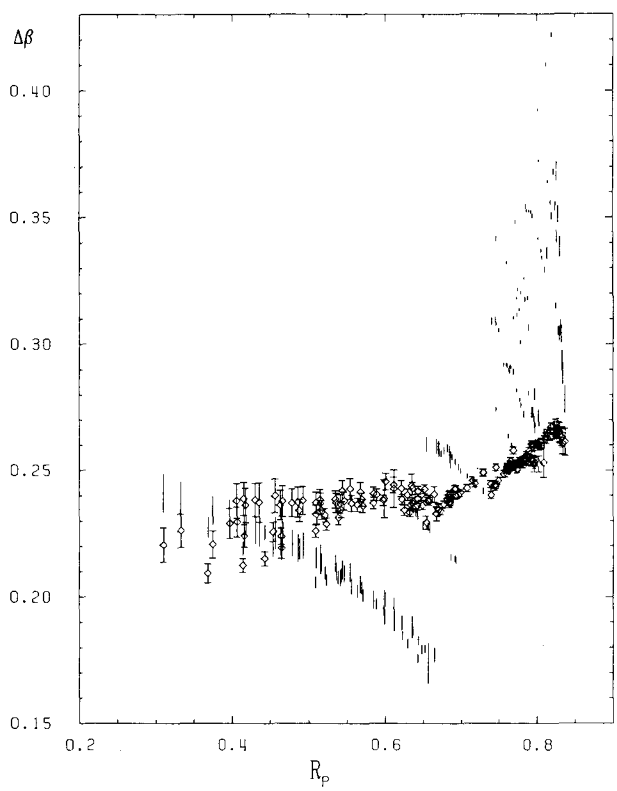

Fig. 4. Values of $\Delta\beta$ for unimproved (bars) and improved (diamonds with error bars) Creutz ratios improved on the 1 loop level at $\beta=2.8$ as function of R_p

● Taking the smallest values of $\Delta\beta$ as a measure of departure from AS, one sees that from $\beta=2.5$ to $\beta=2.8$ the change is from $\Delta\beta=0.20$ (see Fig. 1) to $\Delta\beta=0.225$ (see Fig. 3 and Sect. 4.4), i.e. only 1/3 of the deviation from AS has vanished. This seems to exclude a rapid restauration of AS for Wilson loops with increasing size.

● The plots agree qualitatively among each other apart from $\beta=2.5$ (see Fig. 1). The agreement, however, is not quite as good as it was apparent in [2] where $\beta=2.6$ and $\beta=2.7$ had been compared. This is essentially due to the inclusion of lengths 5 and 6 into the plot at $\beta=2.8$. Nevertheless the main change for increasing β consists in a flow of ratios along the band, together with a weak rise of the left side of the band at small R_p.

Before discussing the different regions of these curves in more detail, the overall properties of the fit to the β-dependence will be explained.

4.2 Fit to the β-dependence

The fit according to (8) is done for the three lattices with $L=24$ at $\beta<2.8$ and for the 24×32-lattice at

Table 3. Fit parameters for Creutz ratios. Abbreviations are explained in the text

Ratio	$\chi_{MC}(l)$ $\beta=2.5$	$\chi_{MC}(l)$ $\beta=2.8$	$\Delta\beta(l)$ $\beta=2.8$	$R_{3,MC}$ $\beta=2.5$	χ_{np}/χ_{MC} $\beta=2.5$	γ_l	χ^2	max. dev.
42|33 / 41|43	0.31067 (14)	0.23621 (4)	0.2671 (4)	0.103 (1)	0.011 (1)	7.3 (12)	4.9	$2\cdot10^{-4}$
42|32 / 41|33	0.27434 (12)	0.20642 (4)	0.2634 (4)	0.113 (1)	0.022 (1)	8.8 (7)	0.3	$5\cdot10^{-5}$
22|22 / 51|11	0.48453 (18)	0.36302 (5)	0.2601 (5)	0.105 (1)	0.031 (1)	8.8 (4)	0.5	$1\cdot10^{-4}$
22 / 31	0.22987 (10)	0.16980 (3)	0.2558 (5)	0.110 (1)	0.037 (1)	8.7 (4)	0.2	$2\cdot10^{-5}$
22|11 / 21|21	0.21382 (10)	0.15468 (3)	0.2577 (6)	0.113 (1)	0.045 (1)	8.4 (3)	0.7	$3\cdot10^{-5}$
32|11 / 31|21	0.38334 (23)	0.27129 (4)	0.2513 (7)	0.126 (1)	0.057 (1)	8.6 (2)	1.6	$9\cdot10^{-5}$
32|11 / 22|21	0.15346 (15)	0.10149 (2)	0.267 (1)	0.147 (2)	0.088 (1)	8.4 (3)	1.7	$9\cdot10^{-5}$
42|21 / 32|31	0.15205 (18)	0.10169 (2)	0.237 (1)	0.161 (2)	0.093 (2)	8.6 (3)	0.6	$4\cdot10^{-5}$
52|31 / 42|41	0.15041 (17)	0.10050 (4)	0.238 (1)	0.160 (2)	0.095 (2)	9.0 (3)	0.3	$8\cdot10^{-5}$
33|22 / 32|32	0.11147 (23)	0.06871 (3)	0.238 (2)	0.167 (2)	0.104 (3)	9.8 (5)	0.5	$6\cdot10^{-5}$
44|33 / 43|43	0.07178 (39)	0.03751 (6)	0.236 (4)	0.218 (6)	0.159 (9)	11.0 (4)	1.4	$2\cdot10^{-4}$
55|44 / 54|54	0.05500 (61)	0.02484 (8)	0.235 (8)	0.283 (21)	0.150 (14)	11.0 (6)	0.1	$4\cdot10^{-5}$
53|32 / 43|42	0.08029 (31)	0.04360 (6)	0.228 (2)	0.220 (5)	0.178 (5)	10.4 (4)	0.2	$4\cdot10^{-5}$
63|32 / 53|42	0.07575 (29)	0.03951 (6)	0.229 (4)	0.226 (6)	0.209 (4)	10.1 (4)	0.4	$8\cdot10^{-5}$
64|43 / 54|53	0.05627 (52)	0.02459 (8)	0.230 (7)	0.257 (16)	0.269 (17)	9.5 (7)	1.0	$2\cdot10^{-4}$
43|22 / 33|32	0.07514 (31)	0.03891 (4)	0.228 (2)	0.238 (4)	0.194 (3)	9.8 (4)	3.1	$2\cdot10^{-4}$
53|22 / 43|32	0.06283 (35)	0.02868 (5)	0.213 (3)	0.294 (6)	0.253 (7)	10.3 (4)	1.8	$1\cdot10^{-4}$
65|44 / 55|54	0.04404 (67)	0.0149 (1)	0.224 (10)	0.366 (41)	0.309 (32)	9.5 (13)	0.8	$3\cdot10^{-4}$

$\beta=2.8$. Consequently there is a break in the (probably very small) finite size effects, nevertheless the fits to a large sample of ratios works exceedingly well. The results are illustrated in Table 3, where the second and third columns give MC-values for the ratios with statistical errors. These values indicate, especially for the first group of ratios, the weak variation of the ratios with β which makes $\Delta\beta$ sensitive to small changes in $c_{p,\,1\,\mathrm{loop}}(l)$. The values for $\Delta\beta$ at $\beta=2.8$, as derived from the scaling test, are listed in column 4. Next the ratios of the $O(g_r^6)$-term to the full MC result are given, being denoted by $R_{3,\mathrm{MC}}$. The increase of this fractions towards the bottom of the table is only partly a consequence of an increase in $g_r^2(l_0)$. This shows that the absorption of higher order term into the running coupling constant is not complete. Column 6 contains the relative contribution of the nonperturbative piece at $\beta=2.5$. It is smaller than the fraction $1-R_P$ defined in (17) by more than a factor 2 which comes from the positive contribution of the third order term in (8). For the exponential slopes γ_l of the nonperturbative contribution, listed in column 7, one notices an increase with increasing nonperturbative fraction.

The quality of the fit is shown in the last two columns. The average $\chi^2 \approx 1.1$ is very reasonable for 1 d.o.f. Also the maximal deviations between fit and MC-values, given in the last column, are quite small. The use of the 24-lattice at $\beta=2.8$ lead essentially to the same fit parameters except for the γ_l which are typically lower by 0.7 than for the larger lattice. The χ^2 is somewhat worse. This variation may give an indication of uncertainties due to finite size effects. It would be possible, of course, to fit the data with 4 free parameters by a pure power series in $1/\beta$, but this would require oscillating expansion terms leading to a complete failure outside the fit interval. The fits, on the contrary, interpolate perfectly to the 12×16-lattice at $\beta=2.55$, and for ratios of small size they extrapolate within 0.5% to the 12-lattice at $\beta=2.35$. For ratios where the nonperturbative contribution is larger, the deviation at $\beta=2.35$ may amount to 10%. Even the data at $\beta=3.5$ on a 16^4-lattice [27] for the smallest ratio, $\frac{22|11}{21|21}$, are reproduced within 0.3%, which is 1.5 s.d. The total variation of the ratio, which is correctly described by the fit, amounts thus to a factor 3. It is clear that the fit parameters will depend on the assumption on the renormalization point l_0. Especially for those ratios where the nonperturbative part χ_{np} amounts to a few percent only, it may be reduced by 50% or more, if l_0 is raised from 1.5 to 2. For other ratios downward changes are typically 20% or less. A larger value of l_0 seems to be unreasonable, so this variation is a good guess for the size of systematic errors. The prime interest here is not to extract physically meaningful numbers for χ_{np} but to show that it is small. For this purpose these systematic errors are small enough.

When available, the ratios at scale $2l$ have been fitted too. It is then possible to define perturbative correction factors $c_{p,\mathrm{fit}}(l)$ by including third order

Table 4. Correction factors and nonperturbative contributions for Creutz ratios

Ratio	$c_{p,\mathrm{tree}}(l)$	$c_{p,l_0}(l)$	$c_{p,\,1\,\mathrm{loop}}(l)$	$c_{p,\mathrm{fit}}(l)$	$\chi_{\mathrm{np}}(l)/\delta A$	$\frac{\chi_{\mathrm{np}}(2l)}{\chi_{\mathrm{np}}(l)}$
$\frac{42\|33}{41\|43}$	1.066	1.096	1.090	1.092 (4)	0.0036 (4)	6.2 (1.2)
$\frac{42\|32}{41\|33}$	1.098	1.137	1.128	1.131 (3)	0.0061 (3)	6.3(4)
$\frac{22}{31}$	1.067	1.115	1.106	1.110 (3)	0.0085 (2)	5.0 (3)
$\frac{22\|11}{21\|21}$	1.060	1.061	1.061	1.067 (3)	0.0096 (2)	4.7 (2)
$\frac{32\|11}{31\|21}$	1.041	1.033	1.034	1.037 (5)	0.0110 (2)	4.6 (2)
$\frac{32\|11}{22\|21}$	0.986	0.895	0.889	0.919 (7)	0.0135 (2)	4.3 (1)
$\frac{42\|21}{32\|31}$	1.050	1.023	1.026	1.017 (9)	0.0142 (3)	4.4 (2)
$\frac{33\|22}{32\|32}$	0.887	0.895	0.894	0.891 (13)	0.0116 (4)	3.5 (4)
$\frac{43\|22}{33\|32}$	0.956	0.944	0.943	0.946 (33)	0.0146 (4)	3.6 (5)

terms in analogy to (12). The results are listed in Table 4 together with the other correction factors. Close agreement is observed between columns 4 and 5 within errors, i.e. mainly on the level of 0.5%. There is one exception, $\frac{32|11}{22|21}$, which will be discussed below. The dependence of $c_{p,\mathrm{fit}}(l)$ on the renormalization point l_0 is of the same order as the statistical errors, if l_0 is varied by half a unit. Column 6 gives the nonperturbative contributions divided by the difference of the areas in the Creutz ratios, δA, at $\beta=2.5$. The increase with loop size is not very large starting from $\frac{22|11}{21|21}$, being roughly consistent with an interpretation of this term as an area term in the Wilson loops. If, however, it were due to a gluon condensate [18], it should increase quadratically with the size of the loops, which is excluded. The same is true for the scale factor under factor 2 scaling, listed as the last item in Table 4. The ratios of the nonperturbative contributions scatter around 4 and are thus compatible with an area term in the Wilson loops, but not with a gluon condensate which would change a factor 16 under scaling. The exponential slopes γ_l, however, are too large for an area term, as this would require $\gamma_l=5.06$ according to perturbative scaling. The values listed in Table 3 typically are above 8.4 in the cases where χ_{np} is determined well. This discrepancy between l-dependence and β-dependence is central to the whole phenomenon of deviations from AS and of scale breaking.

4.3 Properties of individual ratios

I now turn to a more detailed look on individual ratios. With a little ambiguity the ratios can be grouped (at least) into five classes. These classes have quite different correction factors, and, independent from that, different scaling behaviour.

1. The ratios of the first class have at least one length 1 in the denominator and a corresponding length 2 in the numerator. The other l_i are mostly greater than 2. An example is $\frac{42|32}{41|33}$. These ratios have $c_p(l)>1$, a very small nonperturbative fraction, and values* $\Delta\beta>0.26$. Some $\Delta\beta$'s even range up to 0.27, i.e. very close to the asymptotic value $\Delta\beta_{\mathrm{AS}}=0.274$. The dependence of $\Delta\beta$ on R_p is very smooth except for those ratios where loops of size 5×5 (i.e. 10×10 on the big lattice) or larger contribute. This may be due to a large, 2 s.d. fluctuation. The ratio $\frac{22}{31}$ also joins in smoothly.

* Quotations for $\Delta\beta$ will refer to $\beta=2.8$

2. The ratios of the second class are the smallest "quadratic" ratio $\frac{22|21}{21|21}$ and its partners $\frac{32|11}{31|21}$ and $\frac{32|11}{22|21}$. They also have $c_p(l)>1$, and the nonperturbative part is up to 8.8%. The values of $\Delta\beta$ for the last ratio, $\frac{32|11}{22|21}$, lies about 0.015 above the band of class 1, which is most pronounced at $\beta=2.7$ and 2.8. The ratio is not included in the plots of Figs. 1–3 because $c_{p,\mathrm{tree}}(l)$ and $c_{p,\,1\,\mathrm{loop}}(l)$ differ by 9%. It is then amusing to notice that the ratio of the third order terms at scale $2l$ and l, $\chi^{(3)}(2l, 2L)/\chi^{(3)}(l, L)$, is higher by 16% than the correction factor $c_{p,\,1\,\mathrm{loop}}(l)$. Since the third order term contributes about 15% in the ratio, the correction factor coming from the fit to the β-dependence is higher by 3% than $c_{p,\,1\,\mathrm{loop}}(l)$ (see 6th line of Table 4). The use of this new factor would bring $\Delta\beta$ of the ratio much closer to the band of class 1, albeit with some overcorrection.

3. The third class contains the "potential differences" between distance $1a$ and $2a$. They are defined by $\frac{T,2|T-2,1}{T-1,2|T-1,1}$. The first example listed in Table 3 is $\frac{42|21}{32|31}$. The step to lower the time-like extension by one unit at distance $1a$ as compared to $2a$ helps to reduce perturbative finite T-effects. The above ratio in fact agrees within 2.5% with the extrapolated ratio ($T\to\infty$, see next section), whereas the ratio $\frac{42|31}{32|41}$ is off by 6%. The correction factor is still >1, but χ_{np} is now up to 10% with a consequent decrease of $\Delta\beta$. All the ratios give $\Delta\beta\approx0.238$ with very small errors.

4. The fourth class contains the "potential differences" $\frac{T,R|T-2,R-1}{T-1,R|T-1,R-1}$ for distances $R>2a$ and furthermore standard Creutz ratios with $l_i>1a$. They have $c_{p,\,1\,\mathrm{loop}}(l)<1$ with a slow approach to 1 for $l_i\to\infty$ at fixed geometry. The values of $\Delta\beta$ scatter in the range $0.228<\Delta\beta<0.244$. They show a correlation with $c_{p,\,1\,\mathrm{loop}}(l)$ in the sense that an increase of $c_{p,\,1\,\mathrm{loop}}(l)$ goes in parallel with a decrease in $\Delta\beta$. This could be explained by the assumption that χ_{np} has smaller relative lattice artifacts than expressed by $c_{p,\,1\,\mathrm{loop}}(l)$. The quoted values of $\Delta\beta$ then would be upper limits. Unfortunately it is not possible to use additive improvement for these ratios (which would be correct also for zero lattice artifacts in χ_{np}) since in this class $c_{p,\,1\,\mathrm{loop}}(l)<1$ throughout, and the lattice artifacts cannot be cancelled under superposition of ratios with positive coefficients.

5. The fifth class contains ratios in which the perturbative contributions are "oversubtracted" as compared to the two preceeding classes. They are of the form $\frac{T,R|R-1,R-1}{T-1,R|R,R-1}$ with $T>R$, a typical example listed in Table 3 being $\frac{53|22}{43|32}$. They are definitely smaller than the potential difference $V(aR)-V(a(R-1))$.The reason is the opposite of that in class 3: The second "column" of the ratio, containing much smaller lengths than the first one, has larger perturbative contributions which will be subtracted. Consequently R_p decreases for an increasing mismatch between T and R, and at the same time $\Delta\beta$ decreases. The smallest $\Delta\beta$ is reached for $\frac{63|22}{53|32}$ with $\Delta\beta=0.210\pm0.004$. There again is a clear correlation between $\Delta\beta$ and $c_{p,1\,\mathrm{loop}}(l)$, and it is possible that the low values for $\Delta\beta$ have a systematic downward shift. Now for this class of ratios it is possible to perform additive improvement which is indicated in the following.

4.4 Additive improvement and multiple scaling

Additive improvement among ratios with a similar perturbative fraction works in a few cases, if we somewhat relax the condition to use positive coefficients only for the superposition. Thus in the following three examples the ratios entering with a negative coefficient give a small correction only. The ratio χ_1 has an average R_p of 0.70, and the leading ratios of χ_2 and χ_3 belong to class 5. The results for $\Delta\beta$ again refer to $\beta=2.8$.

$$\chi_1=\frac{41|33}{51|32}+0.375\,\frac{52|22}{42|41}-0.098\,\frac{41|32}{51|22}$$
$$\chi_2=\frac{53|22}{43|32}+0.048\,\frac{43|22}{42|32}-0.036\,\frac{43|22}{33|32} \qquad (18)$$
$$\chi_3=\frac{64|33}{54|43}+1.46\,\frac{43|22}{33|32}-0.33\,\frac{53|32}{43|42}.$$

The corresponding value of $\Delta\beta$ are

$$\Delta\beta_1=0.2422\pm0.0011$$
$$\Delta\beta_2=0.2164\pm0.0027 \qquad (19)$$
$$\Delta\beta_3=0.2235\pm0.0042.$$

It is a reasonable guess that the last two values of $\Delta\beta$ are typical for ratios with relatively small perturbative content. Averaging these with the most accurate values of class 4 in Table 3 gives $\Delta\beta=0.225$. This number was quoted in the introduction as characteristic for the smallest value at $\beta=2.8$.

Finally I turn to the question of lattice artifacts for the nonperturbative part χ_{np}. As stated in the last section, the only sensible question at the moment is whether the smallness of χ_{np} at small scales is anomalous in the sense that the present fitting procedure gives much larger χ_{np}'s at larger scales. For the smallest ratio, $\frac{22|11}{21|21}$, scaling is possible by factors 2, 3 and 4 over the whole range of β. From the fit to the β-dependence it is found that χ_{np} increases by factors 4.72 ± 0.14, 8.6 ± 0.7 and 12.2 ± 2.5 resp. under these scale changes (all number refer to $\beta=2.5$). These factors again are well compatible with an area law behaviour of χ_{np}, and they corroborate that for $\frac{22|11}{21|21}$ the smallness of χ_{np} cannot be interpreted as a lattice artifact. On the contrary, since the slopes for the scaled ratios are around 10 or larger, χ_{np} for the scaled ratios will be considerably smaller than for $\frac{22|11}{21|21}$, if a shift in β by $\frac{\xi}{2}\Delta\beta_{\mathrm{AS}}$ is performed for scaling by a factor ξ.

5 The static $q\bar{q}$-potential

Since scaling violations are apparent among many kinds of Creutz ratios, they are expected to show up also in the extrapolation $T\to\infty$ e.g. of the ratios of classes 3 and 4 of the last section. These extrapolated ratios are differences of the static $q\bar{q}$-potential, which is defined by

$$V(R)=-\lim_{T_1,T_2\to\infty}\frac{1}{T_1-T_2}\ln(W(R,T_1)/W(R,T_2)). \qquad (20)$$

From Table 3 it is obvious that the spread in $\Delta\beta$ (at fixed β) is not very large within the classes 3 and 4. The observed differences, however, are significant, and they are rather what can be predicted from the type of scale breaking already assumed, namely from a sum of a scaling perturbative term and of a linearly rising potential contribution with an overly steep β-dependence.

The extrapolation of $W(R,T)$ to $T\to\infty$ in (20) can successfully be performed by the $2N$-parameter fit

$$W(R,T)=\sum_{i=1}^{N}c_i(R)\exp(-\lambda_i(R)\,T). \qquad (21)$$

The smallest exponent, $\lambda_1(R)$, is the potential $V(R)$. At $\beta=2.8$, fits with $N=4$ are absolutely stable. The point with $T=0$ is included which gives 13 data points for 8 free parameters. Even subdividing the data into 15 bins still leads to reasonable fits which allows to define errors for the potential as well for potential differences. The latter ones are smaller than expected from the former ones because of correlations. The only difficulty then is that occasionally the second exponent, $\lambda_2(R)$, moves towards the first one, in which case the first one will come out rather small. Therefore the average of the individual fits is smaller than the fit to the average, although still within errors. A related problem is that the extrapolation (21) may well be unstable against inclusion of further terms, and therefore only upper limits to the potential can be given in principle.

The correction for lattice artifacts is slightly modified with respect to [1, 2]. As done previously, first the difference of the continuum propagator $1/R$ and the bare lattice propagator [1, (3)] is taken which then, however, is multiplied by the renormalized coupling $\alpha_r(l_0)=\frac{3}{4}g_r^2(l_0)/4\pi$. For convenience $l_0=2$ is taken, since the lattice artifacts are only important at $R\leqq 3$. The differences to the procedure used previously, where the coupling has been adjusted to the potential difference between 1a and 2a, is small. The advantage here is the smooth and prescribed β-dependence of the correction. A similar method has been applied in [28]. The corrected potential will be called $V_C(R)$. The question to be addressed here is whether there is an overall representation of the data for all β, exhibiting the scaling behaviour. It has been noted repeatedly that fits of the form

$$V_C(R)=-\alpha_s/R+\text{const}+K(\beta)R \tag{22}$$

do not work down to $R=1a$ and $R=2a$. Fits starting with $R=3a$ fall deeply below the data when extrapolated to $1a$ and $2a$. If the first term in (22) is interpreted as being of perturbative origin, this is a consequence of the logarithmic terms in the gluon exchange potential. Its explanation by a fluctuating string is currently more popular, although phenomenologically unjustified in view of the physical distances involved [1]. In $SU(2)$ the ambiguity is not easy to resolve numerically because the running coupling constant defined below in (24) at $\beta=2.8$ and $R=4a$ becomes equal to $\pi/12$ which is the coefficient α_s in the string picture. I do not think that it is appropriate to chop off the Coulomb term for $R\geqq 3$ and interprete α_s as the effect of a fluctuating string, even if the fit is tolerable. It has been argued in [28] that the Coulomb term and a nonperturbative string piece have to be added. The argument is based on the evidence that the nonperturbative piece seems to be a Lorentz scalar, whereas the perturbative potential is predominantely of vector type [29]. Although this may lead to deep problems with the continuation of the increasing logarithmic terms of the perturbative potential at large R, a fit at moderate R may be successful. I therefore try a representation of the form

$$V_C(R)=V_{2\,\text{loop}}(R,\Lambda_r)+\text{const}+K(\beta)R \tag{23}$$

Here $V_{2\,\text{loop}}(R,\Lambda_r)$ is given by the integral of the perturbative 2-loop force

$$V'_{2\,\text{loop}}(R,\Lambda_r)$$
$$=3/16\,\pi R^2\{\beta_0\ln(R\Lambda_r)^{-2}+\beta_1/\beta_0\ln\ln(R\Lambda_r)^{-2}\}. \tag{24}$$

The parameter Λ_r is given by [19]

$$\Lambda_r=20.78\,\Lambda_{\text{latt}}. \tag{25}$$

A fit with this value leaves, for all β, a definite discrepancy which varies rapidely with R and which is compatible with a Coulomb term. This is in line with the experience from Creutz ratios that also third order terms are necessary in the perturbative expansion. As a convenient parametrization of these, Λ_r is treated as a free parameter. The data can be fitted well with $\Lambda_r=23.5\,\Lambda_{\text{latt}}$ as shown in Fig. 5. In the figure and in the fit an error of 20% of the lattice correction has been added to the statistical errors. Only the statistical errors are included in Fig. 6 where the differences between data and fit are given on an expanded scale. The parameter $K(\beta)$ need not agree with the slope of the potential at large R and therefore should not be associated with the string tension. It describes a parametrization of the potential at intermediate distances (perhaps in the range up to 0.3 Fermi at $\beta=2.8$), in line with our previous experience that the difference between MC-data and perturbation theory can be expressed by an area term in Creutz ratios. The results for $K(\beta)$ are the following:

$$\begin{aligned}\beta&=2.5:\ a^2K(\beta)=0.0224\pm0.0007\\ \beta&=2.6:\ a^2K(\beta)=0.0108\pm0.0004\\ \beta&=2.7:\ a^2K(\beta)=0.0050\pm0.0003\\ \beta&=2.8:\ a^2K(\beta)=0.0030\pm0.0002.\end{aligned} \tag{26}$$

The above errors are purely statistical. The results at $\beta=2.6$ and $\beta=2.7$ are marginally lower than those published in [1, 2] which is due to the slight increase in Λ_r. Taking into account possible finite lattice size effects especially at $\beta=2.7$, which will reduce $K(\beta)$, the values are almost consistent with a decrease by a factor 2 for a change of β by 0.1, i.e. the corresponding $\Delta\beta$ is 0.20. This is somewhat lower than, but compatible with the smallest $\Delta\beta$ found in the analysis of the factor 2 test. But it is definitely larger than what corresponds to the exponents given in Table 2. This is due to the uncertainties of the subtraction of a scaling function from nonscaling data: The larger the subtraction, the steeper the β-dependence of the rest. Since the values of $K(\beta)$ do not follow AS, a quotation of $\sqrt{K}/\Lambda_{\text{latt}}$ will not be given here.

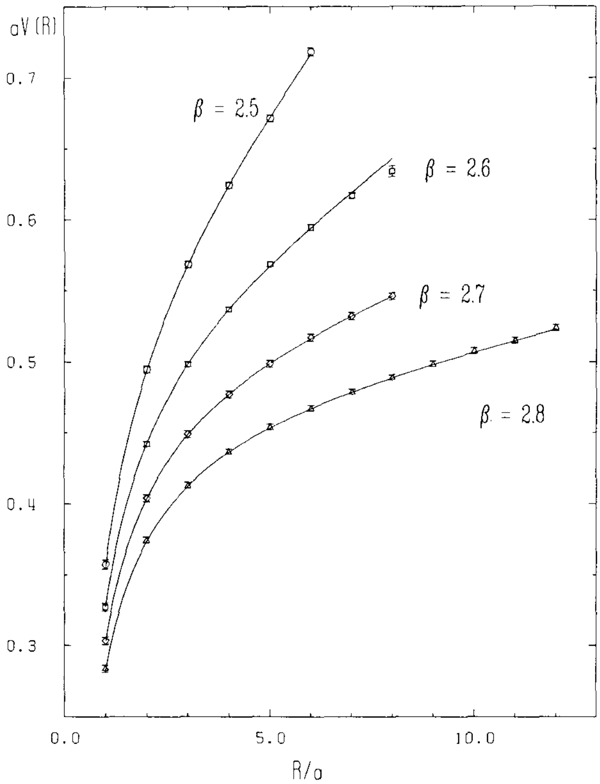

Fig. 5. The corrected static $q\bar{q}$-potential for various β. The curves are the perturbative 2-loop potential (as described in (23). The errors include 20% of the finite a-correction

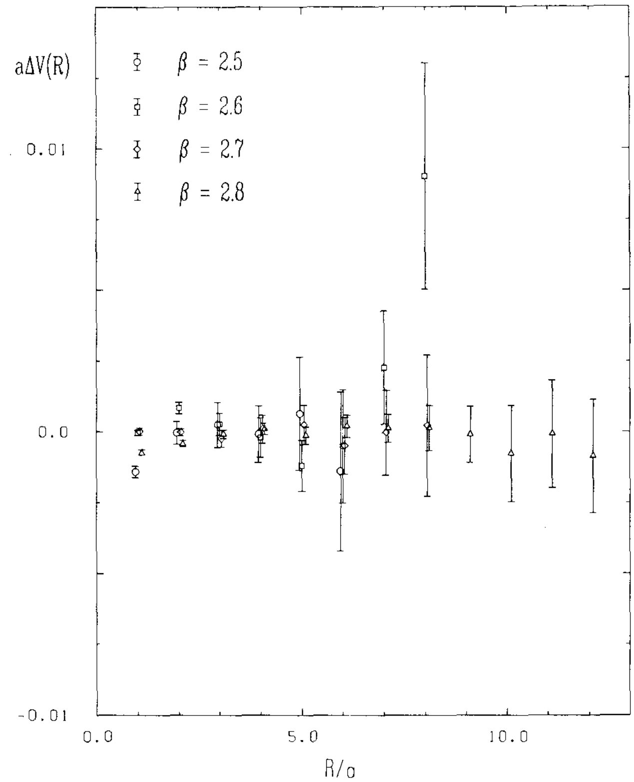

Fig. 6. The difference between the static $q\bar{q}$-potential of Fig. 6 and the fit (23) on an enlarged scale. The errors are purely statistical

Recently, the string tension has been determined from correlations of Polyakov loops on large lattices [30]. The β-dependence is consistent with the above value of $\Delta\beta=0.20$. At $\beta=2.5$ and $\beta=2.6$ the results are higher than the values given above by more than a factor 1.5. Most probably this difference is due to different assumptions about the influence of the perturbative background.

In view of possible finite size effects at large R, the significance of scaling violations in the potential is not as high as for Creutz ratios at finite T. Due to the special form of the potential, the apparent violations are not very large at our values of β. With the parametrization of (23) one predicts variations of $\Delta\beta$ on the order 0.01 between $R=1a$ and $R=2a$, and such variations are visible for potential-like Creutz ratios (see Table 4). The importance of the above consideration lies in demonstrating that the data are consistent with a fit with built-in scaling violations, which will become more pronounced at larger β.

6 Conclusions

From the β-dependence of individual Creutz ratios it follows that at $\beta=2.5$ ratios with $l_{\min}=1$ are dominated by perturbation theory by 90% or more. At $\beta=2.8$ the nonperturbative contribution has dropped to about 1%. This holds, if the perturbation expansion is defined in the running coupling constant $g_r^2(l_0)$, the scale of which is taken half way between the smallest lengths of the ratio. There are arguments that lattice artifacts are successfully corrected for. One comes from the apparent smoothness of $\Delta\beta$ vs R_p in the factor 2 scaling test, the other from the agreement between correction factors including third order contributions on the one hand and the factors obtained in second order perturbation theory on the other hand, as shown in Table 4. For the ratio $\frac{22|11}{21|21}$ scaling can be done for factors 2, 3 and 4, and the results from the β-dependence indicate that the smallness of the nonperturbative contribution observed for $\frac{22|11}{21|21}$ is not a lattice artifact, but persists, after a proper shift in β, also at the larger scales. It will be possible to extend these fits with future MC-data to slightly higher β-values and for more ratios of class 1 and 2 (as defined in Sect. 4). Also scaling by factors 3 will be feasable for ratios with $l=3$, since the determination of ratios at lengths $l \leqq 9$ poses no big problems* for larger β. The perturbative expansion can then be determined with much better accuracy.

Anticipating a confirmation of the present trends we will have to explain the situation that at small l scaling is described correctly by the 2-loop $\bar{\beta}$-function, but not at large l. The behaviour for small l rules out that the bare coupling constant is so large that it leads to a perturbative $\bar{\beta}$-function significantly different from the two-loop expression. Therefore terms not calculable in perturbation theory have to be responsible for these deviations. It is clear that a parametrization of these terms of the form given in Sect. 3 inevitably will lead to scaling violations, if the exponents γ_l are larger than predicted by perturbative scaling for an area term, while the l-dependence is in agreement with such a term. Unless the slopes significantly change close to the present region of β, one has to envisage the situation where all ratios with fixed lengths approach perturbation theory so fast that one cannot assign a physical meaning to the rapidely vanishing terms. Eventually they have to be considered as lattice artifacts altogether. Truly nonperturbative effects like the confining force then could only show up at lengths where the perturbative expansion breaks down. If one takes as a measure for this the point where the perturbative potential, as defined by (24), has a point of inflexion, the lengths are $R=7$ at $\beta=2.5$ and $R=16$ at $\beta=2.8$. To go beyond these distances by Monte Carlo methods will be exceedingly difficult.

Another possibility is that ratios exceeding a certain minimal length stay nontrivial with respect to the β-dependence and to the presence of an area term. Then the small l-region has to be discarded altogether in the sense that the absence of nonperturbative effects is a lattice artifact. This will be hard to understand in view of the smallness of these artifacts in perturbation theory.

Although the present study differs somewhat in conclusions from previous work, there do not seem to be strong discrepancies in MC-data. Deviations from asymptotic scaling of the long distance part of the static potential are well established in $SU(2)$ [30, 31]. They also have been reported for $SU(3)$ [32–34], and it is to be expected that the (not yet analyzed) scaling properties of small sized Creutz ratios also will lead to quasi perturbative behaviour. On the other hand, the topological susceptibility in $SU(2)$ shows excellent asymptotic scaling behaviour [35] in the range $2.4 \leqq \beta \leqq 2.7$, which is a strong indication for scale breaking, if taken together with the deviations from AS of large Creutz ratios. The latter ones should be little affected by the uncertainties of the improvement procedure.

A cautious conclusion is that the present numerical evidence from Creutz ratios, although reasonably accurate, does not give positive support to the existence of a universal $\bar{\beta}$-function describing the scaling properties in the region $\beta \leqq 2.8$. The data are well described by a rapid approach to low order renormalized perturbation theory. It may be worthwhile to study phenomenological realizations of QCD in which at short distances ($R \leqq 0.2$ Fermi say) perturbation theory is exact and not only dominant.

Acknowledgement. The bulk of the new Monte Carlo calculations has been performed on the Siemens vector computer VP200 at the IABG GmbH, Ottobrunn. The author is indebted to the DESY directorium for buying computer time and to the members of the IABG-Rechenzentrum for their effective support. Special thanks in this respect are due to Dipl.-Inf. U. Harms and to Mrs. Dipl.-Inf. E. Koste. Thanks for help in connection with the FPS 164 of the GSI, Darmstadt are due to Mr. A. Popielevicz. Useful discussions with H. Joos, M. Lüscher, J. Kripfganz and R. Sommer are gratefully acknowledged.

* The essential requirement is on computer memory to keep finite lattice size effects small

A. Tables of Creutz ratios for large lattices

Table 5. Creutz ratios of the form $\frac{I, J | I-1, J-1}{I, J-1 | I-1, J}$ at $\beta=2.5$. The lattice size is 24^4

I	$J=2$	3	4	5	6	7	8
2	0.2138 (1)						
3	0.1695 (1)	0.1115 (2)					
4	0.1559 (2)	0.0926 (2)	0.0718 (4)				
5	0.1514 (2)	0.0858 (3)	0.0638 (4)	0.0550 (6)			
6	0.1498 (2)	0.0831 (2)	0.0608 (5)	0.0516 (5)	0.0478 (13)		
7	0.1491 (3)	0.0823 (4)	0.0585 (4)	0.0504 (8)	0.0464 (10)	0.0456 (26)	
8	0.1489 (2)	0.0812 (4)	0.0581 (6)	0.0494 (11)	0.0482 (15)	0.0406 (36)	0.0522 (49)

Table 6. Creutz ratios of the form $\frac{I, J|I-2, J-2}{I, J-2|I-2, J}$ at $\beta=2.5$. The lattice size is 24^4

I	$J=3$	4	5	6	7	8
3	0.6643 (6)					
4	0.5295 (8)	0.3684 (10)				
5	0.4857 (8)	0.3140 (13)	0.2544 (17)			
6	0.4701 (8)	0.2935 (13)	0.2311 (19)	0.2059 (26)		
7	0.4643 (10)	0.2847 (14)	0.2213 (18)	0.1962 (23)	0.1862 (33)	
8	0.4616 (11)	0.2802 (15)	0.2164 (22)	0.1943 (29)	0.1808 (54)	0.1790 (101)

Table 7. Creutz ratios of the form $\frac{I, J|I-1, J-1}{I, J-1|I-1, J}$ at $\beta=2.7$. The lattice size is 24^4

I	$J=2$	3	4	5	6	7	8
2	0.16923 (4)						
3	0.12887 (4)	0.07755 (8)					
4	0.11671 (6)	0.06138 (7)	0.04343 (17)				
5	0.11260 (7)	0.05538 (8)	0.03684 (13)	0.02994 (19)			
6	0.11102 (7)	0.05296 (13)	0.03397 (17)	0.02639 (21)	0.02264 (23)		
7	0.11037 (7)	0.05181 (12)	0.03243 (21)	0.02487 (24)	0.02087 (30)	0.01867 (45)	
8	0.11021 (10)	0.05128 (14)	0.03130 (22)	0.02368 (30)	0.02004 (37)	0.01765 (45)	0.0169 (9)
9				0.02365 (53)	0.01820 (35)	0.01668 (49)	0.0164 (8)
10				0.02274 (58)	0.01945 (68)	0.01683 (74)	0.0135 (9)

Table 8. Creutz ratios of the form $\frac{I, J|I-2, J-2}{I, J-2|I-2, J}$ at $\beta=2.7$. The lattice size is 24^4

I	$J=3$	4	5	6	7	8
3	0.50453 (15)					
4	0.38451 (19)	0.24374 (31)				
5	0.34607 (23)	0.19703 (34)	0.14705 (49)			
6	0.33196 (26)	0.17915 (42)	0.12714 (55)	0.10536 (73)		
7	0.32616 (31)	0.17117 (49)	0.11766 (60)	0.09477 (70)	0.08305 (98)	
8	0.32367 (35)	0.16681 (51)	0.11228 (68)	0.08946 (89)	0.07723 (104)	0.07083 (157)
9			0.10879 (84)	0.08556 (106)	0.07257 (95)	0.06756 (162)
10			0.10670 (105)	0.08403 (141)	0.07115 (137)	0.06337 (137)

Table 9. Creutz ratios of the form $\frac{I, J|I-1, J-1}{I, J-1|I-1, J}$ at $\beta=2.8$. The lattice size is $24^2 * 32^2$

I	$J=2$	3	4	5	6	7
2	0.15468 (3)					
3	0.11661 (2)	0.06871 (2)				
4	0.10533 (2)	0.05383 (4)	0.03751 (5)			
5	0.10153 (4)	0.04842 (4)	0.03136 (5)	0.02484 (7)		
6	0.10006 (4)	0.04613 (4)	0.02868 (7)	0.02171 (8)	0.01866 (10)	
7	0.09940 (6)	0.04502 (4)	0.02719 (6)	0.02023 (8)	0.01673 (8)	0.01495 (11)
8	0.09910 (4)	0.04439 (6)	0.02650 (8)	0.01922 (6)	0.01574 (14)	0.01374 (13)
9	0.09887 (8)	0.04406 (6)	0.02596 (7)	0.01869 (7)	0.01486 (11)	0.01324 (19)
10	0.09883 (6)	0.04391 (7)	0.02560 (8)	0.01820 (15)	0.01469 (17)	0.01250 (20)
11	0.09877 (8)	0.04380 (8)	0.02547 (8)	0.01812 (14)	0.01407 (26)	0.01236 (25)
12	0.09880 (8)	0.04367 (8)	0.02539 (14)	0.01783 (17)	0.01432 (16)	0.01202 (33)

I	$J=8$	9	10	11	12
8	0.01275 (23)				
9	0.01180 (31)	0.01162 (40)			
10	0.01138 (30)	0.01012 (37)	0.01018 (67)		
11	0.01083 (39)	0.01005 (45)	0.00961 (57)	0.01034 (131)	
12	0.01030 (26)	0.01057 (40)	0.00924 (70)	0.00689 (105)	0.00979 (225)

Table 10. Creutz ratios of the form $\frac{I, J|I-2, J-2}{I, J-2|I-2, J}$ at $\beta=2.8$. The lattice size is $24^2 * 32^2$

I	$J=3$	4	5	6	7
2	0.48661 (6)				
3	0.48660 (7)				
4	0.34447 (10)	0.21389 (15)			
5	0.30911 (13)	0.17113 (17)	0.12508 (21)		
6	0.29614 (14)	0.15459 (19)	0.10659 (26)	0.08692 (29)	
7	0.29061 (15)	0.14701 (20)	0.09780 (28)	0.07733 (29)	0.06708 (32)
8	0.28791 (15)	0.14310 (23)	0.09313 (27)	0.07192 (31)	0.06117 (38)
9	0.28642 (19)	0.14091 (25)	0.09037 (25)	0.06851 (31)	0.05758 (50)
10	0.28567 (21)	0.13953 (25)	0.08846 (29)	0.06644 (39)	0.05529 (52)
11	0.28531 (21)	0.13879 (27)	0.08740 (32)	0.06508 (53)	0.05362 (60)
12	0.28505 (22)	0.13834 (32)	0.08682 (39)	0.06434 (48)	0.05277 (66)

I	$J=8$	9	10	11	12
8	0.05519 (49)				
9	0.05153 (60)	0.04797 (77)			
10	0.04892 (74)	0.04493 (90)	0.04204 (105)		
11	0.04708 (82)	0.04239 (107)	0.03996 (97)	0.03973 (163)	
12	0.04550 (80)	0.04175 (88)	0.03947 (131)	0.03608 (191)	0.03392 (290)

References

1. F. Gutbrod: Z. Phys. C – Particles and Fields 30 (1986) 585
2. F. Gutbrod: Phys. Lett. 186B (1987) 389
3. M. Creutz: Phys. Rev. D23 (1981) 1815
4. K.C. Bowler et al.: Nucl. Phys. B257 (FS14) (1985) 155
5. K.C. Bowler et al.: Phys. Lett. 163B (1985) 367
6. K.C. Bowler et al.: Phys. Lett. 179B (1986) 375
7. A. Patel, S. Otto, R. Gupta: Phys. Lett. 159B (1985) 143
8. F. Gutbrod, I. Montvay: Phys. Lett. 136B (1983) 411
9. U. Heller, F. Karsch: Phys. Rev. Lett. 54 (1985) 1765
10. A. Patel, R. Gupta: Nucl. Phys. B251 (FS13) (1985) 789
11. S.A. Gottlieb et al.: Phys. Rev. Lett. 55 (1985) 1958
12. N.H. Christ, A.E. Terrano: Phys. Rev. Lett. 56 (1986) 111
13. D. Toussaint: In: Proc. Symposium Lattice Gauge Theory 1986. New York: Plenum 1987
14. N.H. Christ: Coulumbia preprint CU-TP-372 (1987)
15. A. Hasenfratz, P. Hasenfratz: Phys. Lett. 93B (1980) 165
16. U. Heller, F. Karsch: Nucl. Phys. B251 (F13) (1985) 254
17. G. Parisi, R. Petronzio, F. Rapuano: Phys. Lett. 128B (1983) 418
18. R. Kirschner, J. Kripfganz, J. Ranft, A. Schiller: Nucl. Phys. B210 (FS6) (1982) 567
19. A. Billoire: Phys. Lett. 104B (1981) 472
20. A. Hasenfratz, P. Hasenfratz, U. Heller, F. Karsch: Phys. Lett. 143B (1984) 193
21. A. Hasenfratz, P. Hasenfratz, U. Heller, F. Karsch: Phys. Lett. 140B (1984) 76
22. W. Celmaster, F. Green, R. Gupta, E. Kovács: Comput. Phys. Commun. 36 (1985) 409
23. G. 't Hooft: Nucl. Phys. B153 (1979) 141
24. A. Gonzales-Arroyo, J. Jurkiewicz, C.P. Korthals-Altes: Proceedings of the Freiburg NATO Summer Institute (1981). New York: Plenum
25. M. Lüscher, P. Weisz: Nucl. Phys. B266 (1986) 309
26. C. Michael: Preprint P/86/5/67 (Urbana, 1986)
27. M. Campostrini, G. Curci, A. Di Giacomo, G. Paffuti: Z. Phys. C – Particles and Fields 32 (1986) 377
28. A. Huntley, C. Michael: Nucl. Phys. B286 (FS16) (1987) 211
29. J. Pantaleone, S.-H.H. Tye, Y.J. Ng: Phys. Rev. D33 (1986) 777
30. C. Michael, M. Teper: Preprint Ref. 54/87 (Oxford, 1987)
31. A. Huntley, C. Michael: Nucl. Phys. B270 (FS16) (1986) 123
32. Ph. de Forcrand: In: Lattice Gauge Theory, Proceedings of a NATO workshop in Wuppertal 1985, B. Bunk, K.H. Mütter, K. Schilling (eds.). New York: Plenum Press 1986
33. J. Flower, S.W. Otto: Phys. Rev. D34 (1986) 1649
34. S. Itoh, Y. Iwasaki, T. Yoshié: Phys. Lett. 185B (1987) 390
35. A.S. Kronfeld, M.L. Laursen, G. Schierholz, U.-J. Wiese: DESY 86-082 (July 1986) (to be published in Nucl. Phys. B), and private communication