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SCATTERING THEORY FOR EUCLIDEAN LATTICE FIELDS

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A Haag-Ruelle scattering theory for Euclidean lattice fields is developped.

The main motivation for the investigation of Euclidean Lattice Field Theories (ELFT) is the aim to understand particle physics. It is, however, not obvious at all that ELFT directly describe particles. Usually for the relation to particle physics the continuum limit is used. Continuum Euclidean field theories lead (under rather general circumstances) to quantum field theories on Minkowski space which (again under quite general conditions) describe the behavior of particles. The pertinent quantities in the continuum limit are then approximated on the lattice. This procedure is of course not unique.

A direct particle interpretation of ELFT would be highly desirable. It would lead to an unambiguous definition of physical quantities like cross sections. It also would justify the interpretation of ELFT with a trivial continuum limit as effective theories for particles. In any case ELFT would become a respectable theory which shares some structural properties with continuum quantum field theory and which is better accessible by analytical and numerical methods.

Let us briefly review the status of the particle interpretation of continuum quantum field theory. The basic structural properties which are exploited are local commutativity of space like separated observables as an implementation of Einstein causality, and the spectrum condition. A rather satisfactory analysis can be made in theories without massless particles. In theories with physical massless particles problems with infraparticles occur which are presently not under full control [1].

In purely massive theories one first has to find the single particle states. If they belong to the vacuum sector, i.e. if the particles carry no charge, there exist so-called almost local operators which create the particle states from the vacuum. Using them one constructs the outgoing and incoming multiparticle states via methods of the Haag-Ruelle scattering theory [2]. If the single particle state is not in the vacuum sector, i.e. if the particle is charged, one first has to apply the theory of superselection sectors (see [3] for a review) to construct the group of global gauge transformations and the charged fields. Then one can again apply the methods of the Haag-Ruelle scattering theory for the construction of all scattering states.

There are also some unsolved problems. Besides the already mentioned problem of infraparticles the main open problem is the asymptotic completeness, i.e. the question whether each state is an incoming and an outgoing scattering state. (For recent progress in this problem see [1], [4]).

The main obstruction for performing a corresponding analysis in ELFT is the absence of local commutativity for spacelike separations in the corresponding real time quantum theory. Consider e.g. the theory of a scalar Euclidean field $\varphi(x)$, $x = (x^0, \underline{x}) \in \mathbb{Z}^{d+1}$, $d \geq 1$. Let $\langle . \rangle$ be a translation invariant and reflection positive state. Then, using the transfer matrix formalism, one finds a Hilbert space $\mathcal{H}$, a vector $\Omega \in \mathcal{H}$,

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representing the vacuum, a positive operator $T \leq 1$ (the transfer matrix) and a quantum field $\Phi(\underline{x})$ such that for $x_1^0 \leq \ldots \leq x_n^0$,

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle =$$

$$(\Omega, \Phi(\underline{x_1}) T^{x_2^0 - x_1^0} \ldots T^{x_n^0 - x_{n-1}^0} \Phi(\underline{x_n}) \Omega) \qquad (1)$$

In typical cases $T > 0$ [5]; one then can define the (quantum) Hamiltonian by $H = -\ln T$ and the real time evolved quantum field by

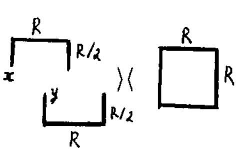
$$\Phi(t, \underline{x}) = \exp(iHt)\, \Phi(\underline{x})\, \exp(-iHt). \qquad (2)$$

Unfortunately H and $\Phi(t, \underline{x})$, $t \neq 0$ seem to be , in general, rather delocalized, even if the state on the Euclidean fields is a Gibbs state with respect to some local Euclidean action.

For the identification of single particle states in massive theories this presents no problem, since they are characterized completely in terms of the energy-momentum spectrum. There are numerous heuristic and numerical results, and rigorous proofs of existence have been given in many cases [6]. There are also examples of charged single particle states: the electrically charged particles in the free-charge phase of the Z(2) gauge-Higgs model [7] and soliton like particles in a variety of models [8].

All the existence proofs rely on an analysis of 2-point functions $G(x,y)$. In the case of charged fields the definition of G is not obvious since it is difficult to find field operators which create the particles from the vacuum (for an attempt see [9]). In [7], following [10] the following definition of G was used (the pictorial notation means a product of the gauge fields along the path (in the unitary gauge)):

$$G(x,y) = \{\lim_{R\to\infty} \langle$$



$$\rangle^{-1}\}^{1/2}. \qquad (3)$$

Note the similarity of G with the vacuum overlap order parameter [10, 11].

Let us now describe the main idea for the construction of scattering states. Euclidean Green functions in massive phases cluster exponentially. These clustering properties show up in some way in the real time Green functions (Wightman functions). The clustering properties of Wightman functions then lead to a weak form of local commutativity which is sufficient for the purposes of scattering theory.

First we investigate the cluster properties of Wightman functions. In a massive phase with unique vacuum the connected Euclidean Green functions decay exponentially:

$$|\langle \varphi(x_1) \ldots \varphi(x_n) \rangle_c| \leq \exp(-mR(x_1, \ldots, x_n)) \qquad (4)$$

where $m > 0$ is the mass gap and $R(x_1, \ldots, x_n)$ is the side lenght of the smallest lattice cube containing $x_1, \ldots, x_n$. For later convenience we choose φ to be bounded which is always to be achieved on a lattice by redefining φ (e.g. by replacing φ by $\arctan\varphi$ for a real field φ). We choose now a function $g(t, \underline{x})$ whose Fourier transform is smooth with compact support, and define cutoff quantum fields Φ_g by

$$\Phi_g(t, \underline{x}) = \int ds \sum_{\underline{y}} g(s, \underline{y}) \Phi(t+s, \underline{x}+\underline{y}) \qquad (5)$$

where Φ is the quantum field corresponding to φ. Let

$$W_g(t_1, \underline{x_1}; \ldots; t_n, \underline{x_n}) =$$

$$(\Omega, \Phi_g(t_1, \underline{x_1}) \ldots \Phi_g(t_n, \underline{x_n}) \Omega) \qquad (6)$$

be the Wightman functions of the cutoff field Φ_g, and let $W_{g,c}$ denote the corresponding connected (sometimes also called truncated) functions. We have the following theorem:

Theorem 1 *For all natural numbers N there is some $C_N = C_N(g) > 0$ such that*

$$|W_{g,c}(t_1, \underline{x_1}; \ldots; t_n, \underline{x_n})| \leq C_N (1 + t^N)/(mR)^N, \qquad (7)$$

where $t = \max_i |t_{i+1} - t_i|$ and $R = R(\underline{x_1}, \ldots, \underline{x_n})$.

We sketch the proof for the simplest nontrivial case $n = 2$, $\tilde{g}$ independent of $\underline{p}$, $\langle\varphi\rangle = 0$ (so the connected and the disconnected 2-point function coincide). Using the definition of W_g and Φ_g one finds

$$W_{g,c}(s,\underline{y};t+s,\underline{x}+\underline{y}) = .$$

$$(\Omega, \Phi(0)e^{iHt}\tilde{g}(H)\tilde{g}(-H)\Phi(\underline{x})\Omega). \quad (8)$$

Let $\tilde{g}(H)\tilde{g}(-H) = \chi(H)$. $\chi(H)\exp(iHt)$ is (in contrast to $\exp(iHt)$) a continuous function of $\exp(-H) = T$ and can therefore be approximated uniformly by polynomials of $\exp(-H)$:

$$\chi(H)\exp(iHt) = \sum_{k=0}^{n} a_k^{(n)} \exp(-nH) + \epsilon(n), \quad (9)$$

with $\|\epsilon(n)\| \to 0$ for $n \to \infty$. Inserting this approximation in (8) one obtains

$$|W_{g,c}(0;t,\underline{x})| \leq$$

$$\sum_{k=0}^{n} |a_k^{(n)}| \; |\langle\varphi(0),\varphi(k,\underline{x})\rangle_c| + \|\epsilon(n)\| \, \langle|\varphi|^2\rangle. \quad (10)$$

If one uses an approximation by Chebyshev polynomials in (9) one gets the estimates

$$\|\epsilon(n)\| \leq C'_N(|t|/n)^N, \quad C'_N > 0, N \in I\!N,$$

$$\sum |a_k^{(n)}| \leq C' \exp(cn), \qquad c, C' > 0. \quad (11)$$

Together with the bound (4) this implies

$$|W_{g,c}(0;t,\underline{x})| \leq C'\exp(cn - mR(0,\underline{x})) + C'_N(|t|/n)^N. \quad (12)$$

Choosing $n = c"R(0,\underline{x})$, $cc" < m$ we obtain the desired bound (7).

With the help of the cluster properties of Wightman functions established in this theorem the construction of scattering states is straightforward. We assume that there is an isolated shell $\{(E(\underline{p}),\underline{p}),\ \underline{p} \in (-\pi,\pi]^d\}$ in the energy-momentum spectrum of $\Phi(0)\Omega$, with a smooth dispersion relation $E(\underline{p})$. We further assume that there are sufficiently many different velocities , i.e. $grad\,E(\underline{p}) \neq grad\,E(\underline{q})$ for almost all $\underline{p} \neq \underline{q}$. We now choose a smooth cutoff function g such that $\tilde{g} = 1$ on the single particle shell and $\tilde{g} = 0$ on the remaider of the energy momentum spectrum. Then $\Phi_g(0)\Omega$ is a single particle state. Now let f be a negative frequency solution of the wave equation corresponding to the dispersion relation $E(\underline{p})$:

$$f(t,\underline{x}) = (2\pi)^{-d}\int d^d\underline{p}\; e^{-i(E(\underline{p})t-\underline{p}.\underline{x})}\hat{f}(\underline{p}), \quad (13)$$

where $\hat{f}$ is smooth. Let $V(f)$ denote the velocity support of f,

$$V(f) = \{grad\,E(\underline{p}),\ \underline{p} \in supp\,\hat{f}\}. \quad (14)$$

$f(t,\underline{x})$ is essentially localized in the kinematically allowed set $\{(t,\underline{v}t),\ \underline{v} \in V(f)\}$. More precisely one has

Proposition 1

$$(i)\ \sum_{\underline{x}} |f(t,\underline{x})| \leq const|t|^{d/2}, \quad (15)$$

$$(ii)\ |f(t,\underline{x})| \leq C_N t^{-N} dist(\underline{x}/t, V_\delta(f))^{-N}, \quad (16)$$

with $V_\delta(f) = \{\underline{v},\ dist(\underline{v},V(f)) \leq \delta\}$, $\delta > 0$, *for all* $N \in I\!N$. *For a proof see for instance* [12].

Consider now the smeared field

$$A_f(t) = \sum_{\underline{x}} \Phi_g(t,\underline{x})f(t,\underline{x}). \quad (17)$$

$A_f(t)\Omega \equiv \Psi(f)$ is a one particle state which does not depend on t. Its momentum space wave function is $\hat{f}(\underline{p})Z(\underline{p})^{1/2}$ where $Z(\underline{p}) = \sum_{\underline{x}}(\Phi_g(0)\Omega, \Phi_g(\underline{x})\Omega)\exp(i\underline{p}.\underline{x})$ and where we used the normalization

$$\langle\ \underline{p}\ |\ \underline{q}\ \rangle = \delta(\underline{p}-\underline{q}) \quad (18)$$

for the improper single particle momentum eigenvectors. Let $f_1,\ldots,f_n$ be smooth negative frequency solutions with nonoverlapping velocities, i.e.

$$V(f_i) \cap V(f_j) = \emptyset, \quad i \neq j. \quad (19)$$

Then the Haag-Ruelle approximants for a multiparticle scattering state with wave function

$$\hat{f}(\underline{p_1},\ldots,\underline{p_n}) = Z(\underline{p_1})^{1/2}\ldots Z(\underline{p_n})^{1/2}\hat{f}_1(\underline{p_1})\ldots\hat{f}_n(\underline{p_n}) \quad (20)$$

are defined by

$$\Psi(f_1,\ldots,f_n;t) = A_{f_1}(t)\ldots A_{f_n}(t)\Omega. \quad (21)$$

Using the Theorem 1 and the Proposition one finds:

Theorem 2 *(i) The limit*

$$\lim_{t\to\pm\infty} \Psi(f_1,\ldots,f_n;t) \equiv \Psi_{\substack{out\\in}}(f_1,\ldots,f_n) \quad (22)$$

exists,

(ii)

$$(\Psi_{\substack{in\\out}}(f_1,\ldots,f_n), \Psi_{\substack{in\\out}}(g_1,\ldots,g_k)) =$$

$$\delta_{nk} \sum_{permutations\ \sigma} (\Psi_{f_1},\Psi_{g_{\sigma(1)}})\ldots(\Psi_{f_n},\Psi_{g_{\sigma(n)}}). \quad (23)$$

Theorem 2 (ii) shows in particular that the space of outgoing, resp. incoming scattering states has the structure of a bosonic Fock space, so the particle is a boson (as expected).

The S-matrix is now given by

$$(2\pi)^{-dn} \int d^d\underline{p_1}\ldots\int d^d\underline{p_n}$$
$$Z(\underline{p_1})^{1/2}\ldots Z(\underline{p_n})^{1/2}$$
$$\overline{\hat{f}_1(\underline{p_1})}\ldots\overline{\hat{f}_k(\underline{p_k})}\hat{f}_{k+1}(\underline{p_{k+1}})\ldots\hat{f}_n(\underline{p_n})$$
$$\langle\underline{p_1}\ldots\underline{p_k}|S|\underline{p_{k+1}}\ldots\underline{p_n}\rangle =$$
$$(\Psi_{out}(f_1,\ldots,f_k),\Psi_{in}(f_{k+1},\ldots,f_n)). \quad (24)$$

There is also an LSZ reduction formula for nonoverlapping velocities. One finds for $grad\,E(\underline{p_i}) \neq grad\,E(\underline{p_j}),\ i\neq j$:

$$\langle\underline{p_1}\ldots\underline{p_k}|S|\underline{p_{k+1}}\ldots\underline{p_n}\rangle =$$
$$Z(\underline{p_1})^{-1/2}(p_1^0 - E(\underline{p_1}))\ldots Z(\underline{p_n})^{-1/2}(p_n^0 - E(\underline{p_n}))$$
$$i^n\int dt_1\ldots dt_n \sum_{\underline{x_1}\ldots\underline{x_n}}$$
$$\exp\{i[\epsilon_1(p_1^0t_1 - \underline{p_1}\cdot\underline{x_1}) + \ldots + \epsilon_n(p_n^0t_n - \underline{p_n}\cdot\underline{x_n})]\}$$
$$(\Omega, T(\Phi(t_1,\underline{x_1})\ldots\Phi(t_n,\underline{x_n}))\Omega)|_{p_1^0=E(\underline{p_1}),\ldots,p_n^0=E(\underline{p_n})}, \quad (25)$$
$$\epsilon_1 = \ldots\epsilon_k = -1,\quad \epsilon_{k+1} = \ldots\epsilon_n = +1.$$

Here T is a time ordering prescription. Actually, there is a large freedom in the choice of T. One possibility is, to make a Fourier transform of the Euclidean correlation functions, perform a Wick rotation and transform back to position space. As Lüscher has shown [13] the result differs from the real time ordering prescription only by exponentially decaying tails which do not affect the validity of the LSZ formula.

Scattering states can also be constructed for particles carrying a gauge charge, i.e. for particles which cannot be created from the vacuum by local fields. Since there is still no theory of the superselection structure of ELFT one has to guess the approximants for scattering states. In the Z(2) gauge-Higgs model one can use the ground states $|\underline{x}_1,\ldots,\underline{x}_n\rangle$ with external charges at prescribed points $\underline{x}_1,\ldots,\underline{x}_n$ constructed in [10]. A Haag-Ruelle approximant for a two-particle scattering state, e.g., is given by

$$\Psi(f_1,f_2;t) =$$
$$\int dt'dt" \sum_{\underline{x},\underline{y},\underline{x}',\underline{y}'} f_1(t,\underline{x})f_2(t,\underline{y})g(t',\underline{x}')g(t",\underline{y}')$$
$$\sigma_3(t+t',\underline{x}+\underline{x}')\sigma_3(t+t",\underline{y}+\underline{y}')|\underline{x}+\underline{x}',\underline{y}+\underline{y}'\rangle. \quad (26)$$

where σ_3 is the Z(2) Higgs quantum field. One can show that in that part of the free charge phase where the existence of charged particles was established the Haag-Ruelle approximants defined as above converge, and one finds that the space of scattering states is a bosonic Fock space. So the charged particle in this model is a boson.

There are also models (in 2+1 spacetime dimensions) where anomalous statistics is expected [8]. The anomalous statistics should show up in an unambiguous way in the scalar products of scattering states.

We conclude that ELFT typically have a full particle structure. In principle, scattering amplitudes can be computed from Euclidean correlation functions. How this can be realized e.g. in Monte Carlo simulations has still to be worked out.

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