THE PRODUCT FORM FOR PATH INTEGRALS ON CURVED MANIFOLDS

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A general and simple framework for treating path integrals on curved manifolds is presented. The crucial point will be a product ansatz for the metric tensor and the quantum hamiltonian, i.e. we shall write $g_{\alpha\beta}=h_{\alpha\gamma}h_{\beta\gamma}$ and $H=(1/2m)h^{\alpha\gamma}p_\alpha p_\beta h^{\beta\gamma}+V+\Delta V$, respectively, a prescription which we shall call "product form" definition. The p_α are hermitian momenta and ΔV is a well-defined quantum correction. We shall show that this ansatz, which looks quite special, is in fact – under reasonable assumptions in quantum mechanics – a very general one. We shall derive the lagrangian path integral in the "product form" definition and shall also prove that the Schrödinger equation can be derived from the corresponding short-time kernel. We shall discuss briefly an application of this prescription to the problem of free quantum motion on the Poincaré upper half-plane.

1. Introduction

Many problems in theoretical physics make it desirable to have a precise and comfortable formulation of path integrals on curved manifolds. Approaches towards a general theory exist due to DeWitt [1], McLaughlin and Schulman [2], Dowker and Mayes [3], Mizrahi [4], Gervais and Jevicki [5], Omote [6], Marinov [7], Lee [8] and Grosche and Steiner [9]. Let us recall first the most important facts.

We start with the generic case where the time dependent Schrödinger equation in some riemannian manifold M with metric $g_{\alpha\beta}$ and line element $ds^2=g_{\alpha\beta}dq^\alpha dq^\beta$ is given by

$$\left(-\frac{\hbar^2}{2m}\Delta_{LB}+V(q)\right)\psi(q;t)=\frac{\hbar}{i}\frac{\partial}{\partial t}\psi(q;t)\ . \tag{1}$$

ψ is some state function, defined in the Hilbert space $L^2(M)$ – the space of all square integrable functions in the sense of the scalar product

$$(f_1,f_2)=\int_M \sqrt{g}\,f_1(q)f_2^*(q)\,dq$$

$[g:=\det(g_{\alpha\beta}),\ f_1,f_2\in L^2(M)]$ and Δ_{LB} is the Laplace–Beltrami operator

$$\Delta_{LB}:=g^{-1/2}\partial_\alpha g^{1/2}g^{\alpha\beta}\partial_\beta=g^{\alpha\beta}\partial_\alpha\partial_\beta+g^{\alpha\beta}(\partial_\alpha \ln\sqrt{g})\partial_\beta+g^{\alpha\beta}{}_{,\alpha}\partial_\beta$$

(implicit sums over repeated indices are understood).

The hamiltonian $H:=-(\hbar^2/2m)\Delta_{LB}+V(q)$ is usually defined in some dense subset $D(H)\subseteq L^2(M)$, such that H is selfadjoint. In contrast to the time independent Schrödinger equation $H\psi=E\psi$, which is an eigenvalue problem, and eq. (1) which are both defined on $D(H)$, the unitary operator $U(T):=e^{-iTH/\hbar}$ describes the time evolution of arbitrary states $\psi\in L^2(M)$ (time-evolution operator); H is the infinitesimal generator of U. The

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[#1] We only consider systems which such a simple structure; see ref. [4] for a generalisation.

time evolution for some state ψ reads: $|\psi(t'')\rangle = e^{-iTH/\hbar}|\psi(t')\rangle$ $(T=t''-t')$. Rewriting the time evolution with $U(T)$ as an integral operator we get

$$\psi(q'';t'') = \int \sqrt{g(q')} K(q'',q';T)\psi(q';t')\,dq' , \tag{2}$$

where $K(T)$ is the celebrated Feynman kernel. Eqs. (1) and (2) are connected. Having an explicit expression for $K(T)$ in (2) one can derive in the limit $T=\epsilon\to 0$ eq. (1). This, on the other hand proves that $K(T)$ is indeed the correct integral kernel corresponding to $U(T)$. A rigorous proof includes, of course, the check of the selfadjointness of H, i.e. $H=H^*$.

It was Feynman's (and Dirac's) genius [10] to see that $K(T)$ can be expressed as a sum over all possible paths connecting the points q' and q'' with weight factor $\exp[(i/\hbar)S(q'',q';T)]$ where S is the action, i.e.

$$K(q'',q';T) = \sum_{\text{all paths}} \exp[(i/\hbar)S(q'',q';T)] . \tag{3}$$

In the case of a euclidean space, where $g_{\alpha\beta}=\delta_{\alpha\beta}$, S is just the classical action,

$$S_{cl} = \int [\tfrac{1}{2}m\dot{q}^2 - V(q)]\,dt = \int \mathcal{L}_{cl}(q,\dot{q})\,dt ,$$

and we get explicitly $[\Delta q^{(j)} := (q^{(j)} - q^{(j-1)}),\ q^{(j)} = q(t_j),\ t_j = t' + j\epsilon,\ \epsilon = (t''-t')/N,\ N\to\infty,\ d=$ dimension of the euclidean space$]$:

$$K(q'',q';T) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon\hbar}\right)^{Nd/2} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dq^{(j)} \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}\Delta^2 q^{(j)} - \epsilon V(q^{(j)})\right)\right] . \tag{4}$$

For a proof see e.g. refs. [11,12].

For an arbitrary metric $g_{\alpha\beta}$ things are unfortunately not so easy. The first formulation for this case is due to DeWitt [1]. His result reads

$$K(q'',q';T) = \int_{\text{DeW}} \sqrt{g}\,\mathrm{D}q(t) \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\tfrac{1}{2}m g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - V(q) + \frac{\hbar^2 R}{6m}\right)dt\right]$$

$$:= \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon\hbar}\right)^{Nd/2} \prod_{j=1}^{N-1} \int \sqrt{g(q^{(j)})}\,dq^{(j)} \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon} g_{\alpha\beta}(q^{(j-1)})\Delta q^{\alpha,(j)}\Delta q^{\beta,(j)}\right.\right.$$

$$\left.\left. - \epsilon V(q^{(j-1)}) + \epsilon\frac{\hbar^2}{6m}R(q^{(j-1)})\right)\right] \tag{5}$$

($R = g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\gamma\beta,\alpha} + \Gamma^\delta_{\alpha\beta}\Gamma^\gamma_{\gamma\delta} - \Gamma^\delta_{\gamma\beta}\Gamma^\gamma_{\alpha\delta})$: scalar curvature; $\Gamma^\alpha_{\beta\gamma} = g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$: Christoffel symbols). Two comments are in order:

(1) Eq. (5) has the form (3) but the corresponding $S=\int\mathcal{L}dt$ is not the classical action, respectively the langrangian $\mathcal{L}$ is not the classical lagrangian $\mathcal{L}_{cl}(q,\dot{q}) = \frac{1}{2}m g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - V(q)$, but rather an effective one:

$$S_{eff} = \int \mathcal{L}_{eff}\,dt \equiv \int (\mathcal{L}_{cl} - \Delta V_{DeW})\,dt . \tag{6}$$

The quantum correction $\Delta V_{DeW} = -(\hbar^2/6m)R$ is indispensible in order to derive from the time evolution (2) the Schrödinger equation (1), see also ref. [13]. The appearance of a quantum correction ΔV is a very general feature for path integrals defined on curved manifolds; but, of course, $\Delta V \sim \hbar^2$ depends on the lattice definition.

(2) A specific lattice definition has been chosen. The metric terms in the action are evaluated at the "prepoint" $q^{(j-1)}$. Changing the lattice definition, i.e. evaluation of the metric terms at other points, e.g. the "postpoint" $q^{(j)}$ or the "midpoint" $\bar{q}^{(j)} := \frac{1}{2}(q^{(j)} + q^{(j-1)})$ changes ΔV, because in a Taylor expansion of the relevant

terms, *all* terms of $O(\epsilon)$ contribute to the path integral. This fact is particularly important in the expansion of the kinetic term in the lagrangian, where we have $\Delta^4 q^{(j)}/\epsilon \sim O(\epsilon)$.

A very convenient lattice prescription is the midpoint definition, which is connected to the Weyl-ordering prescription in the hamiltonian H. Let us discuss this prescription in some detail. First we have to construct momentum operators [14]:

$$p_\alpha = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial q^\alpha} + \frac{\Gamma_\alpha}{2}\right), \quad \Gamma_\alpha = \frac{\partial \ln\sqrt{g}}{\partial q^\alpha}, \tag{7}$$

which are hermitian with respect to the scalar product $(f_1, f_2) = \int f_1 f_2^* \sqrt{g}\, \mathrm{d}q$. In terms of the momentum operators (7) we rewrite H by using the Weyl-ordering prescription [4,8,9] (W=Weyl):

$$H(p,q) = \frac{1}{8m}(g^{\alpha\beta}p_\alpha p_\beta + 2p_\alpha g^{\alpha\beta}p_\beta + p_\alpha p_\beta g^{\alpha\beta}) + \Delta V_{\mathrm{W}}(q) + V(q)\,. \tag{8}$$

In eq. (8) a well-defined quantum correction appears which is given by [4,6,9]

$$\Delta V_{\mathrm{W}} = \frac{\hbar^2}{8m}(g^{\alpha\beta}\Gamma^\delta_{\alpha\gamma}\Gamma^\gamma_{\beta\delta} - R) = \frac{\hbar^2}{8m}[g^{\alpha\beta}\Gamma_\alpha\Gamma_\beta + 2(g^{\alpha\beta}\Gamma_\alpha)_{,\beta} + g^{\alpha\beta}{}_{,\alpha\beta}]\,. \tag{9}$$

Using the Trotter formula $\mathrm{e}^{-\mathrm{i}t(A+B)} = \text{s-lim}_{N\to\infty}(\mathrm{e}^{-\mathrm{i}tA/N}\mathrm{e}^{-\mathrm{i}tB/N})^N$ [12] and the short-time approximation for the matrix element $\langle q''|\mathrm{e}^{-\mathrm{i}\epsilon H/\hbar}|q'\rangle$ one obtains the hamiltonian path integral

$$K(q'',q';T) = [g(q')g(q'')]^{-1/4}\prod_{j=1}^{N-1}\int \mathrm{d}q^{(j)}\prod_{j=1}^{N}\int\frac{\mathrm{d}p^{(j)}}{(2\pi)^d}\exp\left(\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}[\Delta q^{(j)}p^{(j)} - \epsilon\mathscr{H}(p^{(j)},\bar{q}^{(j)})]\right). \tag{10}$$

The effective hamiltonian to be used in the path integral (10) reads

$$\mathscr{H}(p^{(j)},\bar{q}^{(j)}) = \frac{1}{2m}g^{\alpha\beta}(\bar{q}^{(j)})p^{(j)}_\alpha p^{(j)}_\beta + V(\bar{q}^{(j)}) + \Delta V_{\mathrm{W}}(\bar{q}^{(j)})\,. \tag{11}$$

The midpoint prescription arises here in a very natural way, as a consequence of the Weyl-ordering prescription. It is a general feature that ordering prescriptions lead to *specific* lattices [#2]. The lagrangian path integral reads (MP=midpoint):

$$K(q'',q';T) = [g(q')g(q'')]^{-1/4}\int_{\mathrm{MP}}\sqrt{g}\,\mathrm{D}q(t)\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathscr{L}_{\mathrm{eff}}(q,\dot{q})\,\mathrm{d}t\right)$$

$$:= [g(q')g(q'')]^{-1/4}\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{Nd/2}\left(\prod_{j=1}^{N-1}\int\mathrm{d}q^{(j)}\right)\prod_{j=1}^{N}\sqrt{g(\bar{q}^{(j)})}$$

$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2\epsilon}g_{\alpha\beta}(\bar{q}^{(j)})\Delta q^{\alpha,(j)}\Delta q^{\beta,(j)} - \epsilon V(\bar{q}^{(j)}) - \epsilon\Delta V_{\mathrm{W}}(\bar{q}^{(j)})\right)\right]. \tag{12}$$

Eq. (12) is equivalent with (5). This is due to the fact that different lattices define different ΔV.

It is straightforward but tedious (see e.g. ref. [6]) to deduce from the short-time kernel of (12) and the time evolution equation the Schrödinger equation (1).

In our previous publications [9,17,18], we have calculated the path integral for the d-dimensional rotator (including a discussion of some other interesting problems), the path integral on the Poincaré upper half-plane and for Liouville quantum mechanics, and for the d-dimensional pseudosphere, respectively. The midpoint prescription turned out to be a bit bothersome, such that we have always turned to a path integral defined in

[#2] For a general discussion see e.g. refs. [15,16].

a "product form". This was possible because the metric $g_{\alpha\beta}$ in the above examples had the general form $g_{\alpha\beta}(q)=f_\gamma^2(q)\delta_{\alpha\gamma}\delta_{\beta\gamma}$ with functions f_γ ($\gamma=1, ..., d$). We then have changed in (12) the metric expressions as follows,

$$g_{\alpha\beta}(\bar{q}^{(j)})=f_\gamma^2(\bar{q}^{(j)})\delta_{\alpha\gamma}\delta_{\beta\gamma}\to f_\gamma(q^{(j)})f_\gamma(q^{(j-1)})\delta_{\alpha\gamma}\delta_{\beta\gamma}\,. \tag{13}$$

This prescription has to be accompanied by a Taylor expansion in the kinetic energy term $(m/2\epsilon)g_{\alpha\beta}(\bar{q}^{(j)})\Delta q^{\alpha,(j)}\Delta q^{\beta,(j)}$ up to fourth order in Δq. This formulation turned out to be more appropriate to our problems.

Our paper is organised as follows:

In section 2 we shall develop the precise formulation of the "product form"-definition in the path integral. We shall write the metric tensor $g_{\alpha\beta}$ in the form ("product form")

$$g_{\alpha\beta}=h_{\alpha\gamma}h_{\beta\gamma}\,, \tag{14}$$

and the hamiltonian ("product ordering")

$$H=\frac{1}{2m}h^{\alpha\gamma}p_\alpha p_\beta h^{\beta\gamma}+V+\Delta V\,, \tag{15}$$

with a well-defined quantum correction ΔV. We shall show that this ansatz, which looks quite special, is natural for reasonable manifolds M. An expression like (14) for the metric appears e.g. also in lattice gauge theories. $h_{\alpha\beta}$ can be identified in this case with the Maurer–Cartan form σ (see e.g. ref. [19]). In (super) gravity theories $h_{\alpha\beta}$ is also denoted as the "vielbein". We shall also prove that with $K(T)$ in the "product form"-definition the time-dependent Schrödinger equation can be derived.

In section 3 we shall discuss the "product form"-definition for the example of quantum motion on the Poincaré upper half plane.

Section 4 will summarize our results.

2. The product form

In order to develop the "product form"-definition in path integrals we consider the generic case of a classical lagrangian on the d-dimensional manifold M given by $\mathscr{L}_{\rm cl}=\frac{1}{2}mg_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta$. We assume that the metric tensor $h_{\alpha\beta}$ is real and symmetric and has $\operatorname{rank}(g_{\alpha\beta})=d$, i.e. we have no constraints on the coordinates. Thus one can always find a linear transformation C: $q_\alpha=C_{\alpha\beta}y_\beta$ such that $\mathscr{L}_{\rm cl}=\frac{1}{2}m\Lambda_{\alpha\beta}\dot{y}^\alpha\dot{y}^\beta$ with $\Lambda_{\alpha\beta}=C_{\alpha\gamma}g_{\gamma\delta}C_{\delta\beta}$ and where Λ is diagonal. C has the form $C_{\alpha\beta}=u_\alpha^{(\beta)}$ where the $\boldsymbol{u}^{(\beta)}$ ($\beta\in\{1, ..., d\}$) are the eigenvectors of $g_{\alpha\beta}$ and $\Lambda_{\alpha\beta}=f_\gamma^2\delta_{\alpha\gamma}\delta_{\beta\gamma}$ where $f_\alpha^2\neq 0$ ($\alpha\in\{1, ..., d\}$) are the eigenvalues of $g_{\alpha\beta}$. Without loss of generality we assume $f_\alpha^2<0$ for all $\alpha\in\{1, ..., d\}$. (For a time-like coordinate q_α one might have e.g. $f_\alpha^2<0$, but we want to exclude cases like this.) Thus one can always find a representation for $g_{\alpha\beta}$ which reads

$$g_{\alpha\beta}(q)=h_{\alpha\gamma}(q)h_{\beta\gamma}(q)\,. \tag{16}$$

Here the $h_{\alpha\beta}=C_{\alpha\gamma}f_\gamma C_{\gamma\beta}=u_\gamma^{(\alpha)}f_\gamma u_\gamma^{(\beta)}$ are real symmetric $d\times d$ matrices and satisfy $h_{\alpha\beta}h^{\beta\gamma}=\delta_\alpha^\gamma$. Because there exists the orthogonal transformation C eq. (16) yields for the y-coordinate system (denoted by M_y)

$$\Lambda_{\alpha\beta}(y)=f_\gamma^2(y)\delta_{\alpha\gamma}\delta_{\beta\gamma}\,. \tag{17}$$

Eq. (17) includes, of course, the special case $g_{\alpha\beta}=\Lambda_{\alpha\beta}$. The square-root of the determinant of $g_{\alpha\beta}$, $\sqrt{g}$ and the Christoffels Γ_α read in the q-coordinate system (denoted by M_q)

$$\sqrt{g}=\det(h_{\alpha\beta})=:h\,,\quad \Gamma_\alpha=\frac{h_{,\alpha}}{h}\,,\quad p_\alpha=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial q_\alpha}+\frac{h_{,\alpha}}{2h}\right). \tag{18}$$

The Laplace–Beltrami operator expressed in the $h^{\alpha\beta}$ reads on M_q

$$\Delta_{LB}^{M_q}=\left(h^{\alpha\gamma}h^{\beta\gamma}\frac{\partial^2}{\partial q^\alpha\partial q^\beta}+\left(\frac{\partial h^{\alpha\gamma}}{\partial q^\alpha}h^{\beta\gamma}+h^{\alpha\gamma}\frac{\partial h^{\beta\gamma}}{\partial q^\alpha}+\frac{h_{,\alpha}}{h}h^{\alpha\gamma}h^{\beta\gamma}\right)\frac{\partial}{\partial q^\beta}\right] \tag{19}$$

and on M_y

$$\Delta_{LB}^{M_y}=\frac{1}{f_\alpha^2}\left[\frac{\partial^2}{\partial y_\alpha^2}+\left(\frac{f_{\beta,\alpha}}{f_\beta}-2f_{\alpha,\alpha}\right)\frac{\partial}{\partial y_\alpha}\right]. \tag{20}$$

With the help of the momentum operators (18) we rewrite the hamiltonian in the "product-ordering" form (PF=product-form)

$$H=-\frac{\hbar^2}{2m}\Delta_{LB}^{M_q}+V(q)=\frac{1}{2m}h^{\alpha\gamma}(q)p_\alpha p_\beta h^{\beta\gamma}(q)+V(q)+\Delta V_{PF}^{M_q}(q)\,, \tag{21}$$

with the well-defined quantum correction

$$\Delta V_{PF}^{M_q}=\frac{\hbar^2}{8m}\left[4h^{\alpha\gamma}h^{\beta\gamma}{}_{,\alpha\beta}+2h^{\alpha\gamma}h^{\beta\gamma}\frac{h_{,\alpha\beta}}{h}+2h^{\alpha\gamma}\left(h^{\beta\gamma}{}_{,\beta}\frac{h_{,\alpha}}{h}+h^{\beta\gamma}{}_{,\alpha}\frac{h_{,\beta}}{h}\right)-h^{\alpha\gamma}h^{\beta\gamma}\frac{h_{,\alpha}h_{,\beta}}{h^2}\right]. \tag{22}$$

On M_y the corresponding $\Delta V_{PF}^{M_y}$ is given by

$$\Delta V_{PF}^{M_y}=\frac{\hbar^2}{8m}\frac{1}{f_\alpha^2}\left[\left(\frac{f_{\beta,\alpha}}{f_\beta}\right)^2-\frac{4f_{\alpha,\alpha\alpha}}{f_\alpha}+4\frac{f_{\alpha,\alpha}}{f_\alpha}\left(2\frac{f_{\alpha,\alpha}}{f_\alpha}-\frac{f_{\alpha,\alpha}}{f_\beta}\right)+2\left(\frac{f_{\beta,\alpha}}{f_\beta}\right)_{,\alpha}\right]. \tag{23}$$

Note that we have chosen a specific ordering prescription of momentum and position operators in the hamiltonian (21). The expressions (22) and (23) look somewhat circumstantial, so we shall display a special case and the connection to the quantum correction ΔV_W which corresponds to the Weyl-ordering prescription.

(1) Let us assume that $\Lambda_{\alpha\beta}$ is proportional to the unit tensor, i.e. $\Lambda_{\alpha\beta}=f^2\delta_{\alpha\beta}$.

The $\Delta V_{PF}^{M_y}$ simplifies to

$$\Delta V_{PF}^{M_y}=\hbar^2\frac{d-2}{8m}\frac{(4-d)f_{,\alpha}^2+2ff_{,\alpha\alpha}}{f^4}. \tag{24}$$

This implies an important corollary:

Corollary. Assume that the metric has or can be transformed into the special form $\Lambda_{\alpha\beta}=f^2\delta_{\alpha\beta}$. If the dimension of the space is $d=2$, then the quantum correction $\Delta V_{PF}^{M_q}$ vanishes.

An example is the Poincaré upper half-plane see section 3.

(2) A comparison between (22) and (9) gives the connection with the quantum correction corresponding to the Weyl-ordering prescription:

$$\Delta V_{PF}^{M_q}=\Delta V_W+\frac{\hbar^2}{8m}(2h^{\alpha\gamma}h^{\alpha\gamma}{}_{,\alpha\beta}-h^{\alpha\gamma}{}_{,\alpha}h^{\beta\gamma}{}_{,\beta}-h^{\alpha\gamma}{}_{,\beta}h^{\beta\gamma}{}_{,\alpha})\,. \tag{25}$$

In the case of eq. (17) this yields

$$\Delta V_{PF}^{M_y}=\Delta V_W+\frac{h^2}{4m}\frac{f_{\alpha,\alpha}^2-f_\alpha f_{\alpha,\alpha\alpha}}{f_\alpha^4}. \tag{26}$$

These equations often simplify practical applications.

Next we have to consider the short-time matrix element $\langle q''|\exp[-(i/\hbar)TH]|q'\rangle$ in order to derive the path integral formulation corresponding to the ordering prescription (21). We consider position $|q\rangle$ and momentum eigenstates $|p\rangle$ with the property

$$\langle q''|q'\rangle=(g'g'')^{-1/4}\delta(q''-q')\,,\quad \langle q|p\rangle=(2\pi)^{-d/2}\exp[(\mathrm{i}/\hbar)pq]\,. \tag{27}$$

We have for the Feynman kernel for an arbitrary $N\in\mathbb{N}$ [which is due to the half-group property of $U(T)$, i.e. $U(t_1+t_2)=U(t_1)U(t_2)$]:

$$K(q'',q';T)=\langle q''|\exp[-(\mathrm{i}/\hbar)TH]|q'\rangle$$

$$=\left(\prod_{j=1}^{N-1}\int\sqrt{g^{(j)}}\,\mathrm{d}q^{(j)}\right)\prod_{j=1}^{N}\langle q^{(j)}|\exp[-(\mathrm{i}/\hbar)(T/N)H]|q^{(j-1)}\rangle\,. \tag{28}$$

We consider the short-time approximation to the matrix element [$\epsilon=T/N$, $g^{(j)}=g(q^{(j)})$]:

$$\langle q^{(j)}|\exp[-(\mathrm{i}\epsilon/\hbar)H]|q^{(j-1)}\rangle\simeq\langle q^{(j)}|1-(\mathrm{i}\epsilon/\hbar)H|q^{(j-1)}\rangle$$

$$=\frac{[g^{(j)}g^{(j-1)}]^{-1/4}}{(2\pi)^d}\int\exp[(\mathrm{i}/\hbar)p\Delta q^{(j)}]\,\mathrm{d}p-\frac{\mathrm{i}\epsilon}{2m\hbar}\langle q^{(j)}|h^{\alpha\gamma}p_\alpha p_\beta h^{\beta\gamma}|q^{(j-1)}\rangle$$

$$-\frac{\mathrm{i}\epsilon}{\hbar}\langle q^{(j)}|V+\Delta V_{\mathrm{PF}}^{\mathrm{M}_q}|q^{(j-1)}\rangle\,. \tag{29}$$

The matrix element of the potential terms is simple, yielding

$$\langle q^{(j)}|V+\Delta V_{\mathrm{PF}}^{\mathrm{M}_q}|q^{(j-1)}\rangle=\frac{[g^{(j)}g^{(j-1)}]^{-1/4}}{(2\pi)^d}[V(q^{(j)})+\Delta V_{\mathrm{PF}}^{\mathrm{M}_q}(q^{(j)})]\int\exp[(\mathrm{i}/\hbar)p\Delta q^{(j)}]\,\mathrm{d}p\,. \tag{30}$$

The choice of the "post point" $q^{(j)}$ in the potential terms is not unique. A "prepoint", "midpoint" or a "product form" expansion is also legitimate. However, changing from one to another formulation does not alter the path integral, because differences in the potential terms are of $O(\epsilon)$, i.e. of $O(\epsilon)^2$ in the short-time Feynman kernel and therefore do not contribute.

The kinetic term gives

$$\langle q^{(j)}|h^{\alpha\gamma}p_\alpha p_\beta h^{\beta\gamma}|q^{(j-1)}\rangle=h^{\alpha\gamma}(q^{(j)})h^{\beta\gamma}(q^{(j-1)})\int\mathrm{d}p\,\mathrm{d}q\,\langle q^{(j)}|p_\alpha p_\beta|p\rangle\langle p|q\rangle\langle q|q^{(j-1)}\rangle$$

$$=h^{\alpha\gamma}(q^{(j)})h^{\beta\gamma}(q^{(j-1)})\frac{[g^{(j)}g^{(j-1)}]^{-1/4}}{(2\pi)^d}\int\exp[(\mathrm{i}/\hbar)p\Delta q^{(j)}]p_\alpha p_\beta\,\mathrm{d}p\,. \tag{31}$$

Therefore we get for the short-time matrix element ($\epsilon\ll 1$):

$$\langle q^{(j)}|\exp[-(\mathrm{i}\epsilon/\hbar)H]|q^{j-1)}\rangle\simeq\frac{[g^{(j)}g^{(j-1)}]^{-1/4}}{(2\pi)^d}$$

$$\times\int\mathrm{d}p\exp\left(\frac{\mathrm{i}}{\hbar}p\Delta q^{(j)}-\frac{\mathrm{i}\epsilon}{2m\hbar}h^{\alpha\gamma}(q^{(j)})h^{\beta\gamma}(q^{(j-1)})p_\alpha p_\beta-\frac{\mathrm{i}\epsilon}{\hbar}V(q^{(j)})-\frac{\mathrm{i}\epsilon}{\hbar}\Delta V_{\mathrm{PF}}^{\mathrm{M}_q}(q^{(j)})\right). \tag{32}$$

The Trotter formula $\mathrm{e}^{-\mathrm{i}T(A+B)}:=\text{s-lim}_{N\to\infty}(\mathrm{e}^{-\mathrm{i}TA/N}\,\mathrm{e}^{-\mathrm{i}TB/N})^N$ [12] states that all approximations in eqs. (28)–(32) are valid in the limit $N\to\infty$ and we get for the hamiltonian path integral in the "product form" definition [$h_{\alpha\beta}^{(j)}=h_{\alpha\beta}(q^{(j)})$]:

$$K(q'',q';T)=[g(q')g(q'')]^{-1/4}\lim_{N\to\infty}\left(\prod_{j=1}^{N-1}\int dq^{(j)}\times\prod_{j=1}^{N}\frac{dp^{(j)}}{(2\pi)^d}\right)$$
$$\times\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}\left(p\Delta q^{(j)}-\frac{\epsilon}{2m}h^{\alpha\gamma,(j)}h^{\beta\gamma,(j-1)}p_\alpha^{(j)}p_\beta^{(j)}-\epsilon V(q^{(j)})-\epsilon\Delta V_{\rm PF}^{M_q}(q^{(j)})\right)\right].\tag{33}$$

Performing the momentum integrations we get for the lagrangian path integral in the "product form" definition:

$$K(q'',q';T)=\int_{\rm PF}\sqrt{g}\,Dq(t)\exp\left(\frac{i}{\hbar}\int_{t'}^{t''}[\tfrac{1}{2}mh_{\alpha\gamma}h_{\beta\gamma}\dot q^\alpha\dot q^\beta-V(q)-\Delta V_{\rm PF}^{M_q}(q)]\,dt\right)$$

$$:=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{Nd/2}\prod_{j=1}^{N-1}\int\sqrt{g(q^{(j)})}\,dq^{(j)}$$

$$\times\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}h_{\alpha\gamma}^{(j)}h_{\beta\gamma}^{(j-1)}\Delta q^{\alpha,(j)}\Delta q^{\beta,(j)}-\epsilon V(q^{(j)})-\epsilon\Delta V_{\rm PF}^{M_q}(q^{(j)})\right)\right].\tag{34}$$

In the last step we have to check that the Schödinger equation (1) can be deduced from the short-time kernel of eq. (34). This is, as for eqs. (5) and (12), straightforward but tedious. Because one can always transform from the q-coordinates to the y-coordinates, which is a linear orthogonal transformation and thus does not produce any quantum correction in the path integral (34) defined on M_y, we shall use in the following the representation of eq. (17). We restrict ourselves to the proof that the short-time kernels of eqs. (12) and (34) are equivalent, i.e. we have to show ($\bar y=(y''+y')/2$) [#3]:

$$[g(y')g(y'')]^{-1/4}\sqrt{g(\bar y)}\exp\left(\frac{im}{2\epsilon\hbar}\Lambda_{\alpha\beta}(\bar y)\Delta y^\alpha\Delta y^\beta-\frac{i\epsilon}{\hbar}V(\bar y)-\frac{i\epsilon}{\hbar}\Delta V_{\rm W}(\bar y)\right)$$
$$\doteq\exp\left(\frac{im}{2\epsilon\hbar}f_\alpha(y')f_\alpha(y'')\Delta^2y^\alpha-\frac{i\epsilon}{\hbar}V(y'')-\frac{i\epsilon}{\hbar}\Delta V_{\rm PF}^{M_y}(y'')]\right).\tag{35}$$

Clearly, $\exp[-(i\epsilon/\hbar)V(\bar y)]\doteq\exp[-(i\epsilon/\hbar)V(y'')]$ for the potential term. It suffices to show that a Taylor expansion of the g and the kinetic energy terms on the left-hand side of eq. (35) yield an additional potential $\Delta\bar V$ given by

$$\Delta\bar V(y)=\Delta V_{\rm PF}^{M_y}(y)-\Delta V_{\rm W}(y)=\frac{\hbar^2}{4m}\frac{f_{\alpha,\alpha}^2(y)-f_\alpha(y)f_{\alpha,\alpha\alpha}(y)}{f_\alpha^4(y)}.\tag{36}$$

We consider the g-terms on the left-hand side of eq. (35) and expand them in a Taylor-series around y'. This gives ($\xi_\alpha(y''_\alpha-y'_\alpha)$, $f_\alpha(y')\equiv f_\alpha$)

$$[g(y')g(y'')]^{-1/4}\sqrt{g(\bar y)}\simeq1-\frac{1}{8}\frac{f_\gamma f_{\gamma,\alpha\beta}-f_{\gamma,\alpha}f_{\gamma,\beta}}{f_\gamma^2}\xi^\alpha\xi^\beta.\tag{37}$$

Exploiting the path integral identity (see e.g. refs. [2,5,20])

$$\xi^\alpha\xi^\beta\doteq\frac{i\epsilon\hbar}{m}g^{\alpha\beta},\tag{38}$$

we get by exponentiating the $O(\epsilon)$-terms,

$$[g(y')g(y'')]^{-1/4}\sqrt{g(\bar y)}\simeq\exp\left(-\frac{i\hbar\epsilon}{8m}\frac{f_\alpha f_{\alpha,\beta\beta}-f_{\alpha,\beta}^2}{f_\alpha^2f_\beta^2}\right).\tag{39}$$

[#3] We use the symbol $\doteq$ (following DeWitt [1]) to denote "equivalence as far as use in the path integral is concerned".

Repeating the same procedure for the exponential term gives

$$\exp\left(\frac{im}{2\epsilon\hbar}\Lambda_{\alpha\beta}(\bar{y})\xi^{\alpha}\xi^{\beta}\right) \simeq \exp\left(\frac{im}{2\epsilon\hbar}f_{\alpha}(y')f_{\alpha}(y'')\xi^{\alpha}\xi^{\alpha}\right)\left(1-\frac{i\hbar\epsilon}{8m}(f_{\gamma}f_{\gamma,\alpha\beta}-f_{\gamma,\alpha}f_{\gamma,\beta})\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\gamma}\right). \tag{40}$$

Note that we have to respect the "product form" definition in the kinetic term on the right-hand side of eq. (4). We use the path integral identity (see e.g. refs. [2,5,20])

$$\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \doteq \left(\frac{i\epsilon\hbar}{m}\right)^{2}(g^{\alpha\beta}g^{\gamma\delta}+g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\delta}g^{\beta\gamma}) \tag{41}$$

to get

$$\exp\left(\frac{im}{2\epsilon\hbar}\Lambda_{\alpha\beta}(\bar{y})\xi^{\alpha}\xi^{\beta}\right) \simeq \exp\left(\frac{im}{2\hbar\epsilon}f_{\alpha}(y')f_{\alpha}(y'')\xi^{\alpha}\xi^{\alpha}\right)\exp\left(\frac{i\hbar\epsilon}{8m}\frac{f_{\alpha}f_{\alpha,\beta\beta}-f^{2}_{\alpha,\beta}}{f^{2}_{\alpha}f^{2}_{\beta}}+\frac{i\hbar\epsilon}{4m}\frac{f_{\alpha}f_{\alpha,\alpha\alpha}-f^{2}_{\alpha,\alpha}}{f^{4}_{\alpha}}\right). \tag{42}$$

Combining eqs. (39) and (42) yields the additional potential $\Delta\bar{V}$ and eq. (35) is proven. Thus we conclude that the path integral (34) is well-defined and is the correct path integral corresponding to the Schrödinger equation (1).

3. Example

In this section we want to illustrate eq. (34) with an example: the quantum motion on the Poincaré upper half-plane U which is defined by

$$\mathrm{U} := \{\zeta = x+iy \,|\, y>0,\, x\in\mathbb{R}\}. \tag{43}$$

The study of this space (particularly in bounded domains) arises in the Polyakov approach of string theory (see e.g. refs. [21,22]), and in the theory of quantum chaos [23–25].

A detailed discussion of the path integral on U has been given in ref. [17], so we just state the results. The metric in U is given by $g_{\alpha\beta}=\Lambda_{\alpha\beta}=(1/y^{2})\delta_{\alpha\beta}$. It has the form $g_{\alpha\beta}=f^{2}\delta_{\alpha\beta}$ with $f=1/y$. We can immediately apply the corollary of section 2 and deduce that $\Delta V^{\mathrm{M_q}}_{\mathrm{PF}}=0$. Thus the path integral in the "product form" reads on the Poincaré upper half plane:

$$K(x'',x',y'',y';T) = \int\limits_{\mathrm{PF}} \frac{\mathrm{D}x(t)\,\mathrm{D}y(t)}{y^{2}}\exp\left(\frac{i}{\hbar}\int_{t'}^{t''}\frac{\dot{x}^{2}+\dot{y}^{2}}{y^{2}}\,\mathrm{d}t\right)$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{N}\prod_{j=1}^{N-1}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\mathrm{d}x^{(j)}\,\mathrm{d}y^{(j)}}{y^{(j)2}}\exp\left(\frac{im}{2\epsilon\hbar}\sum_{j=1}^{N}\frac{\Delta^{2}x^{(j)}+\Delta^{2}y^{(j)}}{y^{(j)}y^{(j-1)}}\right). \tag{44}$$

The path integral can be calculated (for details of the calculation, especially for the simulaneous space-time transformation which has to be done, see ref. [17]) yielding

$$K(x'',y'',x',y';T) = \frac{1}{\pi^{3}}\int_{-\infty}^{\infty}\mathrm{d}k\int_{0}^{\infty}\mathrm{d}p\,p\sinh\pi p\exp\left(-\frac{iT}{2m\hbar}(p^{2}+\tfrac{1}{4})\right)$$

$$\times\sqrt{y'y''}K_{ip}(|k|y')K_{ip}(|k|y'')\exp[ik(x''-x')] \tag{45}$$

(K_{ν} is a modified Bessel function). The energy dependent Green function $G(E)=\int K(T)\exp[(i/\hbar)TE]\,\mathrm{d}T$ is explicitly given by

$$G(x'', x', y'', y'; E) = \frac{m}{\pi} \mathcal{Q}_{-1/2+ip}(\cosh r) , \tag{46}$$

$p := \sqrt{2mE - \frac{1}{4}} > 0$, $\mathcal{Q}_\nu$ a Legendre function of the second kind, and $r > 1$ is the hyperbolic distance in U, which reads

$$\cosh r = \frac{y''^2 + y'^2 + (x'' - x')^2}{2y'y''} = 1 + \frac{|\zeta'' - \zeta'|^2}{2\,\mathrm{Im}(\zeta')\,\mathrm{Im}(\zeta'')} . \tag{47}$$

For details concerning the wave functions and the connection of this problem to Liouville quantum mechanics consult ref. [17].

4. Summary

In this paper we have presented a general and simple prescription for treating path integrals on cruved manifolds which we call "product form" definition. In order to formulate our prescription we have written the metric tensor $g_{\alpha\beta}$ in the form

$$g_{\alpha\beta}(q) = h_{\alpha\gamma}(q) h_{\beta\gamma}(q) . \tag{48}$$

Then the hamiltonian in the "product ordering" is given by

$$H = -\frac{\hbar^2}{2m} \Delta^{M_q}_{LB} + V(q) = \frac{1}{2m} h^{\alpha\gamma} p_\alpha p_\beta h^{\beta\gamma} + V(q) + \Delta V^{M_q}_{PF}(q) , \tag{49}$$

where the canonical momenta are defined in (18), and the quantum correction reads

$$\Delta V^{M_q}_{PF} = \frac{1}{8m} \left[4h^{\alpha\gamma} h^{\beta\gamma}{}_{,\alpha\beta} + 2h^{\alpha\gamma} h^{\beta\gamma} \frac{h_{,\alpha\beta}}{h} + 2h^{\alpha\gamma} \left(h^{\beta\gamma}{}_{,\beta} \frac{h_{,\alpha}}{h} + h^{\beta\gamma}{}_{,\alpha} \frac{h_{,\beta}}{h} \right) - h^{\alpha\gamma} h^{\beta\gamma} \frac{h_{,\alpha} h_{,\beta}}{h^2} \right] . \tag{50}$$

Starting with the hamiltonian (49), the langrangian path integral in the "product form" definition can be deduced yielding

$$K(q'', q'; T) = \int_{\mathrm{PF}} \sqrt{g}\, \mathrm{D}q(t) \exp\left(\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[\tfrac{1}{2} m h_{\alpha\gamma} h_{\beta\gamma} \dot{q}^\alpha \dot{q}^\beta - V(q) - \Delta V^{M_q}_{PF}(q) \right] \mathrm{d}t \right) , \tag{51}$$

with lattice definition (34). We have stated a corallary, namely if $g_{\alpha\beta}$ reads or can be transformed into the form $\Lambda_{\alpha\beta} = f^2 \delta_{\alpha\beta}$ with some function f and the dimension of the riemannian spaces is $d=2$, then we have $\Delta V^{M_q}_{PF} = 0$. Our example of an application of the "product form" definition has been the quantum motion on the Poincaré upper half-plane U endowed with the hyperbolic metric. In this case we could apply our corollary and found $\Delta V^{M_q}_{PF} = 0$ on U.

Further examples are, as already noted in the introduction, the d-dimensional sphere S^{d-1}, d-dimensional polar coordinates, the pseudosphere Λ^{d-1}, and path integrals in lattice gauge theories. A detailed discussion of these examples is rather lengthy and therefore will be given elsewhere.

In a forthcoming publication we shall apply the "product form" definition also to the path integral problem of the Poincaré disc and the hyperbolic strip.

We think that the direct use of the "product form" definition in path integrals will simplify calculations.

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