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STRONG COUPLING APPROXIMATION OF GEOMETRIC QCD AND THE SEMI-RELATIVISTIC QUARK MODEL

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1. INTRODUCTION

Geometric QCD which describes the quark fields by differential forms ('Dirac-Kähler fields'[1])

$$\begin{aligned} \Phi &= \sum_H \varphi(x,H)dx^H, \quad (1)\\ dx^H &\equiv dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_h}, \quad H=\{\mu_1,\mu_2,\ldots\mu_h\}, \\ &\mu_1<\mu_2<\ldots<\mu_h. \end{aligned}$$

is characterized by the action

$$S=\int\left\{\frac{1}{2g^2}\mathrm{Tr}(F,F)_0+4(\bar\Phi,(d_A-\delta_A+m)\Phi)_0\right\}. \quad (2)$$

As a model it has the great *advantage* that De Rham mapping leads to a systematic lattice approximation. This has positive implications for the discussion of symmetry properties and of topological questions on the lattice. It leads,in a certain formulation, to the lattice QCD of staggered fermion fields:

$$\begin{aligned} \int_{[\bar x,H]}\phi(y,H)dy^H &\xrightarrow{DeRham} \phi(\bar x,H)\\ &= \chi(x,H(x))\equiv\chi(x) \quad (3)\\ x=\bar x+\frac{1}{2}e_H & \end{aligned}$$

(for notations see Fig.1)
with the action

$$\begin{aligned} S &\longrightarrow \frac{\beta}{6}S_W+S_q \quad (4)\\ S_W &= \sum_\Box Tr(2-U(\Box)-U^{-1}(\Box))\\ S_q &= \sum_x(\sum_\mu \tilde\rho_{\mu H(x)}\frac{1}{2}(\bar\chi(x)U(x,\mu)\chi(x+e_\mu)\\ &\quad -\bar\chi(x+e_\mu)U^{-1}(x,\mu)\chi(x)+m\bar\chi(x)\chi(x)). \end{aligned}$$

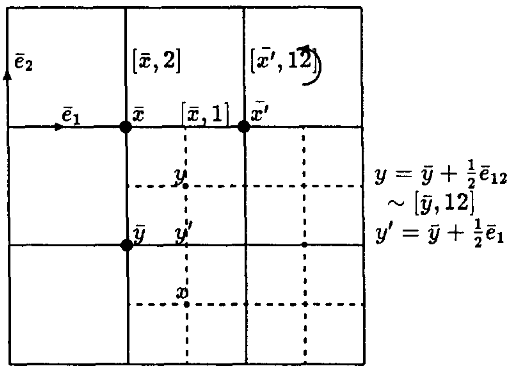

Figure 1: Illustration of the Lattice Notions. Elements of the coarse lattice Γ_{coarse} like points $\bar x$, unit cells $[\bar x,H]$, unit vectors $\bar e_\mu$ are overlined.Points and unit vectors of the fine lattice Γ_{fine} are: x, $e_\mu=\frac{1}{2}\bar e_\mu$ etc.

From the physics point of view geometric QCD has the *disadvantage* that its physical interpretation is not clear. The well known decomposition of Dirac-Kähler fields in four Dirac fields

$$\psi(x)^b_\alpha=\sum_H\phi(x,H)(\gamma^H)^b_\alpha, \quad (5)$$

rises the question of the meaning of the SU(4)-'Susskind flavour' index b=1,2,3,4. Does it denote flavours, families,...? Only the investigation of the dynamics of a geometric standard model can decide between physical sense or nonsense of geometric QCD!

From a QCD with four species of quarks we would expect that it reproduces the results of a corresponding semi-relativistic quark model.[2] This means with respect to the meson spectrum: the calculation of the masses

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of the ground states, angular and radial excitations of the SU(4) singlets and (15)-plets:

$$(F,0^{--}),(F,1^{--}),(F,2^{++}),(F,1^{++}),$$
$$(F,0^{++}),(F,1^{+-}),\ldots \quad F=(1),(15).$$

and their description by Bethe Salpeter amplitudes like:

$$\left\langle |\,\psi^i_\alpha(x)\bar\psi^k_\beta(y)\right| \left.\begin{matrix} M, & F, & s \\ p, & f, & s_3 \end{matrix}\right\rangle = e^{i(p,x+y)/2}\phi\lambda^f_{ik}\Gamma_{\alpha\beta} \tag{6}$$

Here Γ may have the form γ^5, $\gamma^5\gamma^\mu P_\mu$ for pseudo scalar, $\gamma^\mu e^{s_3}_\mu \ldots$ for vector mesons etc. ϕ denotes a radial function.

In this talk we contribute to the discussion of this question by reporting on investigations of
1.The Symmetry of the Problem.
2.'Excited States' in first order strong coupling approximation.
3. Lattice approximation of non-relativistic quantum mechanics simulating the dynamics of strong coupling.

2. THE SYMMETRY OF THE PROBLEM.

Our naive expectation on the structure of the quark model is based on the symmetry of geometric QCD. As one expects for a relativistic theory with 4 degenerate Dirac fields, the action S has the symmetry group $\mathcal{G} \simeq \mathcal{SE} \times SU(4)$; $\mathcal{SE} \equiv$ spinorial Euclidean group. The group theory of the lattice restriction of this symmetry is by now well known from the work of M.F.L.Golterman, J.Smit; [3,4] M.Göckeler,[5] H.Joos, M.Schäfer; [6] G.W.Kilcup, S.R.Sharp, [7], and others. We want to summarize it shortly in our language.

The geometrical and technical understanding of the lattice restriction $\mathcal{G}_L$ of the symmetry gets very much improved by the introduction of Kähler's Clifford product for differntial forms [8]:

$$dx^\mu \vee dx^\nu = g^{\mu\nu} + dx^\mu \wedge dx^\nu$$

1. It defines signs

$$\begin{aligned} dx^H \vee dx^K &= \check\rho_{H,K}\,dx^{H\triangle K}, \\ \Lambda &= H\cap K, \quad H\triangle K = H\cup K - \Lambda, \end{aligned}$$

which appear everywhere in the staggered fermion formalism, e.g. in the action: $\check\rho_{\mu,H(x)} = (-1)^{x_1+\ldots x_{\mu-1}}$
2.The product $\epsilon dx^H \vee \epsilon' dx^K$, $\epsilon,\epsilon' = \pm 1$, defines a group $\mathcal{K}$ of order 2^{dim+1}. ('Kent group',[9] the group of the γ-matrices.)
3. Infinitesimal spinorial and geometrical rotations in the $\mu\nu$-plane can be expressed with help of $S_{\mu\nu} = dx^\mu \wedge dx^\nu$:

$$\delta^s_{\mu\nu}\Phi = \frac{1}{2}S_{\mu\nu}\vee\Phi, \quad \delta^G_{\mu\nu}\Phi = \frac{1}{2}(S_{\mu\nu}\vee\Phi - \Phi\vee S_{\mu\nu}).$$

Flavour transformations are expressed by a constant form u as

$$\Phi' = \Phi \vee u.$$

Geometrical and spinorial rotations differ by flavour transformations! Because the lattice restriction of geometrical rotations follows geometrically from DeRham mapping (in contrast to spinorial rotations), it is advantageous to compose a general element $(f,a,s) =\in SU(4)\times\mathcal{SE}$ by a flavour transformation $\bar f$, a translation: $a = 2b(n_1,n_2,n_3,n_4)$, $n_i \in Z$, and a geometrical rotation R : $[\bar f, a, R(s)] = (\bar f s, a, s)$. Now we can formulate the following proposition on the symmetry of the lattice approximation of geometric QCD:
PROPOSITION 1. The lattice restriction of $\mathcal{G}$ is

$$\mathcal{G}_L = \{[\epsilon d^K, -\frac{1}{2}\bar e_K + a, R] \mid a\in\mathcal{T}_L, R\in W_4\}$$

with the composition law

$$\begin{aligned} &[\epsilon d^K, -\frac{1}{2}\bar e_K + a, R]\circ[\epsilon' d^L, -\frac{1}{2}\bar e_L + a', R'] = \\ &\quad [\epsilon\epsilon'\rho(R, R\circ L)\check\rho_{K,R\circ L} d^{K\triangle R\circ L}, \\ &\quad -\frac{1}{2}(\bar e_K + R\bar e_L) + Ra' + a, RR']. \end{aligned}$$

It is a symmetry group of the free DKE if it acts on staggered fermion fields according to

$$\begin{aligned} ([a]\chi)(y,H(y)) &= \chi(y - a^\mu\bar e_\mu, H(y)), \\ H(y - a^\mu\bar e_\mu) &= H(y), \\ ([R]\chi)(y,H(y)) &= \rho(R,H(y))\chi(R^{-1}y, H(R^{-1}y)), \\ (\epsilon d^K\chi)(y,H(y)) &= \epsilon\check\rho_{H(y),K}\chi(y+e_K, H(y+e_K)). \end{aligned}$$

where the sign $\rho(R,H)$ is the same as in the transformation of the basis differentials of the continuum: $Rdx^H = R^{-1} \vee dx^H \vee R = \rho(R,H)dx^{R^{-1}\circ H}$.

The action $S(\bar{\chi},\chi,U)$, Eq. (2), with gauge interaction is invariant, if $U(x,\mu)$ is transformed as a link field under translations and geommetric rotations, and as

$$(\epsilon d^K U)(x,\mu) = U(x+\frac{1}{2}\bar{e}_K,\mu) \tag{7}$$

under flavour transformations.

Because of the association of the lattice flavour transformation ϵd^K with the translation $[\frac{1}{2}\bar{e}_K,]$ ($\bar{e}_K = \sum_{\mu\in K}\bar{e}_\mu$, $\bar{e}_\mu$ lattice unit vector of the *coarse* lattice Γ_{coarse}) the flavour transformations form a 'non-symmorphous' extension of the translation group:

$\mathcal{FT}_L \not\simeq \mathcal{T}_L \otimes \mathcal{K}_4$, however $\mathcal{FT}_L/\mathcal{T}_L \simeq \mathcal{K}_4$

All irreducible, unitary representations ('irreps') of $\mathcal{G}_L$ are known. [4,6,7] They can be constructed by the Wigner-Mackey procedure. [10] We formulate this result in another proposition, which we explain by examples:

PROPOSITION 2. The irreducible, unitary representations of the symmetry group $\mathcal{G}_L$ of geometric lattice QCD are determined by a 'momentum star St_j', a 'flavour orbit $\Theta_{j,k}$', and the 'reduced spin σ'.

We give these representations for the case of 'mesons' ($(\epsilon,0,1)\to 1$) in a canonical base:

(a)Translations.

$$U(a)\left|\begin{matrix} j, & F, & \sigma \\ p, & L, & n\end{matrix}\right\rangle = e^{i(p,a)}\left|\begin{matrix} j, & F, & \sigma \\ p, & L, & n\end{matrix}\right\rangle \tag{8}$$

with 'momenta' p $:-\pi/2b < p_\mu \leq \pi/2b$,

$p \in St_j = \{R\bar{p}_j | R \in W_4\}$,

e.g. reference momentum $\bar{p}_4 = (0,0,0,E)$,

$St_4 = \{(\pm E,0,0,0),(0,\pm E,0,0),\ldots\}$.

(b) Flavour transformations:

$$U(\epsilon d^K)\left|\begin{matrix} j, & F, & \sigma \\ p, & L, & n\end{matrix}\right\rangle = e^{-i(p,\frac{1}{2}\bar{e}_K)}e^{i\pi(e_L,e_K)}\left|\begin{matrix} j, & F, & \sigma \\ p, & L, & n\end{matrix}\right\rangle \tag{9}$$

with

'flavour character'

$e_L : (e_{1,L},e_{2,L},e_{3,L},e_{4,L}),\ \ e_{\mu,L} = \pm 1$,

$\Theta_{j,F} = \{R\circ e_F | R \in S_j,\ i.eR\bar{p}_j = \bar{p}_j\}$,

e.g.'reference flavour' $e_F = (1,0,0,1)$, for $j=4$, $k=5$,

$\Theta_{4,5} = \{(1,0,0,1),(0,1,0,1),(0,0,1,1)\}$.

(c) Geometric rotations:

$$U(R)\left|\begin{matrix} j, & F, & \sigma \\ p, & L, & n\end{matrix}\right\rangle = \sum_{n'}\left|\begin{matrix} j, & F, & \sigma \\ Rp, & \omega\circ L, & n'\end{matrix}\right\rangle D^\sigma_{n'n}(X) \tag{10}$$

with appropriate Wigner rotations:

$\omega(R,p) \in S_j,\ X(R,L,p) \in S_{j,F}$,

'reduced spin group'

$S_{j,F} = \{R | R \in W_4, R\bar{p}_j, R\circ e_F = e_F\}$

$D^\sigma(X)$ irrep of $S_{j,F}$.

For j, F as above, we get $S_{j,F} \simeq D_4 \times Z_2$with eight 1-dimensional representations: $(1^\pm),(1'^\pm)(1''^\pm),(1'''^\pm)$ and two 2-dimensinal representations:$(2^\pm)$.

For $e_F = (0,0,0,1)$ the reduced spin is given by the 10 irreps of the group W_3 of the cube.

In our treatment of the lattice approximation of geometric QCD, the lattice symmetry group $\mathcal{G}_L$ is geometrically understood as a sub-group of the continuum symmetry $SU(4)\times\mathcal{SE}$. Therefore we may pose the problem of how a restriction of an irrep of $\mathcal{G} : U(\mathcal{G})|_L$ decomposes into irreps of $\mathcal{G}_L$, i.e.one asks for the calculation of the intertwining numbers $I(U(\mathcal{G})|_{\mathcal{G}_L}, U(\mathcal{G}_L))$. Such s calculations are important because of the following hypothesis used in a similar form in the treatment of glue balls [11]:

HYPOTHESIS: A lattice state of a particle characterized by the quantum numbers χ_L of an $\mathcal{G}_L$–irrep can be associated with a continuum particle state characterized by the quantum number χ of an $\mathcal{G}$-irrep, if the $\mathcal{G}_L$-irrep is contained in the $\mathcal{G}$ irrep, i.e. if the intertwining number $I(U^\chi|_L, U^{\chi_L}) \neq 0$).

There are extensive calculations of such intertwining numbers. W.Neudenberger performed such calculations in the framework of the Wigner-Mackey scheme. [12] We want to give two examples of the applications of such calculations.

(1) Splitting of the ground state continuum multiplets in lattice multiplets,(in the cm- system, i.e. for momentum star St_4, and lattice irreps denoted by $[e_F, \sigma]$):

$$\begin{aligned}
&(0^-,15) \to [(0,0,0,1),(1^+)_{W_3}] \oplus [(1,1,1,0),(1^-)_{W_3}] \\
&\oplus[(1,1,1,1),(1^+)_{W_3} \oplus [(1,0,0,0),(1'^-)_{D_4}]) \\
&\oplus[(1,0,0,1),(1'^+)_{D_4}] \oplus [(0,1,1,0),(1'^-)_{D_4} \\
&\oplus[(0,1,1,1),(1'^+)_{D_4}] \\
&(1^-,15) \to [(0,0,0,1),(3^+)_{W_3}] \oplus [1,1,1,0),(3^-)_{W_3}] \\
&\oplus[1,1,1,1),(3^+)_{W_3}] \oplus [(1,0,0,0),(1'^-)_{D_4}] \\
&\oplus[(1,0,0,1),(1^+)_{D_4}] \oplus [(0,1,1,0),(1'^-)_{D_4}] \\
&\oplus[(0,1,1,1),(1^+)_{D_4}] \oplus [(1,0,0,0),(2^+)_{D_4}] \\
&\oplus[(1,0,0,1),(2^-)_{D_4}] \oplus [(0,1,1,0),(2^+)_{D_4}] \\
&\qquad \oplus [(0,1,1,1),(2^-)_{D_4}]. \qquad (11)
\end{aligned}$$

(2) Comparison of the splitting in different momentum stars.

For example we consider the lattice irreps with trivial reduced spin which are contained in the continuum multiplet $(15,1^-)$ for the different momentum stars:

$$\begin{aligned}
St_4: \quad \bar{p}_4 &= (0,0,0,E), \\
St_9: \quad \bar{p}_9 &= (0,0,p,E), \\
St_{11}: \quad \bar{p}_{11} &= (p,p,p,E).
\end{aligned}$$

With the transition from the cm-system (St_4) to other stars the flavour orbit generally splits. From Neudenberger's tables we can read off the splitting pattern for this example

$$(1,0,0,1)_4 \begin{array}{l} \to (1,0,0,1)_9 \oplus (1,0,1,1)_9 \\ \to (1,0,0,1)_{11} \end{array}$$

The dynamical degeneracy of splitted states is an indication of the restauration of Euclidean symmetry.

3. 'EXCITED STATES' IN FIRST ORDER STRONG COUPLING APPROXIMATION

In this Section we discuss the meson spectrum of geometric QCD combining strong coupling approximation with our group theoretical methods.[13]

In order to describe the large variety of mesons we need meson fields representing sufficiently many lattice quantum numbers. We choose so-called multi-link operators

$$\begin{gathered}
\mathcal{M}^{L,\mathcal{F}}(x) = \\
e^{i\pi(e_L + f_{e_{1234}} + e_F, x)}\bar{\chi}(x)\bar{\rho}_{H(x),F}U(\mathcal{F})\chi(x+e_F) \equiv \\
\zeta^{L,\mathcal{F}}(x)\bar{\chi}(x)U(\mathcal{F})\chi(x+e_F) \\
\zeta^{L,\mathcal{F}}(x) = \pm 1 \qquad (12)
\end{gathered}$$

$U(\mathcal{F})$ is the gauge string field along the path $\mathcal{F}$ from x to $x+e_f$. This makes $\mathcal{M}^{L,\mathcal{F}}(x)$ gauge invariant. The sign function on the fine lattice $\zeta^{L,\mathcal{F}}(x)$ is choosen in such a way that $\mathcal{M}^{L,\mathcal{F}}(x)$ has 'natural' covariance properties under the transformations of $\mathcal{G}_L$:

a) Flavour transformations:

$$\left(\epsilon d^k \mathcal{M}^{L,\mathcal{F}}\right)(x) = e^{i\pi(e_L, e_K)}\mathcal{M}^{L,\mathcal{F}}(x+e_K) \qquad (13)$$

b) Geometric Rotations:

$$(R\mathcal{M}^{L,\mathcal{F}})(x) = \rho(R,F)\mathcal{M}^{R^{-1}L,R^{-1}\mathcal{F}}(R^{-1}x) \qquad (14)$$

c) Translations:

$$([a])\mathcal{M}^{L,\mathcal{F}}(x) = \mathcal{M}^{L,\mathcal{F}}(x-a) \qquad (15)$$

Now we have to calculate the meson field propagators

$$\begin{gathered}
\left\langle : \mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(y) : \right\rangle = \\
\int\int \mathcal{D}[U]\mathcal{D}[\chi,\bar{\chi}]\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(y) \\
\times \exp\left(-\frac{\beta}{2N}S_W(U) - S_q(\bar{\chi},\chi,U)\right) \qquad (16)
\end{gathered}$$

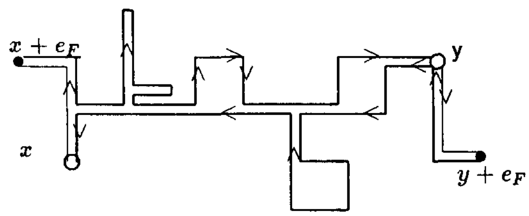
In the evaluation of this integral we follow the procedure:

a) Fermion integration.

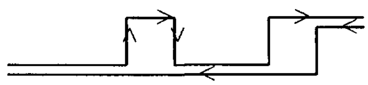
b) Hopping parameter expansion of the quark propagator (and of the fermion determinant in higher order calculations) in the back ground field.

c) Gauge field integration in strong coupling approximation.

d) Resumming of the $q\bar{q}$-paths with help of a renormalized step matrix. This method is due to O.Martin.[14] It is related to Kawamoto's random walk procedure.[15]

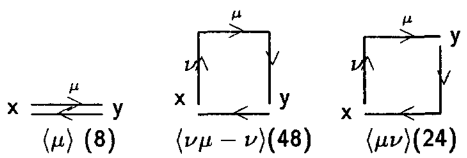


(a) A general $q\bar{q}$-path.

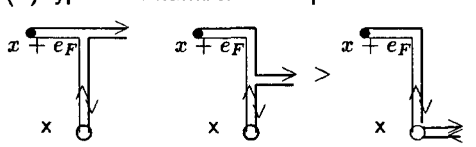


(b) Trunk of path (a)

(c) Examples of dressings of q-paths



(d)Types and numbers of steps.



(e) Screening of multi-link operators with $\mathcal{F}$ from x to $x + e_F$

Figure 2: Structure of $q\bar{q}$-paths.

It is well known that after the first three steps of this procedure we get a representation of the connected propagagator
$\langle: \mathcal{M}(x)\mathcal{M}(y) :\rangle = \langle\mathcal{M}(x)\mathcal{M}(y)\rangle - \langle\mathcal{M}(x)\rangle\langle\mathcal{M}(y)\rangle$
as a sum of paths which screen the multi-link fields. An example of such a path which shows most of the complexities in a first order calculation is shown in Fig.3.a. We express this fact symbolically by

$$\langle: \mathcal{M}(x)\mathcal{M}(y) :\rangle = \frac{1}{m^2}\sum_N^{\infty}\left(\frac{-1}{4m^2}\right)^N T_{xy}(N,\beta). \quad (17)$$

$1/4m^2$ is the hopping parameter of the $q\bar{q}$- propagation. For $\beta = 0$, $T_{xy}(N,0)$ is the number of $q\bar{q}$-paths from x to y enclosing zero area, and consisting of $2N$ $q-$ and $\bar{q}-$links. The evaluation of this expression precedes in several steps. These show similarities to steps appearing in the derivation of the Bethe-Salpeter equation, and to techniques treating random walk problems. [16] We can merely characterize these steps by key words.

(1)*'Renormalization' of the hopping parameter.* Closed detours of the q-lines ($\bar{q}$-lines) -'dressings'- can appear at each point of the $q\bar{q}$-double path. Therefore we can reorder the sum over all $q\bar{q}$-paths by summing over $q\bar{q}$-paths with no dressings ('trunks') and all dressings at each point of the trunk,see Fig.2.b,c. This leads in Eq. (17) to the substitution $2m \rightarrow \alpha$ and $T_{xy}(N,0) \rightarrow B_{xy}(N,0)$, the number of trunks from x to y. A calculation (O.Martin) gives

$$\begin{aligned} a = \alpha^2 &= \alpha_0^2 + \frac{16(d-1)^3}{\alpha_0^2(\alpha_0^2+2d-1)}\beta \equiv a_0 + \beta D, \\ \alpha_0 &= m + \sqrt{m^2+2d-1} \end{aligned} \quad (18)$$

(2) *Generation of trunks by steps. Recursion relations.* Counting all simple $q\bar{q}$-paths from x to y is a random walk problem. Using methods of random walk theory, we introduce the number $B_{x,\hat{\mu}}(N)$ of trunks of lengths N arriving at x from $\hat{\mu}$-direction starting from fixed $y = 0$.The following recursion relation is evident:

$$\begin{aligned} B_{x,\hat{\mu}}(N) &= \sum_{\nu(\neq-\mu)} B_{x-\hat{\mu},\hat{\nu}}(N-1) \\ \equiv B_{x,s}(N) &= \sum_{s'} \hat{M}_{ss'} B_{x-e_s,s'}(N-1) \end{aligned} \quad (19)$$

The general form of this recursion relation describes the generation of first order paths by the first order steps. There are 80 different steps generically indexed by s. These are shown in in Fig.3.d.

(3) *Non-backtracking conditions.* A random walk of $q\bar{q}$-paths would generate dressings which we already considered. In order to avoid double counting, and generate by the recursion relations trunks only, one has to forbid backtracking. This implies $\nu \neq \pm\mu$ in

the sum above. There are similar, more involved non-backtracking conditions in the first order calculation.

(4) *Representation of the propagator by the step matrix.* For the summation over the trunk numbers one uses the Fourier transformation on the fine lattice of the recursion relation. Summing the resulting geometric series leads to

$$\left\langle : \mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(y) : \right\rangle =$$
$$\frac{1}{m^2}\int dp e^{-i(p,x)}\eta^\dagger(p,\mathcal{F})\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\eta(p,\mathcal{F}') \quad (20)$$

with

$$\begin{aligned} M(p) &= (\hat{M}_{s,s'}e^{i(p,e_s)}) \\ &= (e^{ip_\mu}(1-\delta_{\mu,-\nu})) \text{ in zeroth order} \end{aligned}$$

It is $M^L(p) = M(p+\pi e_L)$ as consequance of flavour translation.

(5) *Screening of multi-link fields.* The initial and final step vectors $\eta(p,L,\mathcal{F}'), \eta^\dagger(p,L,\mathcal{F})$ follow from the the superposition of the different screenings compatible with the non-backtracking conditions. (See Fig.3.e)

(6) *Particle content.* The lattice particle states are given by the poles of the Fourier transformation of the propagator, this means by the eigenvalues $\lambda(p)$ of $M^L(p)$ which satisfy the conditions $\lambda(p)+\alpha^2=0$, and belong to eigenvectors which lead to a non-vanishing residue.

(7) *Symmetry decomposition of the step space.* M^L is symmetric under the lattice symmetry group $\mathcal{G}_L$. It decomposes in block matrices in a base which reduces the natural representation $D(g)$ of $\mathcal{G}_L$ in the step space. The result of a decomposition for $\bar{p}=(0,0,0,iE)$ is

for singlet flavour:

$$\begin{aligned} D(g) \simeq\ & 11(1^+)_{W_3} + 10(3^-)_{W_3} + 3(2^+) \\ & +7(2^-)_{W_3} + (1'^+)_{W_3} + (3^+)_{W_3} \\ & +3(3'^-)_{W_3} + 2(3'^+)_{W_3} \end{aligned}$$

for triplet flavour:

$$\begin{aligned} D(g) \simeq\ & 21(1^+)_{D_4} + 10(1^-)_{D_4} + 11(1''^+)_{D_4} \\ & +13(2^-)_{D_4} + (1'^+)_{D_4} + 3(1''^-)_{D_4} \\ & +3(2^+)_{D_4} + 2(1'''^+)_{D_4} \end{aligned}$$

#	$\bar{L}$	σ	$(j^\pi,(n)_{SU(4)})$
1	(1,1,1,1)	$(1^+)_{W_3}$	$(0^-,15)$
2	(1,1,0,1)	$(1^+)_{D_4}$	$(1^-,15)$
3	(0,0,1,1)	$(1^+)_{D_4}$	$(1^-,15)$
4	(0,0,0,1)	$(1^+)_{W_3}$	$(0^-,15)$
5	(0,0,0,1)	$(3^-)_{W_3}$	$(1^+,15)$
6	(1,1,1,0)	$(3^-)_{W_3}$	$(1^-,15)$
7	(0,0,0,0)	$(2^+)_{W_3}$	$(2^-,1)$
8	(1,1,1,1)	$(2^+)_{W_3}$	$(2^+,15)$
9	(0,0,1,0)	$(1^-)_{D_4}$	$(0^-,15)$
10	(1,1,0,1)	$(1^-)_{D_4}$	$(0^+,15)$
11	(0,0,1,0)	$(1'''^+)_{D_4}$	$(2^+,15)$
12	(1,1,0,1)	$(1'''^+)_{D_4}$	$(2^-,15)$
13	(0,0,1,1)	$(2^+)_{D_4}$	$(1^+,15)$
14	(1,1,0,0)	$(2^+)_{D_4}$	$(1^-,15)$

Table 1: Meson states appearing in First Order

The eigenvalues of these sub-matrices are computed by a reduce program.

(8) *Group theoretical particle content.* In order to get the lattice quantum numbers of the particles one has to project the propagator Eq. (20) on the appropriate 'irreducible meson fields':

$$\mathcal{M}^{L,\sigma}_{mm'}(x) = \sum_{s\in S_{4,L}} D^\sigma_{mm'}(s^{-1})\rho(R,H(e_f))\mathcal{M}^{L,s^{-1}\mathcal{F}}(x). \quad (21)$$

The symmetry of $\eta(p,L,\mathcal{F})$ determines the non-vanishing of the residues of the fields, i.e. the quantum numbers of the particle states.

The results of our calculation is summarized in Table 1. The first 4 states are the familiar states always discussed in the literature. [17,18] Our mass calculations agrees within the given order with those calculations. We find also a vanishing mass in first order for the 0^- state, in agreement with the Goldstone picture for this state.

There are 10 additional multiplets which appear for the first time in first order. We call these 'excited states' because

(a) Non-trivial reduced spin, i.e.higher continuum angu-

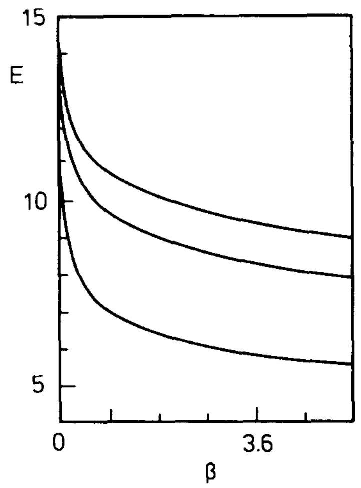


Figure 3: Mass of excited state.

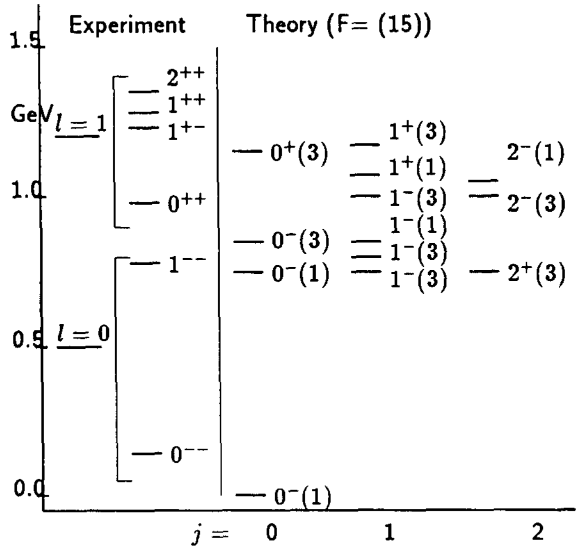


Figure 4: Comparison of the theoretical meson spectrum of first orderstrong coupling approximation with experiment.

lar momentum, indicates internal motion. This might be visualized from the eigen vectors in the step space which describe $q\bar{q}$ separation by one lattice unit.

(b) The masses of these states rise logarithmically to ∞ for $\beta \to 0$, i.e. with increasing string constant. As an example we give the expression for the energy of the state #8 of Table 1. in terms of the renormalized hopping parameter, Eq. (18)

$$\cosh E_8 = 9\frac{aa_0}{\beta} + \frac{3a_0 - 1}{a_0 + 1}. \tag{22}$$

The dependence of E_8 on β for different quark masses mb=0,2,3 is given in Fig.3. below. In Fig.4.we give a comparison of the physical spectrum (for the low lying SU(2)-flavour states with our calculated spectrum for $\beta = 2.7$, $,m_q = 0$). We have choosen for the lattice constant of the coarse lattice 2b= 0.8f, not an unreasonable value. The agreement is very poor. In particular there are many states missing for a complete spectrum of a SU(4) quark model. For the physical ground states (l=0), a comparison of Table 1. with Eq. (11) illlustrates this point. The 'non local' description of the spin structure by staggered fermions requests an higher order strong coupling approximation even for getting only a correct distribution of quantum numbers.

One reason for the poor agreement of the spin structure in our approximation is the fact that'Susskind'-coupling of staggered fermions leads for strong coupling to a strong spin-F-spin coupling.This can be seen from the relativistic spin structure of the B.S.-amplitudes of the ground states, Eq. (6). So we get for the two pseudo scalar mesons of Table 1:

$\#1: \Gamma\lambda^f = \gamma^5(2/\sqrt{5}\lambda^8 + \lambda^6)$,

$\#4: \Gamma\lambda^f = \gamma^5\gamma^\mu P_\mu(-\lambda^7 + \lambda^8)$

where the λ^i denote the $SU(4)$- Gell-Mann matrices.

4. LATTICE APPROXIMATION OF NON-RELATIVISTIC QUANTUM MECHANICS.

In our treatment of the meson spectrum in lattice QCD, the internal relative motion of the $q\bar{q}$-system is described by the step matrix $M(p)$. This matrix becomes more complex, and it describes more states with increasing order of strong coupling approximation. In the last part of our discussion we want to show that we have a similar situation in the lattice approximation of the bound state problem in non-relativistic quantum mechanics. We hope that these remarks give a hint on how one can find the usual bound state equations in the framework of lattice QCD.

The path integral representation of the (Euclidean) Green's function of the Hamiltonean $H = \frac{1}{2m}p^2 + V(x)$ is given by

$$G(x,t;0,0) = \int_{0,0}^{x,t} e^{-\int_0^t V(x(\tau))d\tau} d\mu_w(x(\tau)) \tag{23}$$

Here $d\mu_w(x(\tau))$ denotes the Wiener measure. Now it is well known [19] that the lattice approximation of Brownean motion is the simple random walk on a lattice defined by a transition probability ('step matrix'): $P_0 = 1/2d$ for steps to nearest neighbours, $P_0 = 0$ for other steps. Therefore we consider as the lattice approximation of Eq. (23) for potentials constant on lattice links:

$$G(x,n;0,0) = \int dP_0[\omega] e^{-\sum_{i=1}^{n} V(x_i)} \tag{24}$$

for walks $\omega = (0, x_1, \ldots, x_n = x)$

We want to illustrate this idea with the simplest example for a 1-dim. lattice $\Gamma = \{x | x = bn, n \in Z\}$, namely for a potential

$$V(x) = 0 \text{ for } n = O, \ldots, N, \quad = \infty \text{ for } n < 0, n > 0.$$

For this simple example one can calculate the spectrum explicitly [16]:

$$E_r = 2N^2(1 - \cos(\frac{r}{N+2})), \quad r = 1, \ldots, N+1.$$

It approaches the continuum limit in the following way: (a)For fixed quantum number r it approaches the continuum limit $E_r^c = r^2\pi^2$ as

$$E_r \to \frac{N^2}{(N+2)^2} r^2\pi^2 \text{ for } N \to \infty.$$

(b) With finer lattice we get a finer desciption of internal motion, and together with that more excited states. This is the situation which we found in the strong coupling approximation above.

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