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Calculation of transverse and longitudinal space charge effects within the framework of the fully six-dimensional formalism

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Abstract. In the following report we describe a method for calculating the envelope of a particle bunch in linear coupled storage rings and transport systems in the presence of transverse and longitudinal space charge forces using the (canonical) variables x, p_x, z, p_z, $\sigma = s - v_0 \cdot t$, $p_\sigma = \Delta E/E_0$ of the fully six-dimensional formalism. This work is an extension of earlier calculations on transverse space charge forces [1] to include the synchrotron oscillations. The extension is achieved by defining a 6-dimensional ellipsoid in the $x - p_x - z - p_z - \sigma - p_\sigma$ space. The motion of this ellipsoid under the influence of the external fields and the instantaneous space charge forces can be described by six generating orbit vectors which can be combined into a 6-dimensional matrix $\underline{B}(s)$. This "bunch-shape matrix", $\underline{B}(s)$, contains complete information about the configuration of the bunch. The solution of the equations of motion is carried through in the thin lens approximation. The formalism can also encompass acceleration by cavity fields.

1 Introduction

In [1] we described a technique for calculating the beam envelope in a storage ring when transverse space charge forces are taken into account. The method consists of calculating the motion of the 5-dimensional ellipsoid in the $x - p_x - z - p_z - \Delta p/p$ space of the kinematic variables. The beam envelope at each place in the ring is obtained by projecting this ellipsoid on the $x - z$, $x - p_x$ and $z - p_z$ planes.

In that paper we dealt only with transverse (betatron) oscillations.

The aim of the following report is to generalise these investigations by including the synchrotron oscillations.

To achieve that, additional coordinates $\sigma = s - v_0 \cdot t$ and $\eta = \Delta E/E_0$ which describe the longitudinal motion are introduced. Here σ measures the distance of a particle from the centre of the bunch and η designates the energy variation with respect to the average energy E_0.

With the complete set x, p_x, z, p_z, σ, η we are in a position to provide, in the framework of this 6-dimensional formalism, a linear analytical technique which handles the combined external magnetic and transverse and longitudinal space charge forces in a consistent canonical manner and which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities.

In Chap. 2 the equations of motion are derived by a simultaneous treatment of synchrotron and betatron oscillations taking into account the coupling of the longitudinal and transverse motion where we assume at the beginning that the space charge forces are known. It is shown that these equations may be written in a canonical form if η is chosen as the generalised momentum canonical to σ.

In order to describe the particle distribution in a bunch in Chap. 3 we consider a 6-dimensional ellipsoid in the $x - p_x - z - p_z - \sigma - p_\sigma$ space. The motion of this ellipsoid is described by six generating orbit vectors which are combined into a 6-dimensional "bunch-shape-matrix" $\underline{B}(s)$.

Since this matrix, $\underline{B}(s)$, contains complete information about the configuration of the bunch we are then in a position to estimate the space charge forces. This is done in Chap. 4 by calculating the electric fields in the rest system of the bunch. The space charge forces are then obtained by a Lorentz transformation into the laboratory system of the bunch.

In Appendix A some space charge integrals are investigated.

The solution of the complete equations of motion containing the transverse and longitudinal space charge forces is presented in Chap. 5 in the form of the thin lens approximation. Because these equations are canonical the corresponding transfer matrix is symplectic. Thus well known techniques may be used to investigate the stability of the motion and eigenvector methods may be applied to estimate the tune shifts.

In Appendix B it is shown that the acceleration by cavity fields can also be encompassed in this formalism.

The equations so derived could be used to study both dynamic and stability behaviour of a whole bunch in transport systems and storage rings.

A summary is presented in Chap. 6.

2 The equations of motion

Our investigation of synchro-betatron oscillations in the presence of space charge forces begins with a statement of the equations of motion. We will use the same variables as those in [2]: $x, z, \sigma = s - v_0 \cdot t$ and $\eta = \Delta E/E_0$, where x and z describe the amplitude of transverse motion (betatron oscillations), while σ and η describe the longitudinal (synchrotron) oscillations. Since v_0 designates the average velocity of the particles, the quantity σ describes the longitudinal separation of a particle from the centre of the bunch.

The equations for transverse motion have already been given in [1]. They are:

$$x'' = -G_1 \cdot x + (N + H') \cdot z + 2H \cdot z' + \frac{1}{\gamma_0 \cdot m_0 \cdot v_0^2} \cdot F_x^{\text{self}} + K_x \cdot f(\eta); \tag{2.1a}$$

$$z'' = -G_2 \cdot z + (N - H') \cdot x - 2H \cdot x' + \frac{1}{\gamma_0 \cdot m_0 \cdot v_0^2} \cdot F_z^{\text{self}} + K_z \cdot f(\eta) \tag{2.1b}$$

with

$$N = \frac{1}{2} \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0}; \tag{2.2a}$$

$$H = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot B_s; \tag{2.2b}$$

$$g = \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_z}{\partial x} \right)_{x=z=0} \tag{2.2c}$$

and

$$G_1 = K_x^2 + g; \quad G_2 = K_z^2 - g; \tag{2.3}$$

$$f(\eta) = \frac{p - p_0}{p_0} = \frac{\Delta p}{p^0} = \frac{1}{\beta_0} \cdot \sqrt{(1+\eta)^2 - (m_0 c^2/E_0)^2} - 1 \tag{2.4}$$

and where F_x^{self} and F_z^{self} are the space charge forces in the x and z directions.

In the following we put

$$\frac{1}{\gamma_0 \cdot m_0 \cdot v_0^2} \cdot F_x^{\text{self}} = F_{xx} \cdot x + F_{xz} \cdot z; \tag{2.5a}$$

$$\frac{1}{\gamma_0 \cdot m_0 \cdot v_0^2} \cdot F_z^{\text{self}} = F_{zx} \cdot x + F_{zz} \cdot z \tag{2.5b}$$

where F_{xx}, F_{xz}, F_{zx}, F_{zz} are introduced in Chap. 4. For now, we only need to use the fact that

$$F_{xz} = F_{zx}. \tag{2.6}$$

The quantity $f(\eta)$ in (2.1) can be developed in a power series in η:

$$f(\eta) = f'(0) \cdot \eta + \tfrac{1}{2} \cdot f''(0) \cdot \eta^2 + \cdots \tag{2.7}$$

with

$$f'(\eta) = \frac{1}{\beta_0} \cdot \frac{1+\eta}{\sqrt{(1+\eta)^2 - (m_0 c^2/E_0)^2}} = \frac{1}{\beta_0} \cdot \frac{E}{p \cdot c} = \frac{1}{\beta_0 \cdot \beta}; \tag{2.8a}$$

$$\Rightarrow f'(0) = \frac{1}{\beta_0^2}; \tag{2.8b}$$

$$f''(\eta) = \frac{1}{\beta_0} \cdot \left[(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0} \right)^2 \right]^{-1/2} - \frac{1}{\beta_0} \cdot (1+\eta)^2 \cdot \left[(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0} \right)^2 \right]^{-3/2}$$

$$= -\frac{1}{\beta_0} \cdot \frac{E_0}{E} \cdot \frac{1}{\beta^3 \gamma^2}; \tag{2.9a}$$

$$\Rightarrow f''(0) = -\frac{1}{\beta_0^4 \cdot \gamma_0^2}. \tag{2.9b}$$

To obtain the equations for longitudinal motion we recall that the field in a cavity can be written in terms of σ as

$$\varepsilon_{\text{cavity}} = V(s) \cdot \sin \left[k \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right]; \tag{2.10}$$

$\varphi = 0, \pi$ for protons.

Writing the longitudinal space charge forces as

$$\frac{1}{E_0} F_s^{\text{self}} = F_\sigma(s) \cdot \sigma \tag{2.11}$$

the equation for the variation of η is:

$$\eta' = \frac{e V(s)}{E_0} \cdot \cos \varphi \cdot k \cdot \frac{2\pi}{L} \cdot \sigma + F_\sigma(s) \cdot \sigma. \tag{2.12}$$

The calculation of F_σ is given in Chap. 4.

The variation of $\sigma = s - v_0 \cdot t(s)$ is given by:

$$\sigma' = 1 - v_0 \cdot \frac{dt}{ds};$$

$$dt = \frac{dl}{v};$$

$$dl = ds \cdot [1 + K_x \cdot x + K_z \cdot z + \cdots];$$

$$\Rightarrow \sigma' = 1 - \frac{v_0}{v} \cdot [1 + K_x \cdot x + K_z \cdot z + \cdots]. \tag{2.13}$$

With the relation

$$\frac{v_0}{v} = \beta_0^2 \cdot f'(\eta);$$

$$f'(\eta) = f'(0) + \eta \cdot f''(0) + \cdots$$
$$= \frac{1}{\beta_0^2} - \eta \cdot \frac{1}{\beta_0^4 \cdot \gamma_0^2} + \cdots,$$

from (2.8 and 2.9) we then obtain in linear approximation:

$$\sigma' = \frac{1}{\beta_0^2 \gamma_0^2} \cdot \eta - (K_x \cdot x + K_z \cdot z). \tag{2.14}$$

Equations (2.1, 2.12 and 2.14) provide a complete description of transverse and longitudinal motion in the presence of space charge forces.

To proceed further, it will be useful to write these equations in canonical form:

$$x' = \frac{\partial \hat{H}}{\partial p_x}; \quad p_x' = -\frac{\partial \hat{H}}{\partial x};$$

$$z' = \frac{\partial \hat{H}}{\partial p_z}; \quad p_z' = -\frac{\partial \hat{H}}{\partial z};$$

$$\sigma' = \frac{\partial \hat{H}}{\partial p_\sigma}; \quad p_\sigma' = -\frac{\partial \hat{H}}{\partial \sigma}$$

using the Hamiltonian

$$\begin{aligned} \hat{H} &= \frac{1}{\beta_0^2 \gamma_0^2} \cdot \tfrac{1}{2} p_\sigma^2 - (K_x \cdot x + K_z \cdot z) \cdot p_\sigma \\ &\quad - \frac{1}{2} \cdot \frac{e V(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \sigma^2 - \tfrac{1}{2} F_\sigma(s) \cdot \sigma^2 \\ &\quad + \frac{1}{2\beta_0^2} \cdot \{[p_x + \beta_0^2 \cdot H \cdot z]^2 + [p_z - \beta_0^2 \cdot H \cdot x]^2\} \\ &\quad + \tfrac{1}{2} \beta_0^2 \cdot [G_1 \cdot x^2 + G_2 \cdot z^2 - 2N \cdot xz] \\ &\quad - \tfrac{1}{2} \beta_0^2 \cdot [F_{xx} \cdot x^2 + 2F_{xz} \cdot xz + F_{zz} \cdot z^2]. \end{aligned} \tag{2.15}$$

By eliminating the quantities p_x and p_z from the resulting canonical equations

$$\begin{aligned} x' &= \frac{1}{\beta_0^2} [p_x + \beta_0^2 \cdot H \cdot z]; \\ p_x' &= K_x \cdot p_\sigma + [p_z - \beta_0^2 \cdot H \cdot x] \cdot H \\ &\quad - \beta_0^2 \cdot [G_1 \cdot x - N \cdot z - F_{xx} \cdot x - F_{xz} \cdot z]; \\ z' &= \frac{1}{\beta_0^2} [p_z - \beta_0^2 \cdot H \cdot x]; \\ p_z' &= K_z \cdot p_\sigma - [p_x + \beta_0^2 \cdot H \cdot z] \cdot H \\ &\quad - \beta_0^2 \cdot [G_2 \cdot z - N \cdot x - F_{xz} \cdot x - F_{zz} \cdot z]; \\ \sigma' &= \frac{1}{\beta_0^2 \gamma_0^2} \cdot p_\sigma - (K_x \cdot x + K_z \cdot z); \\ p_\sigma' &= \frac{e V(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \sigma + F_\sigma(s) \cdot \sigma \end{aligned} \tag{2.16}$$

and putting

$$p_\sigma = \eta, \tag{2.17}$$

we recover (2.1, 2.2 and 2.14) with the help of (2.6) provided the linear approximation

$$f(\eta) = \frac{1}{\beta_0^2} \cdot \eta \tag{2.18}$$

is valid.

Since the equations of motion are linear they can be solved in the form

$$\mathbf{y}(s) = \underline{M}(s, s_0) \mathbf{y}(s_0) \tag{2.19}$$

with

$$\mathbf{y} = \begin{pmatrix} x \\ p_x \\ z \\ p_z \\ \sigma \\ p_\sigma \end{pmatrix}.$$

Because the variables $x, p_x, z, p_z, \sigma, p_\sigma$ are canonical, the transfer matrix $\underline{M}(s, s_0)$ is symplectic [1]:

$$\underline{M}^T(s, s_0) \cdot \underline{S} \cdot \underline{M}(s, s_0) = \underline{S} \tag{2.20a}$$

where

$$\underline{S} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.20b}$$

In order to construct the matrix $\underline{M}(s, s_0)$, the quantities F_{xx}, F_{xz}, F_{zx}, F_{zz} and F_σ of (2.16) which describe the self induced space charge forces F_x^{self}, F_z^{self} and F_σ^{self} must be known. This is the topic of the next chapter.

3 The beam envelopes

3.1 The six dimensional ellipsoid in $x - p_x - z - p_z - \sigma - p_\sigma$ space

To obtain the space charge forces we must know the particle distribution.

We will assume that at the start point, s_0, the ensemble is distributed on the surface of a six dimensional ellipsoid in $x - p_x - z - p_z - \sigma - p_\sigma$ space of the form

$$\begin{aligned} &\mathbf{y}(s_0; \varphi, \chi, \delta_I, \delta_{II}, \delta_{III}) \\ &\quad = \cos\varphi \cdot \cos\chi \cdot [\mathbf{y}_1(s_0) \cdot \cos\delta_I + \mathbf{y}_2(s_0) \cdot \sin\delta_I] \\ &\qquad + \cos\varphi \cdot \sin\chi \cdot [\mathbf{y}_3(s_0) \cdot \cos\delta_{II} + \mathbf{y}_4(s_0) \cdot \sin\delta_{II}] \\ &\qquad + \sin\varphi \cdot [\mathbf{y}_5(s_0) \cdot \cos\delta_{III} + \mathbf{y}_6(s_0) \cdot \sin\delta_{III}]. \end{aligned} \tag{3.1}$$

This ellipsoid can be spanned by six linearly independent vectors

$$\mathbf{y}_k = \begin{pmatrix} y_{k1} \\ y_{k2} \\ y_{k3} \\ y_{k4} \\ y_{k5} \\ y_{k6} \end{pmatrix}; \quad (k=1,2,3,4,5,6) \tag{3.2}$$

which are defined by the starting shape of the ellipsoid.
The corresponding vector $\mathbf{y}$ at position, s, is given by

$$\begin{aligned}&\mathbf{y}(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\cos\varphi\cdot\cos\chi\cdot[\mathbf{y}_1(s)\cdot\cos\delta_I+\mathbf{y}_2(s)\cdot\sin\delta_I]\\ &\qquad+\cos\varphi\cdot\sin\chi\cdot[\mathbf{y}_3(s)\cdot\cos\delta_{II}+\mathbf{y}_4(s)\cdot\sin\delta_{II}]\\ &\qquad+\sin\varphi\cdot[\mathbf{y}_5(s)\cdot\cos\delta_{III}+\mathbf{y}_6(s)\cdot\sin\delta_{III}]\end{aligned} \tag{3.3}$$

where

$$\mathbf{y}_k(s)=\underline{M}(s,s_0)\mathbf{y}_k(s_0);\quad (k=1,2,3,4,5,6). \tag{3.4}$$

Thus the ellipsoid remains an ellipsoid.
The beam envelopes can then be obtained by projecting the ellipsoid of (3.3) on the individual phase planes.

3.2 *The projections of the six dimensional ellipsoid*

To define the projections we first of all write the ellipsoid (3.3) in component form:

$$\begin{aligned}&x(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\cos\varphi\cdot\cos\chi\cdot[y_{11}(s)\cdot\cos\delta_I+y_{21}(s)\cdot\sin\delta_I]\\ &\qquad+\cos\varphi\cdot\sin\chi\cdot[y_{31}(s)\cdot\cos\delta_{II}+y_{41}(s)\cdot\sin\delta_{II}]\\ &\qquad+\sin\varphi\cdot[y_{51}(s)\cdot\cos\delta_{III}+y_{61}(s)\cdot\sin\delta_{III}];\end{aligned} \tag{3.5a}$$

$$\begin{aligned}&p_x(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\cos\varphi\cdot\cos\chi\cdot[y_{12}(s)\cdot\cos\delta_I+y_{22}(s)\cdot\sin\delta_I]\\ &\qquad+\cos\varphi\cdot\sin\chi\cdot[y_{32}(s)\cdot\cos\delta_{II}+y_{42}(s)\cdot\sin\delta_{II}]\\ &\qquad+\sin\varphi\cdot[y_{52}(s)\cdot\cos\delta_{III}+y_{62}(s)\cdot\sin\delta_{III}];\end{aligned} \tag{3.5b}$$

$$\begin{aligned}&z(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\cos\varphi\cdot\cos\chi\cdot[y_{13}(s)\cdot\cos\delta_I+y_{23}(s)\cdot\sin\delta_I]\\ &\qquad+\cos\varphi\cdot\sin\chi\cdot[y_{33}(s)\cdot\cos\delta_{II}+y_{43}(s)\cdot\sin\delta_{II}]\\ &\qquad+\sin\varphi\cdot[y_{53}(s)\cdot\cos\delta_{III}+y_{63}(s)\cdot\sin\delta_{III}];\end{aligned} \tag{3.5c}$$

$$\begin{aligned}&p_z(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\cos\varphi\cdot\cos\chi\cdot[y_{14}(s)\cdot\cos\delta_I+y_{24}(s)\cdot\sin\delta_I]\\ &\qquad+\cos\varphi\cdot\sin\chi\cdot[y_{34}(s)\cdot\cos\delta_{II}+y_{44}(s)\cdot\sin\delta_{II}]\\ &\qquad+\sin\varphi\cdot[y_{54}(s)\cdot\cos\delta_{III}+y_{64}(s)\cdot\sin\delta_{III}];\end{aligned} \tag{3.5d}$$

$$\begin{aligned}&\sigma(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\cos\varphi\cdot\cos\chi\cdot[y_{15}(s)\cdot\cos\delta_I+y_{25}(s)\cdot\sin\delta_I]\\ &\qquad+\cos\varphi\cdot\sin\chi\cdot[y_{35}(s)\cdot\cos\delta_{II}+y_{45}(s)\cdot\sin\delta_{II}]\\ &\qquad+\sin\varphi\cdot[y_{55}(s)\cdot\cos\delta_{III}+y_{65}(s)\cdot\sin\delta_{III}];\end{aligned} \tag{3.5e}$$

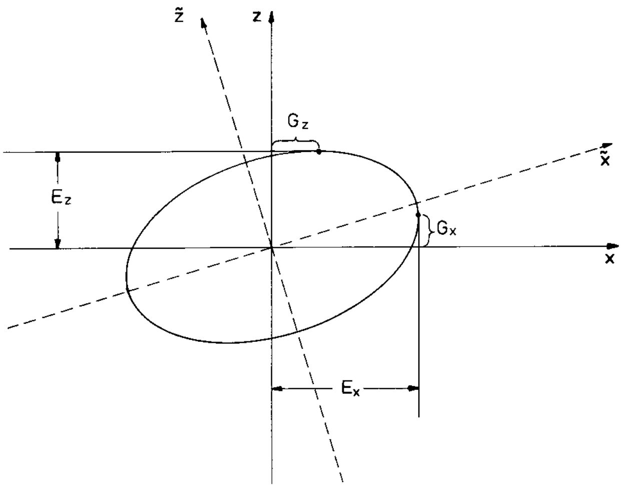

Fig. 1. Beam cross section at position s

$$\begin{aligned}&p_\sigma(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\cos\varphi\cdot\cos\chi\cdot[y_{16}(s)\cdot\cos\delta_I+y_{26}(s)\cdot\sin\delta_I]\\ &\qquad+\cos\varphi\cdot\sin\chi\cdot[y_{36}(s)\cdot\cos\delta_{II}+y_{46}(s)\cdot\sin\delta_{II}]\\ &\qquad+\sin\varphi\cdot[y_{56}(s)\cdot\cos\delta_{III}+y_{66}(s)\cdot\sin\delta_{III}].\end{aligned} \tag{3.5f}$$

The computation of the single projections is then similar to that in [1] in which the functional relationship between pairs of components was investigated.
Since the details of the method have already been given in [1, 3] only a summary will be needed here.

3.2.1 Projection on the $x-z$ plane. We first investigate the projection on the $x-z$ plane. This describes the beam cross section. We will need the maximum amplitude in the x and z directions.

a) Maximum oscillation amplitude in the x direction: Using the relation

$$\mathrm{Max}_{(\varphi)}\{A\cdot\cos\varphi+B\cdot\sin\varphi\}=\sqrt{A^2+B^2}$$

and (3.5a), the largest possible x amplitude is

$$\begin{aligned}&\mathrm{Max}_{(\varphi,\chi,\delta_I,\delta_{II},\delta_{III})}x(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III})\\ &\quad=\sqrt{y_{11}^2+y_{21}^2+y_{31}^2+y_{41}^2+y_{51}^2+y_{61}^2}=E_x(s).\end{aligned} \tag{3.6}$$

This occurs for the values:

$$\cos\delta_I=\frac{y_{11}}{\sqrt{y_{11}^2+y_{21}^2}};\quad \sin\delta_I=\frac{y_{21}}{\sqrt{y_{11}^2+y_{21}^2}};$$

$$\cos\delta_{II}=\frac{y_{31}}{\sqrt{y_{31}^2+y_{41}^2}};\quad \sin\delta_{II}=\frac{y_{41}}{\sqrt{y_{31}^2+y_{41}^2}};$$

$$\cos\delta_{III}=\frac{y_{51}}{\sqrt{y_{51}^2+y_{61}^2}};\quad \sin\delta_{III}=\frac{y_{61}}{\sqrt{y_{51}^2+y_{61}^2}};$$

$$\cos\chi=\frac{\sqrt{y_{11}^2+y_{21}^2}}{\sqrt{y_{11}^2+y_{21}^2+y_{31}^2+y_{41}^2}};$$

$$\sin\chi=\frac{\sqrt{y_{31}^2+y_{41}^2}}{\sqrt{y_{11}^2+y_{21}^2+y_{31}^2+y_{41}^2}};$$

$$\cos\varphi = \frac{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2 + y_{51}^2 + y_{61}^2}};$$

$$\sin\varphi = \frac{\sqrt{y_{51}^2 + y_{61}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2 + y_{51}^2 + y_{61}^2}}. \quad (3.7)$$

The corresponding z coordinate is given by (3.5c) together with (3.9):

$$G_x = \frac{1}{E_x(s)} \cdot \{y_{11} \cdot y_{13} + y_{21} \cdot y_{23} + y_{31} \cdot y_{33} + y_{41} \cdot y_{43} + y_{51} \cdot y_{53} + y_{61} \cdot y_{63}\}. \quad (3.8)$$

b) Maximum oscillation amplitude in the z direction:

Correspondingly, the maximum amplitude in the z-direction is obtained from (3.5c):

$$\mathrm{Max}_{(\varphi,\chi,\delta_I,\delta_{II},\delta_{III})} z(s;\varphi,\chi,\delta_I,\delta_{II},\delta_{III}) = \sqrt{y_{13}^2 + y_{23}^2 + y_{33}^2 + y_{43}^2 + y_{53}^2 + y_{63}^2} = E_z(s). \quad (3.9)$$

The accompanying x-coordinate is then:

$$G_z = \frac{1}{E_z(s)} \cdot \{y_{11} \cdot y_{13} + y_{21} \cdot y_{23} + y_{31} \cdot y_{33} + y_{41} \cdot y_{43} + y_{51} \cdot y_{53} + y_{61} \cdot y_{63}\}. \quad (3.10)$$

Thus

$$E_x \cdot G_x = E_z \cdot G_z. \quad (3.11)$$

c) The boundary curve of the beam cross section.

The projections of the ellipsoid (3.5) are ellipses, and these are described by the three independent quantities E_x, G_x, E_z. The parameter G_z depends on the other three (see (3.11)). In terms of E_x, G_x, E_z, the ellipse can be written as:

$$E_z^2 \cdot x^2 - 2E_x G_x \cdot xz + E_x^2 \cdot z^2 = \varepsilon_{xz}^2 \quad (3.12a)$$

with

$$\varepsilon_{xz} = E_x \cdot \sqrt{E_z^2 - G_x^2}. \quad (3.12b)$$

and where $\pi\varepsilon_{xz}$ is the area of the ellipse.

The half axes E_1 and E_2 of the elliptical beam cross section are:

$$E_{1,2} = \tfrac{1}{2}\{[E_x^2 + E_z^2] \pm \sqrt{[E_x^2 - E_z^2]^2 + 4E_x^2 G_x^2}\} \quad (3.13)$$

and the twist angle θ of the beam is given by:

$$\tan 2\theta = \frac{2E_x G_x}{E_x^2 - E_z^2}. \quad (3.14)$$

3.2.2 Projection on the $x-\sigma$ plane. To find the projection of the ellipsoid (3.3) onto the $x-\sigma$ plane we need (3.5a and 3.5e). Since these have the same general form as (3.5a and 3.5c), we can obtain the projections using exactly the same methods as in the previous section.

The boundary curve of the elliptical projection on the $x-\sigma$ plane is:

$$E_x^2 \cdot \sigma^2 - 2E_\sigma G_\sigma^{(x)} \cdot \sigma x + E_\sigma^2 \cdot x^2 = \varepsilon_{\sigma x}^2 \quad (3.15)$$

with

$$E_\sigma = \sqrt{y_{15}^2 + y_{25}^2 + y_{35}^2 + y_{45}^2 + y_{55}^2 + y_{65}^2}; \quad (3.16a)$$

$$G_\sigma^{(x)} = \frac{1}{E_\sigma} \cdot \{y_{11} \cdot y_{15} + y_{21} \cdot y_{25} + y_{31} \cdot y_{35} + y_{41} \cdot y_{45} + y_{51} \cdot y_{55} + y_{61} \cdot y_{65}\}; \quad (3.16b)$$

$$\varepsilon_{\sigma x} = E_\sigma \cdot \sqrt{E_x^2 - (G_\sigma^{(x)})^2}. \quad (3.16c)$$

The meaning of E_σ and $G_\sigma^{(x)}$ is explained by Fig. 2. $\pi\varepsilon_{\sigma x}$ is the area of the ellipse (3.15).

3.2.3 Projection on the $z-\sigma$ plane. Finally, the projection of the ellipsoid on the $z-\sigma$ plane (see Fig. 3) has the boundary curve:

$$E_z^2 \cdot \sigma^2 - 2E_\sigma G_\sigma^{(z)} \cdot \sigma z + E_\sigma^2 \cdot z^2 = \varepsilon_{\sigma z}^2 \quad (3.17)$$

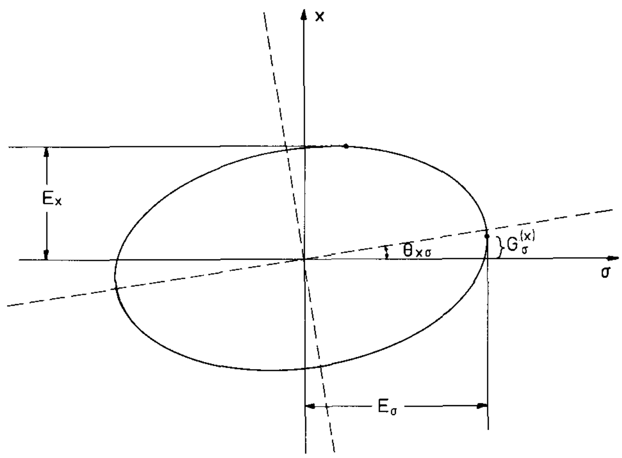

Fig. 2. Projection of the ellipsoid on the $x-\sigma$ plane

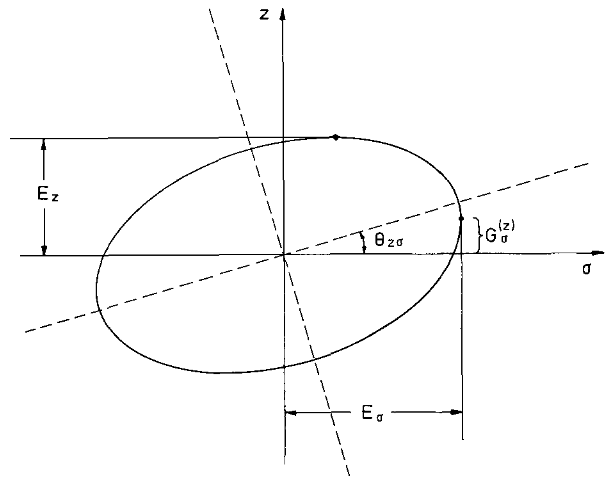

Fig. 3. Projection of the ellipsoid on the $z-\sigma$ plane

with

$$G_\sigma^{(z)} = \frac{1}{E_\sigma} \cdot \{y_{13} \cdot y_{15} + y_{23} \cdot y_{25} + y_{33} \cdot y_{35} + y_{43} \cdot y_{45} + y_{53} \cdot y_{55} + y_{63} \cdot y_{65}\}; \quad (3.18a)$$

$$\varepsilon_{\sigma z} = E_\sigma \cdot \sqrt{E_z^2 - (G_\sigma^{(z)})^2} \quad (3.18b)$$

($\pi \varepsilon_{\sigma z}$ is the area of the ellipse (3.17)).

Now that we have the projections of the ellipsoid on all three planes we can calculate the space charge forces. We assume that the particle distribution is uniform in the $x - z - s$ space.

4 Calculation of space charge forces

In calculating the space charge forces, we assume that the synchro-betatron coupling is small and we therefore ignore the twist angles $\theta_{x\sigma}$ and $\theta_{z\sigma}$ of the bunch with respect to the σ axis (Figs. 2, 3).

Since (see Chap. 2) the variable σ describes the distance of a particle from the centre of the bunch, then E_σ describes the half length of the bunch in the laboratory system. E_1 and E_2 describe the transverse bunch extensions.

The whole bunch can now be represented by a three dimensional ellipsoid which, in a rotated $(\tilde{x}, \tilde{z}, \sigma)$ coordinate system (see Fig. 1), can be described by the equation

$$\frac{\tilde{x}^2}{E_1^2} + \frac{\tilde{z}^2}{E_2^2} + \frac{\sigma^2}{E_\sigma^2} = 1 \quad (4.1)$$

in the laboratory system and by the equation

$$\frac{\tilde{x}^2}{E_1^2} + \frac{\tilde{z}^2}{E_2^2} + \frac{\bar{\sigma}^2}{(\gamma_0 \cdot E_\sigma)^2} = 1 \quad (4.2)$$

in the rest system of the bunch ($\bar{\sigma} = \gamma_0 \cdot \sigma$ describes the longitudinal coordinate in the rest system).

Now for a uniform charge distribution in the (rest system) ellipsoid

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1, \quad (4.3a)$$

the potential inside is given by [4]

$$U = -A \cdot \xi^2 - B \cdot \eta^2 - C \cdot \zeta^2 + D \quad (4.3b)$$

with

$$A = \pi abc \cdot \rho \cdot \int_0^\infty \frac{d\tau}{(a^2 + \tau) \cdot \sqrt{\varphi(\tau)}};$$

$$B = \pi abc \cdot \rho \cdot \int_0^\infty \frac{d\tau}{(b^2 + \tau) \cdot \sqrt{\varphi(\tau)}};$$

$$C = \pi abc \cdot \rho \cdot \int_0^\infty \frac{d\tau}{(c^2 + \tau) \cdot \sqrt{\varphi(\tau)}};$$

$$D = \pi abc \cdot \rho \cdot \int_0^\infty \frac{d\tau}{\sqrt{\varphi(\tau)}};$$

$$\varphi(\tau) = (a^2 + \tau) \cdot (b^2 + \tau) \cdot (c^2 + \tau);$$

(ρ = charge density). (4.3c)

The total charge in the bunch is

$$Q = \frac{4\pi}{3} abc \cdot \rho. \quad (4.4)$$

Thus comparing (4.2 and 4.3) the space charge force in the rest system is

$$F_{\tilde{x}}^{(0)} = \tfrac{3}{2} eQ \cdot I_1 \cdot \tilde{x}; \quad (4.5a)$$

$$F_{\tilde{z}}^{(0)} = \tfrac{3}{2} eQ \cdot I_2 \cdot \tilde{z}; \quad (4.5b)$$

$$F_s^{(0)} = \tfrac{3}{2} eQ \cdot I_3 \cdot \bar{\sigma} \quad (4.5c)$$

with

$$I_1 = \int_0^\infty \frac{d\tau}{(E_1^2 + \tau) \cdot \sqrt{\psi(\tau)}};$$

$$I_2 = \int_0^\infty \frac{d\tau}{(E_2^2 + \tau) \cdot \sqrt{\psi(\tau)}};$$

$$I_3 = \int_0^\infty \frac{d\tau}{(\gamma_0^2 \cdot E_\sigma^2 + \tau) \cdot \sqrt{\psi(\tau)}};$$

$$\psi(\tau) = (E_1^2 + \tau) \cdot (E_2^2 + \tau) \cdot (\gamma_0^2 E_\sigma^2 + \tau). \quad (4.6)$$

The terms I_1, I_2, I_3 can be expressed in terms of elliptical integrals of the second kind.

As shown in Appendix A, if

$$\gamma_0 E_\sigma \gg E_1, E_2$$

I_1, I_2, I_3 can be calculated approximately by analytical means.

We now Lorentz transform to the laboratory frame to obtain the forces

$$F_{\tilde{x}} = \frac{1}{\gamma_0} \cdot F_{\tilde{x}}^{(0)};$$

$$F_{\tilde{z}} = \frac{1}{\gamma_0} \cdot F_{\tilde{z}}^{(0)};$$

$$F_s = F_s^{(0)}$$

so that the space charge forces in the laboratory system are

$$F_{\tilde{x}}^{\text{self}} = \frac{1}{\gamma_0} \cdot \tfrac{3}{2} eQ \cdot I_1 \cdot \tilde{x}; \quad (4.7a)$$

$$F_{\tilde{z}}^{\text{self}} = \frac{1}{\gamma_0} \cdot \tfrac{3}{2} eQ \cdot I_2 \cdot \tilde{z}; \quad (4.7b)$$

$$F_s^{\text{self}} = \tfrac{3}{2} eQ \cdot I_3 \cdot \gamma_0 \cdot \sigma, \quad (4.7c)$$

where in (4.7c) we use the relation (Lorentz contraction):

$$\bar{\sigma} = \gamma_0 \cdot \sigma.$$

The components of space charge force in x and z

directions are

$$F_x^{\text{self}} = \cos\theta\cdot F_{\tilde{x}} - \sin\theta\cdot F_{\tilde{z}} = \frac{1}{\gamma_0}\cdot\tfrac{3}{2}eQ\cdot[x\cdot(I_1\cos^2\theta + I_2\sin^2\theta) + z\cdot\sin\theta\cos\theta\cdot(I_1 - I_2)]; \quad (4.8a)$$

$$F_x^{\text{self}} = \sin\theta\cdot F_{\tilde{x}} + \cos\theta\cdot F_{\tilde{z}} = \frac{1}{\gamma_0}\cdot\tfrac{3}{2}eQ\cdot[x\cdot\sin\theta\cos\theta\cdot(I_1 - I_2) + z\cdot(I_1\sin^2\theta + I_2\cos^2\theta)]. \quad (4.8b)$$

By comparing (4.7c) and (4.8a, b) with (2.11) and (2.5a, b) we finally obtain the coefficients F_{xx}, F_{xz}, F_{zx}, F_{zz} and F_σ:

$$F_{xx} = \tfrac{3}{2}eQ\cdot\frac{1}{\gamma_0^2\cdot m_0 v_0^2}\cdot[I_1\cos^2\theta + I_2\sin^2\theta]; \quad (4.9a)$$

$$F_{zz} = \tfrac{3}{2}eQ\cdot\frac{1}{\gamma_0^2\cdot m_0 v_0^2}\cdot[I_1\sin^2\theta + I_2\cos^2\theta]; \quad (4.9b)$$

$$F_{xz} = \tfrac{3}{2}eQ\cdot\frac{1}{\gamma_0^2\cdot m_0 v_0^2}\cdot\sin\theta\cos\theta\cdot(I_1 - I_2) = F_{zx}; \quad (4.9c)$$

$$F_\sigma = \tfrac{3}{2}eQ\cdot\frac{\gamma_0}{E_0}\cdot I_3. \quad (4.9d)$$

The angle θ is defined by (3.14) and the quantities I_1, I_2 and I_3 by (4.6).

In particular, we see that (4.9c) reproduces (2.6) which was used to derive the Hamiltonian (2.15).

Equation (4.9) can now be used together with (2.16) (and with the help of (3.6, 3.8, 3.9, 3.16a, 3.13 and 3.14)) to obtain explicit forms for the (canonical) equations of motion under the influence of both external and space charge effects.

Remark. In (4.8) the effects of orbit curvature described in [5] are not included but the linear part of these additional forces could be easily incorporated. Linear wakefield effects could be taken into account in the same way.

5 Solution of the equations of motion

The solution of these equations will be obtained in transport matrix form. We write

$$\frac{d}{ds}\mathbf{y} = \underline{A}(s)\cdot\mathbf{y} \quad (5.1)$$

with

$$A_{12} = \frac{1}{\beta_0^2};$$
$$A_{13} = +H;$$
$$A_{21} = -\beta_0^2\cdot[G_1 - F_{xx} + H^2];$$
$$A_{23} = +\beta_0^2\cdot[N + F_{xz}];$$
$$A_{24} = +H;$$
$$A_{26} = K_x;$$
$$A_{31} = -H;$$
$$A_{34} = \frac{1}{\beta_0^2};$$
$$A_{41} = +\beta_0^2\cdot[N + F_{xz}];$$
$$A_{42} = -H;$$
$$A_{43} = -\beta_0^2\cdot[G_2 - F_{zz} + H^2];$$
$$A_{46} = K_z;$$
$$A_{51} = -K_x;$$
$$A_{53} = -K_z;$$
$$A_{56} = \frac{1}{\beta_0^2\cdot\gamma_0^2};$$
$$A_{65} = \frac{eV(s)}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\varphi + F_\sigma;$$
$$A_{ik} = 0 \text{ otherwise.} \quad (5.2)$$

We solve (5.1) using thin lens approximation [1].

Using notation similar to that of [1], we obtain for the transfer matrix

$$\underline{M}(s+\Delta s, s) = \underline{M}_D\left(s+\Delta s, s+\frac{\Delta s}{2}\right)\cdot[\underline{1} + \underline{C}(s)\cdot\Delta s]\cdot\underline{R}(\Delta\Theta)\cdot\underline{M}_D\left(s+\frac{\Delta s}{2}, s\right) \quad (5.3)$$

with

$$\underline{C}(s) = \underline{A}(s) - \underline{D} - \underline{F}; \quad (5.4a)$$

$$\underline{D} = \frac{1}{\beta_0^2}\cdot\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\gamma_0^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad (5.4b)$$

$$\underline{F} = H\cdot\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad (5.4c)$$

$$\underline{M}_D(s+l, s) = \underline{1} + l\cdot\underline{D}; \quad (5.4d)$$

(transfer matrix for a simple drift space of length 1);

$$\underline{R}(\Delta\Theta) = \begin{pmatrix} \cos\Delta\Theta & 0 & +\sin\Delta\Theta & 0 & 0 & 0 \\ 0 & \cos\Delta\Theta & 0 & +\sin\Delta\Theta & 0 & 0 \\ -\sin\Delta\Theta & 0 & \cos\Delta\Theta & 0 & 0 & 0 \\ 0 & -\sin\Delta\Theta & 0 & \cos\Delta\Theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.4e)$$

$(\Delta\Theta = H\cdot\Delta s)$.

As in [1], this thin lens form is symplectic and the matrix $\underset{\sim}{A}$ depends on the shape of the bunch. The latter depends on the generating orbit vectors $\mathbf{y}_k$ $(k=1,2,3,4,5,6)$ which change during the motion of the bunch according to the equation [1]

$$\mathbf{y}_k(s+\Delta s)=\underset{\sim}{M}(s+\Delta s,s)\cdot\mathbf{y}_k(s). \tag{5.5}$$

Periodic solutions must be obtained by self consistent iteration [1].

Finally, we point out that the 6 equations (5.5) for the generating vectors $\mathbf{y}_k$ $(k=1-6)$ can be handled in a compact way by introducing the "bunch-shape matrix"

$$\underset{\sim}{B}=(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{y}_4,\mathbf{y}_5,\mathbf{y}_6) \tag{5.6}$$

so that

$$\underset{\sim}{B}(s+\Delta s,s)=\underset{\sim}{M}(s+\Delta s,s)\cdot\underset{\sim}{B}(s). \tag{5.7}$$

Acceleration by a cavity field is described in Appendix B.

6 Summary

We have investigated the influence of longitudinal and transverse space charge forces on the motion of charged particles in storage rings and transport systems by a simultaneous treatment of synchrotron and betatron oscillations.

The motion is described in terms of the fully six-dimensional formalism with the canonical variables $x, p_x, z, p_z, \sigma=s-v_0\cdot t, p_\sigma=\Delta E/E_0$.

In order to describe the bunch we have introduced a 6-dimensional ellipsoid in the $x-p_x-z-p_z-\sigma-p_\sigma$ space represented by the "bunch-shape matrix", $\underset{\sim}{B}(s)$, which contains as columns, six independent orbit vectors. As in [1], this matrix $\underset{\sim}{B}(s)$ contains complete information about the configuration of the bunch at the point s and can be obtained by matrix multiplication with the transfer matrix $\underset{\sim}{M}$.

In thin lens approximation the matrix takes a simple form which can be conveniently coded for computer.

The equations so derived are valid for arbitrary velocity v_0 (below and above transition energy).

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Appendix A: calculation of the space charge integrals I_1, I_2, I_3

In order to calculate the space charge integrals I_1, I_2, I_3 of (4.6) we assume that

$$E_3=\gamma_0 E_\sigma \gg E_1, E_2. \tag{A.1}$$

In this case, for I_1 we may write (approximately):

$$I_1\approx\frac{1}{E_3}\cdot\int_0^\infty\frac{d\tau}{(E_1^2+\tau)\cdot\sqrt{(E_1^2+\tau)\cdot(E_2^2+\tau)}}. \tag{A.2}$$

With the substitution

$$t^2=\tau+E_2^2 \tag{A.3}$$

we get

$$\begin{aligned}I_1&\approx\frac{2}{E_3}\cdot\int_{E_2}^\infty\frac{dt}{\sqrt{(t^2+E_1^2-E_2^2)^3}}\\&=\frac{2}{E_3}\cdot\frac{1}{E_1^2-E_2^2}\cdot\left[\frac{t}{\sqrt{t^2+E_1^2-E_2^2}}\right]_{E_2}^\infty\\&=\frac{2}{E_3}\cdot\frac{1}{E_1(E_1+E_2)}.\end{aligned} \tag{A.4}$$

I_2 can be obtained in the same way:

$$I_2\approx\frac{2}{E_3}\cdot\frac{1}{E_2(E_1+E_2)}. \tag{A.5}$$

Finally, also using assumption (A.1):

$$I_3\approx\frac{1}{E_3^3}\cdot\int_0^{E_3}\frac{d\tau}{\sqrt{(E_1^2+\tau)\cdot(E_2^2+\tau)}} \tag{A.6}$$

or with the substitution (A.3):

$$\begin{aligned}I_3&\approx\frac{2}{E_3^3}\cdot\int_{E_2}^{E_3}\frac{dt}{\sqrt{t^2+E_1^2-E_2^2}}\\&=\frac{2}{E_3^3}\cdot\left[\ln\left(t+\sqrt{t^2+E_1^2-E_2^2}\right)\right]_{E_2}^{E_3}\\&=\frac{2}{E_3^3}\cdot\ln\frac{E_3+\sqrt{E_3^2+E_1^2-E_2^2}}{E_1+E_2}.\end{aligned} \tag{A.7}$$

Appendix B: the acceleration process

In the solution of the equations of motion we assumed that the average energy E_0 remained constant i.e. that the cavity phase was either set at 0 or π so that no acceleration took place (see (2.10)).

To describe the acceleration process we now consider the case where

$\sin\varphi\neq 0,\pi.$

In linear approximation, the cavity field $\varepsilon_{\text{cavity}}$ varies as

$$\varepsilon_{\text{cavity}}=V(s)\sin\varphi+V(s)\cos\varphi\cdot k\cdot\frac{2\pi}{L}\cdot\sigma, \tag{B.1}$$

and if the cavity is point like at $s=s_0$:

$$V(s)=\hat{V}\cdot\delta(s-s_0) \tag{B.2}$$

the energy gain is

$$\begin{aligned}&E(s_0+0)-E(s_0-0)\\&\quad=e\hat{V}\sin\varphi+e\hat{V}\cos\varphi\cdot k\cdot\frac{2\pi}{L}\cdot\sigma(s_0-0).\end{aligned} \tag{B.3}$$

The average energy gain is thus

$$\Delta E_0 = e\hat{V}\cdot\sin\varphi, \tag{B.4}$$

so that we can put:

$$E_0(s_0+0) = E_0(s_0-0) + \Delta E_0; \tag{B.5}$$

$$p_0(s_0+0) = \sqrt{\frac{1}{c^2}\cdot E_0^2(s_0+0) - m_0 c^2}; \tag{B.6}$$

$$v_0(s_0+0) = \frac{c^2\cdot p_0(s_0+0)}{E_0(s_0+0)}. \tag{B.7}$$

Writing the variable

$$\sigma(s) = s - v_0\cdot t(s)$$

in the form

$$\sigma(s) = v_0(s)\cdot[t_0(s) - t(s)]$$

($t_0(s)$ is the time for the synchronous particle) and recalling that $t_0(s)$ and $t(s)$ are continuous functions:

$$t_0(s_0+0) = t_0(s_0-0);$$
$$t(s_0+0) = t(s_0-0)$$

we then obtain

$$\sigma(s_0+0) = \frac{v_0(s_0+0)}{v_0(s_0-0)}\cdot\sigma(s_0-0). \tag{B.8}$$

Furthermore, using (B.3, B.4 and B.5) we find that

$$\eta(s_0+0) = \frac{E(s_0+0) - E_0(s_0+0)}{E_0(s_0+0)}$$
$$= \frac{1}{E_0(s_0+0)}\cdot\left[E_0(s_0-0)\cdot\eta(s_0-0) + e\hat{V}\cdot\cos\varphi\cdot k\cdot\frac{2\pi}{L}\cdot\sigma(s_0-0)\right]. \tag{B.9}$$

For the variables x, x', z, z' of the transverse motions we have (see [2]):

$$x(s_0+0) = x(s_0-0); \tag{B.10a}$$

$$x'(s_0+0) = \frac{p_0(s_0-0)}{p_0(s_0+0)}\cdot x'(s_0-0); \tag{B.10b}$$

$$z(s_0+0) = z(s_0-0); \tag{B.10c}$$

$$z'(s_0+0) = \frac{p_0(s_0-0)}{p_0(s_0+0)}\cdot z'(s_0-0). \tag{B.10d}$$

Equations (B.8, B.9 and B.10) can now be collected together in matrix form to give

$$\mathbf{y}(s+0)\underline{M}_{\text{cavity}}(s_0+0, s_0-0)\cdot\mathbf{y}(s-0) \tag{B.11}$$

where

$$\underline{M}_{\text{cavity}}(s_0+0, s_0-0) = ((M_{ik}));$$
$$M_{11} = 1;$$
$$M_{22} = \frac{p_0(s_0-0)}{p_0(s_0+0)};$$
$$M_{33} = 1;$$
$$M_{44} = M_{22};$$
$$M_{55} = \frac{v_0(s_0-0)}{v_0(s_0+0)};$$
$$M_{65} = \frac{e\hat{V}}{E_0(s_0+0)}\cdot\cos\varphi\cdot k\cdot\frac{2\pi}{L};$$
$$M_{66} = \frac{E_0(s_0-0)}{E_0(s_0+0)};$$
$$M_{ik} = 0 \quad \text{otherwise} \tag{B.12}$$

and where $E_0(s_0+0)$, $p_0(s_0+0)$, $v_0(s_0+0)$ can be taken from (B.5, B.6 and B.7).

In particular we get for the generating vectors $\mathbf{y}_k(s)$ of the 6-dimensional ellipsoid:

$$\mathbf{y}_k(s_0+0) = \underline{M}_{\text{cavity}}(s_0+0, s_0-0)\cdot\mathbf{y}_k(s_0-0); \tag{B.13}$$

In the variables (x, x', z, z') the transfer matrix

$$\underline{M}_{\text{cavity}}(s_0+0, s_0-0)$$

is no longer symplectic and transverse damping occurs in x, x', z, z' space. For a symplectic treatment of the acceleration process within the framework of a non-linear theory see [6 and 7].

The transfer matrices for the magnetic lenses remain as in Chap. 5.

References

1. I. Borchardt, E. Karantzoulis, H. Mais, G. Ripken: Z. Phys. C, to be published
2. D.P. Barber, G. Ripken, F. Schmidt: DESY 87-36
3. G. Ripken: Untersuchungen zur Strahlführung und Stabilität der Teilchenbewegung in Beschleunigern und Storage-Ringen unter strenger Beruecksichtigung einer Kopplung der Betatron-schwingungen; DESY R1-70/04
4. O.D. Kellog: Foundations of potential theory. Berlin, Heidelberg, New York: Springer 1967
5. The effects of orbit curvature on interparticle forces have recently been discussed in: R. Talman: Phys. Rev, Lett. 56 (1986) 1429; A. Piwinski: CERN/LEP-TH/85/43; M. Bassetti: CERN/LEP-TH/86-13; M. Bassetti, D. Brandt: CERN/LEP-TH/86-04
6. G. Ripken: DESY 85-84
7. F. Schmidt: Phd. Thesis: DESY HERA 88-02