# FERMIONIC JACOBIAN AND GAUGE INVARIANCE IN THE CHIRAL SCHWINGER MODEL

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Received 1 March 1988, revised manuscript received 16 April 1988

The jacobian for a finite gauge transformation of the fermion fields in the chiral Schwinger model is calculated In contrast to the results published before this jacobian is suitable for the construction of a gauge invariant fermionic quantum theory

The chiral Schwinger model [1] (chiral $QED_2$) has become a popular tool frequently used for the demonstration of ideas concerning anomalies (See ref [2] and references therein ) The reason is twofold the model is exactly solvable and it is consistent in spite of the apparent anomaly [1] The consistency relies on a nonzero value of a regularization parameter $a$ which reflects the ambiguity in the treatment of chiral fermions In the path integral approach, where the anomaly originates in the gauge noninvariance of the fermionic measure [3], $a$ depends on the regularization of the jacobian belonging to a chiral gauge transformation of the fermions This is not unique, since there is no requirement for gauge invariance, contrary to the nonchiral case This has been put to question [4,5], but explicit regularization prescriptions have been given which are able to introduce such an arbitrary $a$ [6–9] [#1] Therefore the consistency of the model is established by now

This can be understood as a consequence of gauge invariance In fact, it has been shown that the procedure of quantizing the gauge field automatically leads to a gauge invariant quantum theory [11–13] This is achieved by a Wess–Zumino scalar field which can be viewed upon as the (surviving) gauge degree of freedom contained in the gauge field For gauge invariance the gauge variation of the Wess–Zumino action has to cancel the abovementioned jacobian of the fermionic measure It has been proven by general arguments that this procedure works [11–13] Also, gauge invariance of the chiral Schwinger model has been demonstrated at the level of a purely bosonic theory [14,15] Of course, in the chiral Schwinger model it should also be possible to show explicitly that the gauge variation of the Wess–Zumino action cancels the fermionic jacobian This has not been done up to now

Indeed, if one would try to do so by taking the Wess–Zumino action and the regularization of the fermionic measure from the literature, one would fail The reason is that some jacobians are incorrect [7,9,10] and that the correct ones [6,8] are not suitable for the design of a gauge invariant quantum theory Therefore an explicit construction of a gauge invariant version of the chiral Schwinger model at the fermionic level is still lacking The present letter is going to fill this gap

The classical fermionic action reads

$$S[\bar\psi,\psi,A]=\int \bar\psi\gamma^{\mu}[\mathrm{i}\partial_{\mu}+eA_{\mu}P_{\mathrm{L}}]\psi\,\mathrm{d}^{2}x\,, \qquad (1)$$

where the conventions of ref [2] are used Under a gauge transformation with group element $h=\exp(-\mathrm{i}\alpha)$, the fermions transform according to

$$\psi^{h}=\exp(-\mathrm{i}\alpha P_{\mathrm{L}})\psi,\quad \bar\psi^{h}=\bar\psi\exp(\mathrm{i}\alpha P_{\mathrm{R}}) \qquad (2)$$

The jacobian with respect to this transformation is not unity [3], but

[1] Supported by Bundesministerium für Forschung und Technologie, 05 4HH 92P(3), Bonn, Fed Rep Germany

[#1] Ref [10] also tries to introduce a free parameter $\xi$ However, the authors used a regularization operator which effectively is independent of $\xi$ Hence the jacobian cannot depend on $\xi$, either



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$$d\psi^h\, d\bar{\psi}^h = d\psi\, d\bar{\psi}\, J[A, h] , \quad (3)$$

because the $\gamma_5$ part does not cancel. Let the effective action $\tilde{W}[A]$ be the result of integrating out the fermion fields

$$\exp(i\tilde{W}[A]) = \int d\psi\, d\bar{\psi}\, \exp(iS[\bar{\psi}, \psi, A]) \quad (4)$$

Then the Wess–Zumino action is defined by the difference

$$\alpha_1[A, g^{-1}] = \tilde{W}[A^{g^{-1}}] - \tilde{W}[A] \quad (5)$$

with the transformed gauge field

$$A_\mu^{g^{-1}} = A_\mu + e^{-1}\partial_\mu\theta, \quad g = \exp(-i\theta) \quad (6)$$

Now the claim is that the quantum theory defined by the generating functional

$$Z = \int dA\, dg\, \delta(f(A, g))\Delta_f[A, g]\, d\psi\, d\bar{\psi}$$

$$\times \exp\left(i\left\{\int -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}d^2x + W[A, g]\right\}\right), \quad (7)$$

$$W[A, g] \equiv \tilde{W}[A] + \alpha_1[A, g^{-1}] , \quad (8)$$

is gauge invariant [11–13]. This is fulfilled if $W[A, g]$ is gauge invariant. Therefore we calculate

$$\exp(iW[A^h, gh]) = \int d\psi\, d\bar{\psi}$$

$$\times \exp(i\{S[\bar{\psi}^{h^{-1}}, \psi^{h^{-1}}, A] + \alpha_1[A^h, (gh)^{-1}]\})$$

$$= \exp(iW[A, g])(J[A, h^{-1}])^{-1}$$

$$\times \exp(-i\alpha_1[A, h]) , \quad (9)$$

where we used gauge invariance of the classical action and the one-cocycle condition for the Wess–Zumino action

$$\alpha_1[A^h, (gh)^{-1}] = \alpha_1[A, g^{-1}] - \alpha_1[A, h] \quad (10)$$

Hence the theory is gauge invariant, if

$$J[A, h^{-1}] = \exp(-i\alpha_1[A, h]) \quad (11)$$

In the chiral Schwinger model, however, none of the existing explicit regularization prescriptions for the calculation of $J$ [6–9] fulfills this condition, not even for infinitesimal transformations. This makes a new calculation of the jacobian necessary.

The jacobian of an infinitesimal transformation $\psi \to \psi^h$, $h = \exp(-i\delta\alpha)$, can be calculated by the method of Fujikawa [3], appropriately adjusted to the present case. After a Wick rotation to euclidean space, the sum $\sum_n \varphi_n^+ \gamma_5 \varphi_n$ has to be evaluated, where the $\varphi_n$ form a complete set of eigenfunctions. Since the sum is ill-defined, it has to be regularized. This is done by suppressing the large eigenvalues

$$\sum_n \varphi_n^+ \gamma_5 \varphi_n \to \lim_{M\to\infty} \sum_n \varphi_n^+ \gamma_5 \exp[-(\lambda_n/M)^2]\varphi_n \quad (12)$$

Which operator do these eigenvalues correspond to? This is the point where the arbitrariness enters into the regularization procedure. In the vector case, the requirement for vector current conservation fixes the operator to be the covariant derivative which appears in the classical action. Here such a requirement cannot be satisfied, gauge invariance would demand that the chiral current is conserved, which is impossible. Hence we are free to choose (in Minkowski space) [6–9]

$$D\!\!\!/ = \gamma^\mu[\partial_\mu - ie(rA_\mu^+ + sA_\mu^-)] ,$$

$$A_\mu^\pm = \tfrac{1}{2}(g_{\mu\nu} \pm \epsilon_{\mu\nu})A^\nu \quad (13)$$

As was pointed out in ref. [9], the corresponding covariant derivative in euclidean space

$$D\!\!\!/_E = \gamma^\mu[\partial_\mu - ie((r+s)g_{\mu\nu} + i(r-s)\epsilon_{\mu\nu})A^\nu] \quad (14)$$

is not hermitean. This can be cured by an analytical continuation of $r-s$ to imaginary values [9]. Then the calculation of the jacobian is standard [3] and leads to the result (in Minkowski space)

$$J[A, \exp(-i\delta\alpha)]$$

$$= 1 + (ie/2\pi)\int d^2x\, \delta\alpha\, \epsilon^{\mu\nu}\partial_\mu(rA_\nu^+ + sA_\nu^-) \quad (15)$$

This agrees with the result of refs. [6,8], though there the authors did not care about hermiticity. Unfortunately, ref. [9], where this has been taken into account, contains a sign error in the infinitesimal jacobian.

What we really need is the jacobian for a finite transformation. This can be derived from eq. (15) by an iteration procedure [16]. Here two dimensions offer another speciality: there are two possibilities to iterate the gauge field. In the step $\alpha \to \alpha + \delta\alpha$ the regulator has to contain the actual gauge field, which differs from the original one by a transformation with $\exp(i\alpha)$. Since in the fermionic action only $A^-_\mu$ occurs, one might think that only $A^-_\mu$ has to be iterated. This has been done in ref. [6] for the chiral Schwinger model and in ref. [8] for its nonabelian extension. Certainly, because $\not{A}P_L=\not{A}^-$ is true only in two dimensions, the iteration of $A^-$ alone is a two-dimensional speciality. In any other dimension the only chance is to iterate the complete gauge field. Therefore this seems to be reasonable in two dimensions, too. More than that, only the latter procedure is able to satisfy eq. (11), as will be shown below.

In order to confront the finite jacobians with each other, both procedures will be presented. If only $A^-$ is iterated, the jacobian (denoted by $J_1[A, \exp(-i\alpha)]$) is the solution of the differential equation which is implicitly given by

$$J_1[A, \exp\{-i(\alpha+\delta\alpha)\}]=J_1[A, \exp(-i\alpha)] \times J[A+e^{-1}\partial^-\alpha, \exp(-i\delta\alpha)]. \tag{16}$$

$J_1$ is easily calculated to be

$$\ln J_1[A, \exp(-i\alpha)]=\frac{1}{4\pi}\int[-\tfrac{1}{2}s_1\alpha\Box\alpha +2e\alpha\partial^\mu(r_1A^+_\mu-s_1A^-_\mu)]\mathrm{d}^2x. \tag{17}$$

In the other case, where the complete gauge field is iterated, the differential equation reads

$$J_2[A, \exp\{-i(\alpha+\delta\alpha)\}]=J_2[A, \exp(-i\alpha)] \times J[A+e^{-1}\partial\alpha, \exp(-i\delta\alpha)], \tag{18}$$

which has the solution

$$\ln J_2[A, \exp(-i\alpha)]=\frac{1}{4\pi}\int[\tfrac{1}{2}(r_2-s_2)\alpha\Box\alpha +2e\alpha\partial^\mu(r_2A^+_\mu-s_2A^-_\mu)]\mathrm{d}^2x. \tag{19}$$

The effective action $\tilde{W}[A]$ can be calculated as the jacobian for the transformation with $\alpha=-(\sigma-\rho)$ [5], where $\sigma$ and $\rho$ determine the gauge field

$$A_\mu=e^{-1}(\partial_\mu\sigma+\epsilon_{\mu\nu}\partial^\nu\rho). \tag{20}$$

The parameters $r$ and $s$ are adjusted in such a way that $\tilde{W}[A]$ is the same for both prescriptions, namely the effective action given in ref. [1]

$$\begin{aligned}\tilde{W}[A]&=(1/i)\ln J_1[A, \exp\{i(\sigma-\rho)\}]\\ &=(1/i)\ln J_2[A, \exp\{i(\sigma-\rho)\}]\\ &=(e^2/8\pi)\int[aA_\mu A^\mu-A_\mu(g^{\mu\alpha}+\epsilon^{\mu\alpha})\\ &\quad\times(\partial_\alpha\partial_\beta/\Box)(g^{\beta\nu}-\epsilon^{\beta\nu})A_\nu]\mathrm{d}^2x,\end{aligned} \tag{21}$$

where now $J_1$ and $J_2$ are given by

$r_1=\tfrac{1}{2}a,\quad s_1=1\quad\Rightarrow$

$$\ln J_1[A, \exp(-i\alpha)]=\frac{1}{4\pi}\int[-\tfrac{1}{2}\alpha\Box\alpha +2e\alpha\partial^\mu(\tfrac{1}{2}aA^+_\mu-A^-_\mu)]\mathrm{d}^2x, \tag{22}$$

$r_2=\tfrac{1}{2}a,\quad s_2=1-\tfrac{1}{2}a\quad\Rightarrow$

$$\ln J_2[A, \exp(-i\alpha)]=\frac{1}{4\pi}\int\{\tfrac{1}{2}(a-1)\alpha\Box\alpha +2e\alpha\partial^\mu[\tfrac{1}{2}aA^+_\mu-(1-\tfrac{1}{2}a)A^-_\mu]\}\mathrm{d}^2x. \tag{23}$$

$J_1$ corresponds to the iteration procedure of refs. [6,8]. As far as the bosonized version of the chiral Schwinger model is concerned, it is sufficient to have only one free parameter to reflect the regularization ambiguity, because the other one can be absorbed into the gauge coupling constant $e$. In the ordinary Schwinger model [16] as well as in the chiral Schwinger model [5] with $a=0$ it turned out that the effective action is one half of the exponent of the infinitesimal jacobian where $\delta\alpha$ is just replaced by the appropriate finite transformation. This result has been adopted for the chiral Schwinger model with $a\neq0$ as well [7,9]. It is, however, related to the fact that the gauge field which is going to be "rotated away" coincides with the regulator field. This is not true for $a\neq0$, such that the finite jacobians presented in refs. [7,9] are incorrect. This can easily be seen from the coefficient of the mass term. It is given by the coefficient of the $A^+$ term in the infinitesimal jacobian, which cannot be changed by the iteration, because $\sigma-\rho$ only depends on $A^-$. Hence there is no modification of the mass term coming from the iteration procedure, especially no factor $\tfrac{1}{2}$.

Starting from the effective action (21) it is straightforward to calculate the Wess–Zumino action

$$\alpha_1[A, g^{-1}]=\frac{1}{4\pi}\int\{\tfrac{1}{2}(1-a)\theta\Box\theta -2e\theta\partial^\mu[\tfrac{1}{2}aA^+_\mu-(1-\tfrac{1}{2}a)A^-_\mu]\}\mathrm{d}^2x, \tag{24}$$

where $g=\exp(-i\theta)$ A comparison of eq (24) with eqs (22) and (23) shows that $J_2$ satisfies eq (11) and $J_1$ does not Hence the method to iterate $A^-$ only is not appropriate for the construction of a gauge invariant quantum theory containing fermion fields For this purpose there is only one possibility to regularize the fermionic jacobian use $\frac{1}{2}aA_\mu^+ + (1-\frac{1}{2}a)A_\mu^-$ as a regulator field and iterate the complete gauge field to build up a finite transformation out of infinitesimal ones This results in eq (23)

I would like to thank G Kramer for valuable discussions

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