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GENERAL STATISTICS

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We discuss statistical systems with infinitely many degrees of freedom. In distinction to conventional statistical theories we do not make any a priori identification between operators of the system and given observables. In particular, these systems are not formulated in a preexisting spacetime. Space and time should arise as dynamical structures of the system. It is shown that all such systems are equivalent and the theory is therefore unique. The action loses its fundamental meaning. The choice of an action corresponds to a "coordinate choice" on how the system is parametrized. The symmetry of the action consists of infinite-dimensional volume-conserving general coordinate transformations. The expectation values of operators are finite by definition. The emphasis of the work to be done lies in the identification of structures between operators which are reflected by our observation of space, time, particles etc. String theories are contained in these structures.

1. Introduction

We are used to describing physics as motion or evolution of matter in space and time. Although general relativity and quantum mechanics have weakened the concept of absolute properties of space and time we still formulate our theories in a preexisting spacetime manifold. This is unsatisfactory in two respects. First, spacetime manifests itself only through the motion of matter. (Here matter includes gravitational fields like the graviton.) Spacetime without matter is unobservable and seems not to make sense. Spacetime should be understood as a property of matter rather than a preexisting category. Second, for a preexisting spacetime manifold we have to specify its dimension. This introduces an unexplained number. Higher dimensional unification [1] and string theories [2] have prepared the view that it may be possible to formulate the same theory in different dimensions. We adopt the attitude that a fundamental unified theory should explain the dimension as a dynamical property. The same holds for the topology of the spacetime manifold. In particular, the dynamics should allow the transition between different topologies so that topology loses its absolute role.

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String theories are an important step in this direction. It seems, however, that several distinct string theories are consistent. As in quantum field theory the basic concepts which characterize a string theory are the degrees of freedom, the action and the associated symmetries. This points to another problem of fundamental unification: which action, which symmetry to choose? In our opinion, in a fundamental unified theory the action should lose its basic importance. It should be possible to describe the same physics by different actions. The choice of an action should correspond to a parametrization of the theory, like the choice of coordinates for a manifold. On the other hand, no free choice should be left for the selection of symmetries. They should appear as a necessary consequence of the formulation of the theory, with a uniquely determined group structure.

If space, time, action and symmetries lose their basic role, what remains? What concepts are left to formulate the fundamental unified theory? If there is no a priori meaning of space and time, observables like energy, mass and spin of a particle etc. can also have no a priori meaning. More generally, we do not want to assume a priori the existence of observables with given properties. The theory should tell us which structures exist so that we can attempt a posteriori to identify them with observables. We will use, as the only assumptions, that the fundamental unified theory is statistical in nature and that the number of degrees of freedom describing our world is infinite.

The aim of this paper is to describe the general setting of such a theory. In sect. 2 we introduce a probabilistic system for N continuous degrees of freedom s with statistical weight $p(s)$. We emphasize that general statistics deals not with the expectation value of particular operators $v(s)$ for a given $p(s)$, but rather with the possible general abstract structures between operators. In appendix A we establish that these structures are independent of the topology of the N-dimensional manifold parametrized by s. In sect. 3 we consider variable transformations $s \to s'$. We show (in appendix B) that operator structures do not depend on the choice of the statistical weight $p(s)$. All systems with given N are equivalent. We find the symmetry group leaving p invariant, namely the N-dimensional volume-element-conserving general coordinate transformations sgen_N. This symmetry group does not depend on the choice of p. In sect. 4 we introduce the action $S = -\ln p$ and briefly discuss the question of transformations with anomalies. In sect. 5 we define general statistics for infinitely many degrees of freedom by the use of sequences of finite systems with $N \to \infty$. The structures among operators do not depend on the choice of the sequence and general statistics is therefore unique. Among the many possible orderings of infinitely many degrees of freedom we choose one corresponding to a free p-dimensional field theory. The functional measure is shown to include a sum over different topologies of the p-dimensional manifold. For $p = 2$ this ordering corresponds to a bosonic string theory. As a byproduct of the uniqueness of general statistics all bosonic field theories in arbitrary dimensions, which are regularized by sequences with $N \to \infty$, can be mapped into each other.

2. General statistics for a finite number of degrees of freedom

Let us begin with one degree of freedom parametrized by a real variable $s \in \mathbb{R}$. A probability density $p(s)$ is a positive nonvanishing function defined for all s. We also require p to be continuous and infinitely often differentiable everywhere with finite integral

$$Z = \int_{-\infty}^{\infty} \mathrm{d}s\, p(s). \tag{2.1}$$

A function $v(s)$ specifies an operator if the mean value,

$$\langle v \rangle_p = Z^{-1} \int \mathrm{d}s\, p(s) v(s), \tag{2.2}$$

is defined and finite. (For the moment we take continuous functions $v(s)$ and we also request $\langle |v| \rangle_p$ to be finite. We generalize this to a wider class of operators at the end of this section.) We postulate that all one can measure or observe can be described in terms of expectation values $\langle v \rangle_p$ of operators.

The probability density contains all the information on the expectation values of any arbitrary given set of operators v_i. Inversely, it can be reconstructed from the expectation values of (infinitely many) operators. One could think that any definition of the system requires a specification of the probability density $p(s)$. This is true, however, only if a given operator $v(s)$ corresponds to some preidentified observable (like magnetization, for example, in usual statistics). As long as we do not know if an observable quantity corresponds to the mean value of $v(s)$ or some other operator $\tilde{v}(s)$, our system contains less information. In particular we are free to reparametrize the system

$$\begin{aligned} &s \to t(s), \\ &p(s) \to p'(t) = (\mathrm{d}s/\mathrm{d}t)\, p(s(t)), \\ &v(s) \to v'(t) = v(s(t)), \end{aligned} \tag{2.3}$$

with $t(s)$ some monotonically increasing infinitely often differentiable one-to-one map $\mathbb{R} \to \mathbb{R}$. The new probability density p' can be used equally well as p for a search of structures among operators. We will show in sect. 3 that *all* probability densities are equivalent. We are therefore free to choose any $p(s)$ we like. This implies that no information is contained in the particular form of the probability density.

We are interested in general *structures* among operators. They are independent of a particular choice of p. For a very simple example we consider operators with the property

$$\lim_{s \to \pm\infty} v_i(s) = 0. \tag{2.4}$$

For such operators the products $v_i v_j \ldots v_k$ are again operators and we can study correlations. We define positive operators with finite support $\hat{v}_i$ by taking $\hat{v}_i(s)$ to vanish outside a given (open) finite interval I_i and $\hat{v}(s) > 0$ within I_i. They have the property

$$\langle \hat{v}_i \hat{v}_j \rangle_p = 0 \quad \Leftrightarrow \quad I_i \cap I_j = \varnothing . \tag{2.5}$$

This can be used to implement the topological and ordering structure of $\mathbb{R}$ among these operators. Similarly, we may represent the topological structure of the circle S^1 by choosing, among the operators with

$$\lim_{s \to +\infty} v_i(s) = \lim_{s \to -\infty} v_i(s) = c , \tag{2.6}$$

those which are positive in a not necessarily finite interval and which vanish elsewhere. If a certain structure among operators is identified one can make predictions of the type: "if the expectation values of a certain set of operators $v_1, v_2, \ldots$ are known, then the expectation value of some other operator v_0 is predicted in a certain range". In our example $\langle \hat{v}_1 \hat{v}_2 \rangle_p = 0$ and $\langle \hat{v}_1 \hat{v}_3 \rangle_p = 0$ imply $\langle \hat{v}_1 \hat{v}_2 \hat{v}_3 \rangle_p = 0$. Such predictions reflect the general properties of a structure.

More generally, we may pick an arbitrary p and search for abstract structures among the operators. We hope that a generalization to infinitely many degrees of freedom leads to complex structures like space, time, fields, gauge interactions and spinors which will allow an identification of operators with observables. (For the time being we cannot be sure if such an identification will be unique or if several or many candidate structures emerge.) Mathematically, a given p defines a linear map

$$V_p^0 \xrightarrow{p} \mathbb{R} , \qquad v(s) \to \langle v \rangle_p , \tag{2.7}$$

on the vector space of p-integrable functions, $V_p^0 = \{ v(s) \colon \mathbb{R} \to \mathbb{R}$, continuous and $\langle |v| \rangle_p$ defined$\}$. General statistics deals with the general structures of the map (2.7). They are, as we will see, independent of p. Several generalizations are straightforward:

(i) For a finite number of degrees of freedom N the probability density is a map $\mathbb{R}^N \to \mathbb{R}_+$. Similarly the space V_p^0 consists of operators $v(s)$: $\mathbb{R}^N \to \mathbb{R}$ for which $\langle v \rangle_p$ is defined in analogy to eq. (2.2).

(ii) If a sequence of operators v_n has a well-defined limit

$$\lim_{n \to \infty} \langle v_n \rangle_p = \langle v \rangle_p , \tag{2.8}$$

we formally define $v(s) = \lim_{n \to \infty} v_n(s)$ as an operator. The space of operators V_p is extended to include all such limiting operators. Therefore V_p contains functions with discontinuities or poles and distributions. Also the integral $\int_{c_0}^{\infty} \mathrm{d}s\, p(s) v(s)$ may not exist anymore for all finite c_0. If a function $v(s)$ is approached by different sequences v_n, and $\langle v \rangle_p$ depends on the choice of the sequence, the operator $v(s)$ needs a regularization. This means that the sequence v_n must be specified for its definition. In summary we only require that the function $v(s)$ is defined and is continuous in $\mathbb{R}$ except for a set of points with measure zero and that the integral $\langle v \rangle_p$ is finite. Generalization for distributions and for N degrees of freedom is understood.

(iii) We may weaken our assumptions on p and require it only to be a finite-times differentiable or piecewise continuous, etc. All definitions should be adapted so that p appears as a limiting case for a sequence of infinitely often differentiable p_n. This procedure should also be used to extend our discussion to probability densities which may have zeroes or poles.

(iv) Consider a sequence of probability densities p_n with

$$\lim_{n \to \infty} \int \mathrm{d}s p_n(s) = \lim_{n \to \infty} Z_n \to \infty , \tag{2.9}$$

where Z_n is finite for every finite n. The limit defines a system with Z infinite. Operators v_n are properly defined for every finite n. The operators of the system with infinite Z correspond to those sequences v_n which have a finite limit

$$\langle v \rangle_p = \lim_{n \to \infty} Z_n^{-1} \int \mathrm{d}s p_n(s) v_n(s) . \tag{2.10}$$

This limit will be understood if we use eq. (2.2) for infinite Z. In general a system with infinite Z needs a specification of the sequence p_n by which it obtains as a limiting case. This sequence defines the *regularization* of the system.

(v) We may use a different topology for the manifold parametrized by the variable s, for instance S^1 or a finite interval I instead of $\mathbb{R}$. We show in Appendix A that this can be mapped into an equivalent system with topology $\mathbb{R}$. This generalizes for N degrees of freedom. A system defined on an N-dimensional manifold with arbitrary topology is equivalent to a system defined on $\mathbb{R}^N$. Our original definition for N degrees of freedom $s \in \mathbb{R}^N$ can therefore be used without any loss of generality.

3. Transformations and symmetries

Consider a system with N continuous degrees of freedom, $(s^1, s^2, \dots, s^N) \in \mathbb{R}^N$, characterized by a strictly positive and infinitely often differentiable probability density $p(s^u)$. As before, the expectation values of operators $v(s^u)$ are defined in dependence on $p(s)$

$$\langle v \rangle_p = Z^{-1} \int \mathrm{d}^N s\, p(s^u) v(s^u), \tag{3.1}$$

$$Z = \int \mathrm{d}^N s\, p(s^u). \tag{3.2}$$

Let us now make a change of coordinates

$$s^u = f^u(t^v), \tag{3.3}$$

with f an infinitely often differentiable and invertible one-to-one map $\mathbb{R}^N \to \mathbb{R}^N$ with positive jacobian

$$\hat{f} = \mathrm{Det} \frac{\partial f^u}{\partial t^v} > 0. \tag{3.4}$$

The partition function Z remains invariant provided we also use a new probability density

$$Z = \int \mathrm{d}^N t\, p'(t),$$

$$p'(t^v) = \hat{f} p(f^u(t^v)) \equiv f(p). \tag{3.5}$$

The expectation values of transformed operators

$$v'(t^v) = v(f^u(t^v)) \equiv f(v), \tag{3.6}$$

evaluated in the new system, are the same as before

$$\langle v' \rangle_{p'} = Z^{-1} \int \mathrm{d}^N t\, p'(t^v) v'(t^v) = \langle v \rangle_p. \tag{3.7}$$

There is no difference between coordinates s and t and instead of t we can also use the coordinates s. We will stick from now on to a version where *variable transformations* act on p and v with coordinates s kept fixed

$$p(s) \to p'(s) = \hat{f} p(f(s)), \tag{3.8}$$

$$v(s) \to v'(s) = v(f(s)),$$

$$\langle v' \rangle_{p'} = \langle v \rangle_p. \tag{3.9}$$

The variable transformations form a group, the group of N-dimensional general coordinate transformations (diffeomorphisms) gen_N. An operator $v(s)$ transforms as a scalar field, $p(s)$ is a scalar density of weight -1 (c.f. eq. (3.8)) and $\partial_u v$ transforms as a covariant vector, $(\partial_u v)(s) \to (\partial_v v)(f(s))(\partial f^v/\partial s^u)$.

Under infinitesimal transformations, $f^u(s) = s^u - \xi^u(s)$, these quantities change (for v differentiable) as

$$v \to v + \delta v, \qquad \delta v = -\xi^u \partial_u v,$$
$$\partial_u v \to \partial_u v + \delta(\partial_u v), \qquad \delta(\partial_u v) = \partial_u \delta v = -\partial_u \xi^v \partial_v v - \xi^v \partial_v \partial_u v,$$
$$p \to p + \delta p, \qquad \delta p = -\xi^u \partial_u p - \partial_u \xi^u p = -\partial_u(\xi^u p). \tag{3.10}$$

One should note that not every infinitely often differentiable infinitesimal function $\xi(s) = \tilde{\xi}(s)\,\mathrm{d}t$ leads to an acceptable finite transformation $f(s)$: $\mathbb{R}^N \to \mathbb{R}^N$. Some functions $\tilde{\xi}(s)$ correspond to finite maps defined only in a finite interval within $\mathbb{R}^N$ and others map $\mathbb{R}^N$ into such a finite interval (c.f. appendix A). Such transformations are not invertible group elements of gen_N. The functions $\tilde{\xi}(s)$ generating gen_N are therefore constrained. For an arbitrary variable transformation $f(s)$ we can find a continuous sequence of variable transformations $\tilde{f}(t, s)$, $0 \leqslant t \leqslant 1$, with

$$\tilde{f}(0, s) = f(s), \qquad \tilde{f}(1, s) = s, \tag{3.11}$$

$$\frac{\partial \tilde{f}^u}{\partial t}(t, s) = \tilde{\xi}^u(\tilde{f}(t, s)). \tag{3.12}$$

The choice of the sequence is not unique and a finite transformation $f(s)$ can be generated by different $\tilde{\xi}(s)$. On the other hand every $\tilde{\xi}(s)$ defines a unique $\tilde{f}(t, s)$ as a solution of the initial-value problem for the differential equation (3.12), provided we restrict $\tilde{\xi}(s)$ such that, for a finite interval around $t = 1$, $\tilde{f}(t, s)$ remains finite for arbitrary initial values $s = \tilde{f}(1, s)$. This condition specifies the generators of gen_N.

A *symmetry* is a transformation leaving the probability density invariant, $p'(s) = p(s)$. Symmetries relate expectation values of operators for a given p

$$\langle v \rangle_p = \langle v' \rangle_p. \tag{3.13}$$

All operators that can be obtained from a given operator v by symmetry transformations must have the same expectation value. The symmetries contained in the variable transformations (3.8) form a subgroup sgen_N of gen_N, namely the group of all variable transformations leaving the volume element p invariant

$$\partial_u(\xi^u p) = 0. \tag{3.14}$$

In contrast, we call the variable transformations with $p'(s) \neq p(s)$ the *proper variable transformations*. After identification of the symmetries they belong to $\mathrm{gen}_N/\mathrm{sgen}_N$. Arbitrary probability densities $p(s)$ and $p'(s)$ with the same finite $Z = \int ds^N p = \int ds^N p'$ can be transformed into each other by a suitable proper variable transformation. (A proof of this statement is given in appendix B.) This establishes a one-to-one correspondence between the operators of both systems. For a finite ratio $Z(p)/Z(p') = \alpha$ we can trivially rescale p' by multiplication with α without changing the expectation values of operators. For finite N all systems with finite Z are therefore equivalent! The symmetry groups $\mathrm{sgen}_N(p)$ and $\mathrm{sgen}_N(p')$ are isomorphic (cf. appendix C). A choice of p can be understood as a parametrization or a "choice of coordinates" for the system. This equivalence can be extended to infinite Z. The system with infinite Z is defined by a sequence of p_n with finite Z_n. Since for two sequences p_n and p_n' there is a one-to-one correspondence between operators for every n, the two systems with $p = \lim_{n\to\infty} p_n$ and $p' = \lim_{n\to\infty} p_n'$ are equivalent. This holds for $Z(p')$ finite or infinite. In particular the systems with infinite Z are equivalent to systems with finite Z. A similar argument holds for sequences p_n leading to a limiting p which may not be defined, strictly positive, continuous or infinitely often differentiable in the whole $\mathbb{R}^N$. Without loss of generality we will therefore use the requirements on p stated at the beginning of this section. In conclusion the systems with arbitrary p are all equivalent. They are uniquely characterized by the number of degrees of freedom N!*.

The concept of variable transformations should be generalized to functions f which are only defined on $\mathbb{R}^N$ except for a set of points (hypersurfaces) with measure zero. Also, the image of f is allowed to cover $\mathbb{R}^N$ except for a zero-measure set. We still require $\tilde{f} > 0$ and f infinitely often differentiable within the whole range of its definition. According to eq. (3.8) the probability density $p'(s)$ and the operators $v'(s)$ are only defined in the range of definition of f. This does not affect the definition of expectation values and the equality (3.9). The symmetries sgen_N are extended correspondingly. If $p'(s) = p(s)$ for all s where f is defined, we can extend the definition of $p'(s)$ to the whole $\mathbb{R}^N$ in an obvious manner. We actually need this extended version of sgen_N if we first formulate our system on S^N (or some other topology different from $\mathbb{R}^N$) and then study the symmetries of this system in the equivalent version formulated on $\mathbb{R}^N$. The allowed generators $\tilde{\xi}(s)$ of sgen_N should be extended correspondingly. It is possible to represent a general-

* It also seems possible to formulate a system on a discrete basis with $s = \pm 1$. A continuous degree of freedom is represented by infinitely many discrete degrees of freedom. On the other hand, infinitely many discrete degrees of freedom can be ordered in different ways so as to represent N continuous degrees of freedom with arbitrary N. By this generalization the system even becomes independent of N. One may choose a constant probability density $p = 1$ so that all values of the discrete variables are equally weighted. General statistics concerns the structures appearing in the possible orderings of information with infinitely many bits. The structure of continuous degrees of freedom implements notions like neighborhood, continuity etc.

ized $\tilde{\xi}$ as the limit of a sequence $\tilde{\xi} = \lim_{n\to\infty} \tilde{\xi}_n$ with $\tilde{\xi}_n$ defined as before for every finite n.

It is instructive to study the symmetry group sgen_N for a few examples. For $N = 1$ one obtains from eq. (3.14)

$$\frac{\mathrm{d}}{\mathrm{d}s}(\xi p) = 0, \qquad \tilde{\xi}(s) = -\frac{a}{p(s)}. \tag{3.15}$$

The symmetry transformation

$$f(s) = z^{-1}(a + z(s)),$$
$$z(s) = \int_{s_0}^{s} \mathrm{d}s\, p(s), \tag{3.16}$$

is easily computed for a few examples

$$\text{(i)} \quad p = 1, \qquad f = s + a,$$

$$\text{(ii)} \quad p = s^2 + 1, \qquad f = \begin{cases} s + a & \text{for } |s|, |a| \ll 1, \\ s, & \text{for } |s| \gg 1, |a|, \end{cases}$$

$$\text{(iii)} \quad p = (s^2 + 1)^{-1/2}, \qquad f = \begin{cases} s + a, & \text{for } |s|, |a| \ll 1, \\ s \exp a, & \text{for } s \gg 1, \mathrm{e}^{-a}, \\ s \exp(-a), & \text{for } s \ll -1, -\mathrm{e}^{a}, \end{cases}$$

$$\text{(iv)} \quad p_n = \frac{1}{\pi} \frac{n^2}{n^2 + s^2}, \qquad f = n \operatorname{tang}\left[\frac{a\pi}{n} + \operatorname{arctang}\frac{s}{n}\right]. \tag{3.17}$$

For the first example, sgen_1 is the standard translation group. For the second, f acts as a modified local translation for which the s-dependent translation parameter vanishes at large $|s|$. In example three the symmetry acts similarly for small $|s|$, but for large $|s|$ it becomes a scaling of s by a constant factor. In all three cases Z is infinite and sgen_1 has the structure of the abelian noncompact translation group T_1. The situation is somewhat different for the last example with finite Z. The function $f(s)$ is not defined for all $s \in \mathbb{R}$ (except for $a = 0$ where $f(s) = s$). It diverges for

$$s \to s_c = n \operatorname{tang}\left\{\tfrac{1}{2}\pi(2m + 1 - 2a/n)\right\}. \tag{3.18}$$

The open interval $(-\infty, s_c)$ is mapped on the interval $(n \operatorname{tang}\{\frac{1}{2}\pi(1 + 2a/n)\}, \infty)$ whereas the image of (s_c, ∞) is $(-\infty, n \operatorname{tang}\{\frac{1}{2}\pi(1 + 2a/n)\})$. This is an example of the generalized symmetry transformations defined on $\mathbb{R}$ except a point which we have discussed before. We also note that f is periodic in a with period $2n$. The symmetry sgen_1 corresponds now to the compact translation group on a circle. This is not surprising since $p_n(s)$ can be obtained from an equivalent system defined on a circle with radius n/π ($|s| \leqslant n$) and $p = \frac{1}{2}$. (The equivalence map corresponding to eq. (A.1) is $f(s) = (2n/\pi)\,\mathrm{arctang}(s/n)$.) The p_n in example (iv) form a sequence which approaches $p = 1/\pi$ (example (i)) in the limit $n \to \infty$ ($Z_n = n$). Correspondingly, the symmetry transformations approach constant translations for $|s|,|a| \ll n$. For $n \to \infty$ the period of a diverges and the translation group becomes noncompact. This illustrates the equivalence of systems defined on S^1 or $\mathbb{R}$, with Z finite or infinite.

For $N \geqslant 2$ the symmetry group sgen_n is infinite dimensional. We may illustrate this by considering, for $N = 2$ and Z finite, a particular infinite-dimensional subgroup of sgen_2. We take

$$p = p(r^2), \qquad r^2 = (s^1)^2 + (s^2)^2, \qquad Z = \int \mathrm{d}s^1\, \mathrm{d}s^2\, p = 1. \tag{3.19}$$

This probability density is invariant under r-dependent rotations

$$\begin{aligned} f^1 &= \cos\varphi(r^2)s^1 + \sin\varphi(r^2)s^2, \\ f^2 &= \cos\varphi(r^2)s^2 - \sin\varphi(r^2)s^1. \end{aligned} \tag{3.20}$$

We can take for $\varphi(r^2)$ any arbitrary infinitely often differentiable function of r^2. (The value $\varphi(0)$ and the limit $\varphi(r^2 \to \infty)$ should be finite.) One always has the jacobian $\hat{f} = 1$. For constant φ one recovers the rotation group SO(2). The infinite-dimensional symmetry of r-dependent rotations is easily generalized for arbitrary N. We denote this subgroup of sgen_N by $\widetilde{\mathrm{so}}_N$.

The N-dimensional translations T_N are easily represented if p is a product of functions each depending only on one coordinate. We may therefore choose a standard probability density where both T_N and $\widetilde{\mathrm{so}}_N$ are realized in a simple way

$$p_\lambda = \exp(-\lambda r^2) = \prod_{u=1}^{N} \exp\big(-\lambda (x^u)^2\big). \tag{3.21}$$

As for any finite Z, the translations are compact and defined on $\mathbb{R}^N$ except for hypersurfaces of measure zero

$$\begin{aligned} f^u(s) &= w_\lambda^{-1}\big(a^u + w_\lambda(s^u)\big), \\ w_\lambda(s) &= \int_0^s \mathrm{d}s \exp(-\lambda s^2). \end{aligned} \tag{3.22}$$

(Here we take w_λ^{-1} periodic with period $(\pi/\lambda)^{1/2}$.) Additional symmetries in sgen_N can be generated by a combination of translations with r-dependent rotations. In the limit $\lambda \to 0$ one approaches $p_\lambda \to 1$ with Z infinite. In this limit sgen_N consists of all functions $f(s)$ with unit jacobian $\hat{f} = 1$. In particular, it contains the special linear transformations

$$f = As, \qquad \det A = 1. \tag{3.23}$$

The group $\mathrm{SL}(N, \mathbb{R})$ contains the Lorentz-type subgroup $\mathrm{SO}(1, N-1)$ as well as $\mathrm{SO}(N)$. Since the orbits of $\mathrm{SO}(1, N-1)$ in $\mathbb{R}^N$ are not compact, this group is realized in a different (compact) way for finite Z ($\lambda > 0$).

To conclude this section we mention that there are symmetries of p beyond sgen_N. The reflexion R^i

$$\begin{aligned} p(s^1, \ldots, s^i, \ldots, s^N) &\to p(s^1, \ldots, -s^i, \ldots, s^N), \\ v(s^1, \ldots, s^i, \ldots, s^N) &\to v(s^1, \ldots, -s^i, \ldots, s^N), \end{aligned} \tag{3.24}$$

becomes a symmetry for all p which are symmetric in s^i. Since nonsymmetric p are equivalent to symmetric p a generalized reflexion symmetry is present for arbitrary p. We may also define symmetries where the transformation properties of operators $v(s)$ depend on p. They are discussed in appendix C. We can use them to enlarge the symmetry from sgen_N to gen_N.

4. Action

Let us now choose a description with a given probability density p. Since p is positive and finite in $\mathbb{R}^N$ we introduce the *action* S

$$\exp(-S(s)) = p(s). \tag{4.1}$$

As we have seen in sect. 3 the choice of S is arbitrary. We will choose S such that it reflects most easily the symmetries we are interested in. Let us concentrate on those symmetry transformations which can be defined by their action on the variables s without explicit reference to the probability density p. These *symmetries of the action* are the transformations $\mathrm{sgen}_N(p)$ plus discrete reflexions of the type (3.24). We repeat that the symmetry group sgen_N does not depend on the choice of S.

Let us distinguish two types of symmetries of the action. *Scalar symmetries* have unit jacobian $\hat{f} = 1$. They form the subgroup $\mathrm{sgen}_N^0(S)$. Under these transformations S transforms as a scalar and therefore in the same way as any operator v

$$\begin{aligned} S'(s) &= S(f(s)), \\ \delta_0 S &= -\xi^u \partial_u S. \end{aligned} \tag{4.2}$$

If S is expressed in terms of some operators $v(s)$ we can immediately conclude the sgen_N^0 transformation properties of S from the transformation properties of v (in particular for $v = s^u$). The *nonscalar symmetries* have $\hat{f} \neq 1$. The transformation law for S has an additional part related to the jacobian

$$S'(s) = S(f(s)) - \ln \hat{f}(s), \qquad \delta S = \delta_0 S + \delta_{\mathrm{A}} S, \qquad \delta_{\mathrm{A}} S = \partial_u \xi^u \tag{4.3}$$

The splitting into scalar and nonscalar symmetries depends on the choice of S since sgen_N^0 corresponds to the transformations with unit jacobian leaving a given S invariant. Since all S are equivalent the distinction between scalar and nonscalar symmetries has no consequences for the structures of operators; both are genuine symmetries of the action. For practical purposes, however, it is often convenient to choose S so that the symmetries of interest are represented as scalar symmetries. A transformation $f(s)$ with $\delta_0 S = 0$ ($S(f(s) = S(s))$ may be called a *classical symmetry* for historical reasons. A classical symmetry is a true symmetry ("quantum symmetry") only if it is a scalar symmetry. If $\delta_{\mathrm{A}} S \neq 0$ the classical symmetry has an *anomaly* [3]. The anomaly $\delta_{\mathrm{A}} S$ depends only on f (4.3) and the expectation value $\langle f(v) \rangle_p$ differs from $\langle v \rangle_p$ in a well-defined way*. (This leads to anomalous Ward identities in quantum field theory.)

If we choose the standard action corresponding to eq. (3.21)

$$S = \lambda \sum_u (s^u)^2 = \lambda r^2, \qquad \lambda > 0, \tag{4.4}$$

the subgroup $\widetilde{\mathrm{so}}_N$ of r^2 dependent rotations is a scalar symmetry. The translations are represented as nonscalar symmetries with

$$\delta_{\mathrm{A}}^u S = -2a^u \lambda x^u \exp\left(\lambda (x^u)^2\right) \tag{4.5}$$

The classical symmetries of arbitrary s^u dependent rotations are in general anomalous. This generalizes by trivial rescaling to

$$S = \sum_u \lambda_u (s^u)^2. \tag{4.6}$$

5. Infinitely many degrees of freedom

So far we have treated probabilistic systems with a finite number of degrees of freedom N. The general properties of these systems are related to the properties of multidimensional integrals. For a description of the real world one expects an enormous number of degrees of freedom to be relevant. We are therefore interested

* We note that the anomaly could be absorbed by using the generalized S-dependent transformation law for operators (C.7). In this formulation anomalous classical symmetries become true symmetries, but operators transform as densities instead as scalars. This is inconvenient for a study of correlations since the symmetry transformation does not commute with the operation of forming products of operators.

in the limit $N \to \infty$. Consider a sequence of finite systems with increasing N. Denote the probability densities of these systems by $p^{(N)}$. For every finite system the notion of an operator $v^{(N)}$ is well defined by the requirement that $\langle v^{(N)} \rangle_{p^{(N)}}$ should exist. Operators of the infinite system obtain if the limit $\lim_{N\to\infty} \langle v^{(N)} \rangle_{p^{(N)}}$ exists for a suitable sequence of finite-system operators $v^{(N)}$. For the infinite system an operator v and its expectation value $\langle v \rangle_p$ is always defined by such a sequence

$$\langle v \rangle_p = \lim_{N\to\infty} \langle v^{(N)} \rangle_{p^{(N)}} . \tag{5.1}$$

Consider, first, sequences where $Z^{(N)}$ is finite for every $p^{(N)}$ as long as N is finite. (For the infinite system the limit $Z = \lim_{N\to\infty} Z^{(N)}$ does not necessarily exist.) All such infinite systems are equivalent. Indeed, if we have two sequences for probability densities $p^{(N)}$ and $\tilde{p}^{(N)}$ we know that for every given finite N there is a one-to-one mapping between the two systems. Every sequence of operators $v^{(N)}$ for the first system is mapped into a sequence $\tilde{v}^{(N)}$ corresponding to the second system and vice versa. There is therefore also a one-to-one correspondence between operators v and $\tilde{v}$ for the infinite systems. The restriction to finite $Z^{(N)}$ is not important. We have seen in sect. 3 that for a finite number of degrees of freedom N the systems with Z finite or infinite are also equivalent. We may therefore equally well use an equivalent sequence with infinite $Z^{(N)}$ for a definition of the infinite system. This completes the argument that general statistics for an infinite number of degrees of freedom is unique. We can parametrize the system as the limit $N \to \infty$ for any arbitrary sequence of systems with N degrees of freedom. For the finite system the variable s can parametrize any arbitrary N-dimensional manifold K^N with arbitrary topology. The choice of the probability density $p^{(N)}$ is also arbitrary with $Z^{(N)}$ finite or infinite. All such parametrizations are equivalent. General statistics has no free parameter ("coupling constant") on which physics could depend!

There are many possibilities to order an infinite number of degrees of freedom. As an example let us order the s^u as complex variables (s, t real)

$$z^i_{n_1, n_2, \dots, n_p} = s^i_{n_1, n_2, \dots, n_p} + i t^i_{n_1, n_2, \dots, n_p} ,$$
$$i = 1, \dots, \tilde{d}, \qquad n_j = -M, \dots, M ,$$
$$N = 2\tilde{d}(2M+1)^p . \tag{5.2}$$

We consider a gaussian action in the standard form (4.6)

$$S = \hat{S} + S_0 , \tag{5.3}$$

$$\hat{S} = \sum_i \sum_{\{n\}} \left(n_1^2 + n_2^2 + \cdots + n_p^2 \right) z^i_{n_1, n_2, \dots, n_p} \left(z^i_{n_1, n_2, \dots, n_p} \right)^* , \tag{5.4}$$

$$S_0 = a_M^2 \sum z^i_{0,0,\dots,0} \left(z^i_{0,0,\dots,0} \right)^* . \tag{5.5}$$

Then Z is finite for every finite M

$$Z=\int \mathscr{D}\mathscr{M}\exp(-S)=\hat{Z}Z_0, \tag{5.6}$$

$$\mathscr{D}\mathscr{M}=\prod_i\prod_{\{n\}} \mathrm{d}s^i_{n_1,n_2,\ldots,n_p}\,\mathrm{d}t^i_{n_1,n_2,\ldots,n_p}, \tag{5.7}$$

$$\hat{Z}=\left(\prod_{\{n\}}{}_0 \frac{\pi}{\left(n_1^2+n_2^2+\cdots+n_p^2\right)}\right)^{\tilde{d}}, \tag{5.8}$$

$$Z_0=\left(\frac{\pi}{a_M^2}\right)^{\tilde{d}}. \tag{5.9}$$

(The product $\prod_0$ in eq. (5.8) does not include $\{n\}=(0,0,\ldots,0)$.) Choosing

$$a_M^2=\pi^{(2M+1)^p}\left(\prod_{\{n\}}{}_0\left(n_1^2+n_2^2+\cdots+n_p^2\right)\right)^{-1}, \tag{5.10}$$

one obtains $Z=1$ independent of M.

We could interpret the $\{n\}$ as lattice sites and eq. (5.3) would describe an unusual p-dimensional lattice theory for $\tilde{d}$ complex fields defined on lattice sites and without kinetic terms. As an alternative interpretation we may choose a functional representation

$$X^i(\sigma^1,\sigma^2,\ldots,\sigma^p)=(2\pi)^{-p/2}\sum_{\{n\}} z^i_{n_1,n_2,\ldots,n_p}\exp\left(i\left(n_1\sigma^1+\cdots+n_p\sigma^p\right)\right). \tag{5.11}$$

The coordinates σ^α parametrize a p-dimensional torus $\mathrm{T}^p(-\pi\leqslant\sigma^\alpha\leqslant\pi)$ and X^i are complex functions on T^p with appropriate periodicity properties. The action (5.4) now reads

$$\hat{S}=\int \mathrm{d}^p\sigma\sum_{i=1}^{\tilde{d}}\sum_{\alpha=1}^{p}\frac{\partial X^i}{\partial\sigma^\alpha}\left(\frac{\partial X^i}{\partial\sigma^\alpha}\right)^*$$

$$=\int \mathrm{d}^p\sigma\,\partial_\alpha X^i(\partial^\alpha X_i)^*. \tag{5.12}$$

In the limit $M\to\infty(N\to\infty)$ every function on the torus can be obtained for suitable $z^i_{\{n\}}$. This includes functions not defined everywhere on T^p. The $X^i(\sigma)$ can have poles and discontinuities or not be differentiable on a set of hypersurfaces with dimension $\leqslant p-1$. Distributions are also generated. Of course, all such "irregular" functions necessarily need an infinite number of terms in the expansion (5.11). We define the functional measure by eq. (5.7)

$$Z = \int \mathscr{D}X(\sigma) \exp - \left\{ \hat{S}[X(\sigma)] + S_0 \right\}. \tag{5.13}$$

This specifies a free-field theory for $\tilde{d}$ complex massless scalar fields on a p-dimensional torus.

The regularization for all operators is defined by the limit of a suitable sequence with increasing M. For $M \to \infty$ the action becomes independent of $z^i_{0,0,\dots,0}(S_0 \to 0)$ and the factor Z_0 (5.9) diverges to the infinite volume of $\mathbb{R}^{2\tilde{d}}$. All volume-conserving coordinate transformations of the $z^i_{0,0,\dots,0}(\hat{f} = 1)$ become symmetries of the action. These include the $2\tilde{d}$-dimensional Poincaré transformations $P_{2\tilde{d}}$. As discussed in sect. 3 the $P_{2\tilde{d}}$ transformations are also present for finite M, but in a modified (compact) form. The action S_0 can be considered as a regulator for a probability density independent of $z^i_{0,0,\dots,0}$. We also can choose a formulation in terms of real periodic functions $X^i(\sigma)$. For this we can either consider the $\tilde{d}$ complex functions as $2\tilde{d}$ real functions or we impose the identification $X^i(\sigma) = (X^i(\sigma))^*$, $z^i_{-n_1,-n_2,\dots,-n_p} = (z^i_{n_1,n_2,\dots,n_p})^*$ in order to obtain $\tilde{d}$ real functions. For an even number of real functions both procedures are equivalent. These orderings of variables are a possible way to implement the structure of bosonic fields.

The concept of a p-dimensional manifold parametrized by σ^α (the torus T^p in our case) reflects a particular ordering of variables. A priori it has no dynamical role. From our previous discussion we know that systems with arbitrary numbers of components $\tilde{d}$ in arbitrary dimensions p are equivalent. To make the transition from one system to another we first rescale all variables such that the action has the form

$$S = \sum_i \sum_{\{n\}} \pi z^i_{\{n\}} \left(z^i_{\{n\}} \right)^*. \tag{5.14}$$

(In this form the symmetry $\widetilde{\mathrm{so}}_\infty$ is most manifest.) We then reorder the variables in groups corresponding to different $\tilde{d}$ and p and scale them again to obtain eqs. (5.4) and (5.5) for the new ordering. In the same way we may order the variables so that they correspond to an expansion in spherical harmonics on the sphere S^p or some expansion on $\mathbb{R}^p$. By suitable transformations we can also obtain massive and interacting field theories. As a consequence of the uniqueness of general statistics all bosonic field theories which can be regularized by a sequence with increasing N are equivalent and can be transformed into each other.

There is another way to visualize the independence on the topology of the p-dimensional manifold. Let us concentrate on the infinite system ordered on the two-dimensional torus ($p = 2$) according to eq. (5.11). The functional measure contains functions which become independent of σ^2 for a given value σ^1_0. For these functions we can identify the circle $(\sigma^1 = \sigma_0^1, \sigma^2)$ with a point and cut the torus there. The topology becomes a sphere S^2. When "regular" functions on S^2, like spherical harmonics, are represented in this way on the torus they will, in general, not be continuous and infinitely often differentiable for $\sigma^1 = \sigma_0^1$. An infinite number of terms in the expansion (5.11) is needed. This generalizes to oriented, compact, closed connected two-dimensional manifolds K^γ with arbitrary genus γ. We parametrize K^γ by coordinates ϑ^α such that the whole manifold except some set of lines and points is represented by a corresponding open region of $\mathbb{R}^2$: $(\vartheta^1, \vartheta^2) \in \mathscr{R} \subseteq \mathbb{R}^2$. Let us also choose an arbitrary metric $g_{\alpha\beta}$ on K^γ. (The determinant g of $g_{\alpha\beta}$ may vanish or diverge on the boundaries of $\mathscr{R}$ but not inside $\mathscr{R}$.) Consider now the normalized eigenfunctions $Y_n(\vartheta^\alpha)$ of the laplacian. (The Y_n can be used for a generalized harmonic expansion on K^γ.)

$$D^\alpha D_\alpha Y_n = -m_n^2 Y_n, \tag{5.15}$$

$$\int_{\mathscr{R}} d^2\vartheta\, g^{1/2} g^{\alpha\beta} \partial_\alpha Y_n \partial_\beta Y_{n'}^* = m_n^2 \delta_{nn'}. \tag{5.16}$$

To be properly defined on K^γ the Y_n must fulfill certain conditions (periodicity etc.) on the boundaries of $\mathscr{R}$. By a suitable change of coordinates $\vartheta \to \vartheta'(\vartheta)$ the metric can always be made proportional to the unit matrix in the whole image $\mathscr{R}'$ of $\mathscr{R}$

$$g'_{\alpha\beta}(\vartheta') = \exp\varphi(\vartheta')\,\delta_{\alpha\beta}. \tag{5.17}$$

The new functions $Y_n'(\vartheta') = Y_n(\vartheta(\vartheta'))$ obey

$$\int_{\mathscr{R}'} d^2\vartheta'\, g'^{1/2} g'^{\alpha\beta}\, \partial_\alpha Y_n' (\partial_\beta Y_{n'}')^* = \int_{\mathscr{R}'} d^2\vartheta'\, \delta^{\alpha\beta}\, \partial_\alpha Y_n' (\partial_\beta Y_{n'}')^*$$
$$= m_n^2\, \delta_{nn'}. \tag{5.18}$$

We next define a complex coordinate $z' = \vartheta^{1\prime} + i\vartheta^{2\prime}$ and note that the form (5.17) of the metric is preserved by analytic transformations. We use them to map $\mathscr{R}'$ into the interval I^2 parametrized by σ^α $(z' \to z = \sigma^1 + i\sigma^2, -\pi < \sigma^\alpha < \pi)$, where we allow that certain lines or points are cut out from I^2. The functions

$$\tilde{Y}_n(\sigma) = Y_n[\vartheta(\vartheta'(\sigma))], \tag{5.19}$$

can now be interpreted as functions on the torus. Similar to the Fourier harmonics they obey

$$\int_{T^2} d^2\sigma\, \delta^{\alpha\beta}\, \partial_\alpha \tilde{Y}_n\, \partial_\beta \tilde{Y}_{n'}^* = m_n^2\, \delta_{nn'}. \tag{5.20}$$

On the boundary of I^2, however, the functions $\tilde{Y}_n$ inherit the boundary conditions implied by the topology of K^γ rather than the standard periodicity conditions on the torus. Similarly, if $\mathscr{R}'$ is mapped onto I^2, except some points or lines in the interior of the interval, the $\tilde{Y}_n$ may be discontinuous, not differentiable or subject to boundary conditions on these points and lines. Nevertheless, the $\tilde{Y}_n$ are contained in the functional measure $\mathscr{D}X(\sigma)$ for $M \to \infty$. Again their Fourier expansion will involve infinitely many $z^i_{\{n\}}$. For the infinite system the functional measure effectively includes a sum over topologies with arbitrary genus! This partly generalizes to $p > 2$ where the functional measure also effectively includes a sum over different topologies. Unlike in two dimensions, we are not guaranteed, however, that by a suitable choice of coordinates $\sigma(\vartheta)$ we can transform an arbitrary metric into the form $g^{1/2} g^{\alpha\beta} = \delta^{\alpha\beta}$ on I^p. The functions contained in eq. (5.11) can only be properly represented on metric spaces for which this is the case.

For $p = 2$ the action (5.12) describes an euclidean version of the bosonic string theory [2]. For the critical dimension, string theories are believed to lead to an interesting consistent quantum theory for matter propagating in space and time. Since, for the infinite system, the functional measure (5.7) already contains a sum over topologies with arbitrary genus, one may speculate that the appropriate weight factors for integrals over Teichmüller parameters [4] are generated automatically. One is tempted to propose the sequence $M \to \infty$ defined by eqs. (5.3) and (5.7) as a nonperturbative definition of the bosonic string theory. Of course this has to be supplemented by a specification of suitably regularized operators (vertex operators etc.). Based on the two-dimensional fermion – boson equivalence there are good prospects for a generalization which includes fermions. We observe that we have identified a huge symmetry sgen_∞ which is much larger than the symmetries of string theories discussed so far. The implications of this symmetry need to be explored. One may hope that they lead to an understanding of spacetime symmetries in string theories. Nevertheless, our setting is more general than string theories. A systematic classification of structures between operators which could represent space, time, spinors etc. would be of great value. Within general statistics the next step has to be done in this direction.

In conclusion we propose a theory, general statistics, whose only ingredients are its statistical character and the infinite number of degrees of freedom. In this paper our main emphasis concerns the conceptual layout, the uniqueness and the symmetries of the theory. All quantities are well defined and the expectation values of operators are finite by definition. So far, fundamental unification searched for the "right theory." In our approach the theory is unique and the task is rather to find the "right operators," namely to find those structures which permit a correct identification of operators with observables. This task is very complex and it is a priori not guaranteed that concepts like time or spinors are contained in our formulation. The fact that our approach contains the bosonic string theory gives us hope that physics can indeed be described by the extremely simple basic concepts of general statistics. We are fully aware, however, that there is still a long way to go before making direct contact with experiment and observation.

Appendix A

In this appendix we argue that the topology of the manifold parametrized by the variable s is irrelevant. For an N-dimensional manifold K^N with arbitrary topology and arbitrary probability density $p(s)$ we can always find a mapping on an equivalent system with new variables t parametrizing $\mathbb{R}^N$. The new system is characterized by a probability density $p'(t)$. Every operator of the original system is mapped into an operator of the new system with equal expectation value. Structures between operators are therefore preserved.

First we note that we can remove a finite set of points from $\mathbb{R}$ without changing the expectation values of operators. As discussed in sect. 2 the operators $v(s)$ need not be continuous functions. We also admit functions defined everywhere on $\mathbb{R}$ except a finite set of points (for example with poles) provided the integral $\langle v\rangle_p$ exists. There are therefore no additional constraints on $v(s)$ for the resulting open intervals. We can arbitrarily "cut" $\mathbb{R}$ into pieces or "glue" pieces together. Similarly we can remove from $\mathbb{R}^N$ any arbitrary hypersurface with dimension smaller or equal $N-1$. (For the limit where $v(s)$ becomes a distribution one should first perform the cuts and then perform the limit in function space.) Consider now a system defined on a finite interval, for example $|s| \leqslant 1$. Again, we can cut out a finite number of points, in particular the boundary points. The interval becomes then open and has the same topology as $\mathbb{R}$. To map the open interval onto $\mathbb{R}$ we can use any surjective map $\mathrm{I} \to \mathbb{R}$, for example

$$t = \mathrm{tang}\left(\tfrac{1}{2}\pi s\right) = f^{-1}(s)\,. \tag{A.1}$$

The probability density of the new system is

$$p'(t) = \frac{\mathrm{d}s}{\mathrm{d}t} p(f(t))\,. \tag{A.2}$$

(We require $\mathrm{d}s/\mathrm{d}t > 0$.) In our example a constant density on I induces on $\mathbb{R}$

$$p(s) = 1 \to p'(t) = \frac{2}{\pi}(1+t^2)^{-1}\,. \tag{A.3}$$

We now can map every operator $v(s)$ on an operator $v'(t)$

$$v(s) \to v'(t) = v(f(t))\,, \tag{A.4}$$

with $\langle v'\rangle_{p'} = \langle v\rangle_p$. Since the inverse map $\mathbb{R} \to \mathrm{I}$ is also defined the two systems are equivalent.

This generalizes easily. For the topology of the circle S^1 we first cut out a point so that it becomes an open interval I. Since the operators $v(s)$ are not necessarily continuous there are no additional constraints on the $v(s)$ allowed I. (Only for continuous periodic functions on S^1 should the functions on I have the same value on both endpoints.) The interval I is then mapped onto $\mathbb{R}$. Every N-dimensional manifold K^N with arbitrary topology is reduced to an N-dimensional open interval I^N (with topology $\mathbb{R}^N$) by cutting or gluing appropriate lower dimensional manifolds. There are always suitable functions mapping I^N onto $\mathbb{R}^N$. (For spheres we can cut out the poles and use standard projective maps.) We finally should mention that there is no need to restrict this procedure to manifolds. We may also consider "manifolds with edges," like the cube, which are often called orbifolds [5]. By cutting out the edges the orbifolds are completely equivalent to the "noncompact manifolds" considered in ref. [6]. Like any other manifold they are equivalent to $\mathbb{R}_N$.

It may sometimes be useful to apply the inverse procedure and map $\mathbb{R}^N$ on a compact space (e.g. S^N). Depending on the map and the behavior of $p(s)$ at the boundary of $\mathbb{R}^N$ the new density $p'(t)$ may have zeroes or poles at certain isolated points (or hypersurfaces with dimension $\leqslant N-1$) even for $p(s)$ strictly positive and finite for all $s \in \mathbb{R}^N$. This is not problematic in view of the regularization by sequences p_n discussed in sect. 2. It also demonstrates that a general restriction to continuous nonzero $p(s)$ is not justified.

Appendix B

In this appendix we show that an arbitrary probability distribution $p'(s)$ can be obtained from a given arbitrary $p(s)$ through a proper variable transformation f

$$p'(s) = \hat{f} p(f(s)), \tag{B.1}$$

provided $Z = \int ds^N p(s) = \int ds^N p'(s) < \infty$. Both p and p' are nonvanishing, finite, positive functions $\mathbb{R}^N \to \mathbb{R}_+$ which are continuous and $k-1$ times differentiable ($k \geqslant 1$) in the whole $\mathbb{R}^N$. The function f must be a continuous, k times differentiable, invertible one to one map $\mathbb{R}^N \to \mathbb{R}^N$ with positive Jacobian $\hat{f} > 0$. Let us first demonstrate this for $N = 1$. According to eq. (B.1) we have to find a function $f(s)$ with positive derivative $df/ds > 0$, so that

$$\frac{df}{ds} = \frac{p'(s)}{p(f(s))}. \tag{B.2}$$

Eq. (B.2) defines a differential equation for f which is solved by introducing the integrals $z(s)$, $z'(s)$ fulfilling

$$\frac{dz}{ds} = p, \qquad \frac{dz'}{ds} = p'. \tag{B.3}$$

Then the function f is found in terms of the inverse of z

$$f(s)=z^{-1}(z'(s)).\tag{B.4}$$

Eq. (B.3) defines z and z' only up to a constant which is fixed* by the requirement that f is a one-to-one map $\mathbb{R}\to\mathbb{R}$

$$\lim_{s\to\infty} f(s)\to\infty \quad\Rightarrow\quad \lim_{s\to\infty} z(s)=\lim_{s\to\infty} z'(s),$$
$$\lim_{s\to-\infty} f(s)\to-\infty \quad\Rightarrow\quad \lim_{s\to-\infty} z(s)=\lim_{s\to-\infty} z'(s).\tag{B.5}$$

Note that the inverse of eq. (B.5), as well as the existence of z^{-1}, follows from the fact that z and z' are monotonically increasing ($p, p'>0$). Eq. (B.5) gives two conditions for the difference of the two integration constants in eq. (B.3). They are compatible only if the partition function Z is the same for both densities ($Z=z(\infty)-z(-\infty)=z'(\infty)-z'(-\infty)$). Indeed, Z may be viewed as the total volume corresponding to the density p which is of course invariant with respect to gen_1. The positivity and finiteness of the "determinant" $\mathrm{d}f/\mathrm{d}s$ follow from the positivity of p and p'. (For arbitrary functions p and p' the difference in the number of zeroes would be a gen_1 invariant.) Differentiability of $f(s)$ follows from differentiability of p, p' and $p>0$. As an example, the two densities

$$p(s)=\tfrac{1}{2}\cosh^{-2}s,\qquad p'(s)=\frac{1}{\pi}(1+s^2)^{-1},\tag{B.6}$$

are related by the transformation

$$f(s)=\tfrac{1}{2}\ln\frac{1+(2/\pi)\arctan s}{1-(2/\pi)\arctan s}.\tag{B.7}$$

For a generalization to arbitrary N we first note that an arbitrary probability density of the form

$$p(s)=b_i(s^1,s^2,\dots,s^{i-1},s^i)\,a_{i+1}(s^{i+1})\,a_{i+2}(s^{i+2})\dots a_N(s^N),\tag{B.8}$$

can be transformed by gen_N into

$$\tilde{p}(s)=b_{i-1}(s^1,s^2,\dots,s^{i-1})\,a_i(s^i)\,a_{i+1}(s^{i+1})\dots a_N(s^N).\tag{B.9}$$

* A common additive constant for z and z' is irrelevant.

Here b_{i-1} is obtained from b_i by

$$\int ds^i b_i(s^1, s^2, \ldots, s^i) = b_{i-1}(s^1, s^2, \ldots, s^{i-1}) \int ds^i a_i(s^i) . \tag{B.10}$$

This follows from the case $N=1$ by treating all s^u except s^i as fixed parameters. We use a transformation $f^i = f^i(s^1, \ldots, s^i), f^u = x^u$ for $u \neq i$ to bring the functional dependence of p on $d\mathrm{sx}^i$ into an arbitrary form $\sim a_i(s^i)$ for all values of the "parameters." The s^i dependence can then be factored out. We can therefore pick two specific transformations f_1, f_2

$$f_1(p) = p_1 , \qquad f_2(p') = p_2 , \tag{B.11}$$

with p_1 and p_2 probability distributions in the direct product form

$$p_{1,2} = \prod_{j=1}^{N} a_j^{1,2}(s^j) . \tag{B.12}$$

These can be treated in complete analogy with the $N=1$ case and there exists always a proper variable transformation f_3 establishing a one-to-one correspondence between arbitrary p_1 and p_2 with $Z(p_1) = Z(p_2)$. The transformation $f_2^{-1} f_3 f_1$ maps p onto p'. (Of course this transformation and the choice of f_1, f_2, f_3 are not unique. A symmetry transformation can be applied at every step without changing the result $p \to p'$. Also the choice of p_1 and p_2 is arbitrary.)

Appendix C

In this appendix we show that the symmetries sgen_N for two probability densities $p(s)$ and $p'(s)$ are isomorphic. We also extend the symmetry $\mathrm{sgen}_N(p)$ to a symmetry $\mathrm{gen}_N(p)$. According to eq. (3.15) the embedding of sgen_N into gen_N depends on p since it consists of transformations leaving a given probability density p invariant. The proper variable transformations induce an isomorphism between $\mathrm{sgen}_N(p)$ and $\mathrm{sgen}_N(p')$ if $Z(p) = Z(p')$ with finite Z. Indeed, if we pick a suitable proper variable transformation $f(p) = p'$ it establishes a one-to-one correspondence between $g \in \mathrm{sgen}_N(p)$ and $g' \in \mathrm{sgen}_N(p')$ by $g(p) = p$, $g' = fgf^{-1}$, $g'(p') = p'$. (For $Z(p') = \alpha Z(p)$ we first multiply p' with α in order to establish the isomorphism.)

In addition to the proper-variable transformations contained in gen_N there are other transformations whose action on the density p and the operators v can be defined such that $\langle v' \rangle_{p'} = \langle v \rangle_p$. These transformations can also be used to relate different probability densities (with different Z). The simplest example is a constant rescaling of p

$$p \to p' = \alpha p , \qquad v \to v' = v ,$$
$$Z \to Z' = \alpha Z . \tag{C.1}$$

This is a special case of a more general transformation which is linked to the fact that the splitting of the product vp into v and p is not unique. Consider an infinitely often differentiable operator $w(x)>0$, i.e. $\langle w\rangle_p \in \mathbb{R}_+$. We can then define the transformation

$$p' = pw \equiv w(p),$$
$$v' = v\frac{\langle w\rangle_p}{w} \equiv w(v),$$
$$Z' = \langle w\rangle_p Z. \tag{C.2}$$

The inverse transformation

$$w^{-1}(p') = \frac{p'}{w} = p,$$
$$w^{-1}(v') = v'w\left\langle \frac{1}{w}\right\rangle_{p'} = v, \tag{C.3}$$

is always defined for the density $p' = pw$ but not necessarily for p. We can use the *scalings* w to construct a one-to-one map between operators in the system with probability density p and operators in a system with p' provided the ratio

$$\frac{Z(p')}{Z(p)} = \left\langle \frac{p'}{p}\right\rangle_p, \tag{C.4}$$

is finite and does not vanish ($w = p'/p$). The scalings induce another isomorphism between the symmetries of p and p'. For an arbitrary symmetry $g, g(p) = p, \langle g(v)\rangle_p = \langle v\rangle_p$, the transformation $g' = wgw^{-1}$ is a symmetry of $p' = w(p)$ ($g'(p') = p', \langle g'(v')\rangle_{p'} = \langle v'\rangle_{p'}$). For $\langle p'/p\rangle_p \in \mathbb{R}_+$ the scalings w establish again the equivalence of p and p'.

How are the isomorphisms of symmetries generated by scalings w related to those from proper-variable transformations? Let g' be an element of $\mathrm{sgen}_N(p')$. For arbitrary $p(s)$ and $p'(s)$ with $w(s) = p'(s)/p(s)(Z(p')/Z(p) \in \mathbb{R}_+)$ the transformation $g_w = w^{-1}g'w$ is a symmetry of p but it is in general *not* contained in gen_N. An operator $v(s)$ transforms as

$$g_w(v) = \frac{w(s)}{w(g'(s))}v(g'(s)),$$
$$g_w(p) = p, \tag{C.5}$$

or, using the fact that g' is an element of $\mathrm{sgen}_N(p')$,

$$g_w(v)=\hat{g}'\frac{p(g'(s))}{p(s)}v(g'(s)).\tag{C.6}$$

Indeed, we can use a combination of variable transformations and scalings to extend the symmetry of a probability density p from $\mathrm{sgen}_N(p)$ to $\mathrm{gen}_N(p)$. The transformation group $\mathrm{gen}_N(p)$ is isomorphic to gen_N but the transformation law of p and v corresponding to a given function $f(s)$ differs from gen_N

$$f_p(p)=p'=p,$$

$$f_p(v)=v'=\frac{\hat{f}p(f(s))}{p(s)}v(f(s)).\tag{C.7}$$

The groups gen_N and $\mathrm{gen}_N(p)$ have a common subgroup $\mathrm{sgen}_N(p)$. The symmetries in $\mathrm{gen}_N(p)/\mathrm{sgen}_N(p)$ are associated via eqs. (C.5) and (C.6) to symmetries $\mathrm{sgen}_N(p')$ for suitable p'.

The transformations in $\mathrm{gen}_N(p)$ are not the only possible symmetries. Any transformation $p\to p$, $v\to v'$ with

$$\int \mathrm{d}s^N pv=\int \mathrm{d}s^N pv',\tag{C.8}$$

may be called a symmetry and $\mathrm{gen}_N(p)$ is only a subset of all symmetry transformations $\mathrm{sym}_N(p)$ defined by eq. (C.8).

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